Freezing effect in a network of oscillators coupled to thermostats of finite energy

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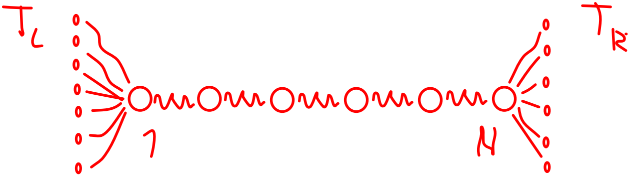


$(q, p) \in \mathbb{R}^{2N}$ — coordinates and moments of the oscillators from the chain

• Initial conditions of the thermostats are distributed accordingly to the Gibbs measure with temperatures $T_L > 0$ and $T_R > 0$:

$$\mathcal{D}(q,p)(t) \to \mu \quad \text{as} \quad t \to \infty$$

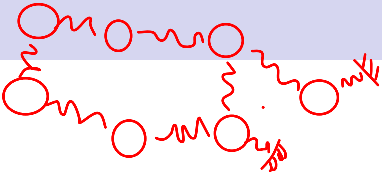
where μ is a unique stationary measure of the system (Eckmann, Pillet, Rey-Bellet '99). **Exchange of energy** between thermostats in the stationary regime. Chain is a **conductor**.

• Initial conditions are deterministic and such that the initial energies of the thermostats satisfy $\mathcal{E}_L(0) < \infty$ and $\mathcal{E}_R(0) < \infty$:

$$(q,p)(t) \to (q^{\pm}, 0) \quad \text{as} \quad t \to \pm\infty$$

where $q^{\pm}$ are critical points of some effective potential.

Freezing effect: $\mathcal{E}_L(t) \to \mathcal{E}_L^{\infty}$, $\mathcal{E}_R(t) \to \mathcal{E}_R^{\infty}$ as $t \to \infty$, where $\mathcal{E}_L^{\infty} \neq \mathcal{E}_R^{\infty}$.



Oscillators are situated in vertices of a finite undirected graph $\mathcal{G}$.

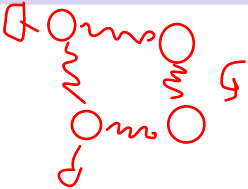
$$\mathcal{L}^O(q,\dot{q}) = \sum_{j\in\mathcal{G}} \Big(\frac{\dot{q}_j^2}{2} - U_j(q_j)\Big) - \frac{1}{2}\sum_{i,j\in\mathcal{G}:\, i\sim j} V_{ij}(q_i - q_j),$$

where $(q,\dot{q}) = (q_j,\dot{q}_j)_{j\in\mathcal{G}} \in \mathbb{R}^{2|\mathcal{G}|}$, U_j, V_{ij} are smooth real functions and $V_{ij}(q_i - q_j) = V_{ji}(q_j - q_i)$.

Equations of motion:

$$\ddot{q}_j = -U_j'(q_j) + \sum_{i\in\mathcal{G}:\, i\sim j} V_{ij}'(q_i - q_j), \qquad (q_j,\dot{q}_j)(0) = (q_{0j},\dot{q}_{0j}).$$

Thermostats



Oscillators from a set

$$\Lambda \subset \mathcal{G}$$

are coupled with thermostats.

Each thermostat is an infinite dimensional linear Lagrangian system given by a continual collection of independent harmonic oscillators parametrized by their internal frequency ν:

$$\mathcal{L}_m^T(\xi_m, \dot{\xi}_m) = \frac{1}{2}\int_{-\infty}^{\infty} \dot{\xi}_m^2(\nu) - \nu^2 \xi_m^2(\nu)\, d\nu, \qquad m \in \Lambda,$$

where $\xi_m(\nu, t), \dot{\xi}_m(\nu, t) \in \mathbb{R}$ — coordinate and velocity of the oscillator with internal frequency ν.

Equations of motion:

$$\ddot{\xi}_m(\nu, t) = -\nu^2 \xi_m(\nu, t), \quad (\xi_m, \dot{\xi}_m)(\nu, 0) = (\xi_{0m}, \dot{\xi}_{0m})(\nu), \quad \nu \in \mathbb{R}$$

The coupling between oscillators and thermostats is linear and given by interaction potentials

$$V_m^{int}(q_m, \xi_m) = -q_m \int_{-\infty}^{\infty} \kappa_m(\nu)\xi_m(\nu)\, d\nu, \qquad m \in \Lambda,$$

where κ_m are real smooth functions, decaying at infinity sufficiently fast.

Lagrangian:

$$\mathcal{L}(q,\xi,\dot{q},\dot{\xi}) = \mathcal{L}^O(q,\dot{q}) + \sum_{m\in\Lambda} \mathcal{L}^T_m(\xi_m,\dot{\xi}_m) - \sum_{m\in\Lambda} V^{int}_m(q_m,\xi_m),$$

Equations of motion:

$$\ddot{q}_j = -U'_j(q_j) + \sum_{i\in\mathcal{G}\colon i\sim j} V'_{ij}(q_i - q_j) + \delta_{j\Lambda}\int_{-\infty}^{\infty} \kappa_j(\nu)\xi_j(\nu)\,d\nu,$$

$$\ddot{\xi}_m(\nu) = -\nu^2\xi_m(\nu) + \kappa_m(\nu)q_m, \qquad m\in\Lambda,\ \nu\in\mathbb{R},\ j\in\mathcal{G},$$

$$(q,\xi,\dot{q},\dot{\xi})(0) = (q_0,\xi_0,\dot{q}_0,\dot{\xi}_0).$$

- *For any $m \in \Lambda$ we have $\kappa_m(\nu) = 0 \Leftrightarrow \nu = 0$.*

Let $K_m := \int_{-\infty}^{\infty} \frac{\kappa_m^2}{\nu^2}\, d\nu < \infty$.
An **effective potential**:

$$V^{eff}(q) := \sum_{j \in \mathcal{G}} U_j(q_j) + \frac{1}{2} \sum_{i \sim j} V_{ij}(q_i - q_j) - \sum_{m \in \Lambda} \frac{K_m q_m^2}{2}$$

- *The effective potential V^{eff} has only isolated critical points and satisfies $\left|V^{eff}(q)\right| \to \infty$ as $|q| \to \infty$.*

The energy of the m-th thermostat:

$$\mathcal{E}_m(\xi, \dot{\xi}) := \frac{1}{2} \int_{\infty}^{\infty} \dot{\xi}_m^2(\nu) + \nu^2 \xi_m^2(\nu)\, d\nu.$$

- *The initial energy of thermostats is finite: $\mathcal{E}(\xi_0, \dot{\xi}_0) < \infty$.*

Proposition.
The system has a unique solution $q(t), \xi(\nu, t)$ and this solution is defined for all $t \in \mathbb{R}$.
The total energy $E(q, \xi, \dot{q}, \dot{\xi})(t) = E(q_0, \xi_0, \dot{q}_0, \dot{\xi}_0) < \infty$.
The functions $q_k(t)$, $\mathcal{E}_m(q, \xi, \dot{q}, \dot{\xi})(t)$ are bounded.

Proof: local existence theorem + apriori bounds

Total energy:

$$E(q, \xi, \dot{q}, \dot{\xi}) = \sum_{j \in \mathcal{G}} \frac{\dot{q}_j^2}{2} + \sum_{m \in \Lambda} \int_{-\infty}^{\infty} \frac{\dot{\xi}_m^2}{2}\, d\nu + V^{eff}(q)$$
$$+ \sum_{m \in \Lambda} \int_{-\infty}^{\infty} \frac{\nu^2}{2} \Big(\xi_m - \frac{\kappa_m q_m}{\nu^2} \Big)^2 d\nu.$$

We construct a set $\Lambda_\Gamma \subset \mathcal{G}$ by inductive procedure:

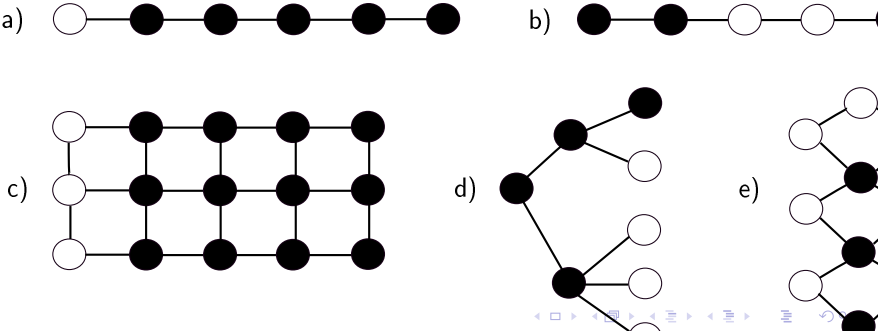
– $\Lambda_\Gamma := \Lambda$

– Λ_Γ^1 consists of vertices $j \in \Lambda_\Gamma$ for which there exists a *unique* vertex $n(j) \in \mathcal{G} \setminus \Lambda_\Gamma$ adjacent to j

– Add the vertices $n(j)$ to $\Lambda_\Gamma := \cup_{j \in \Lambda_\Gamma^1} n(j) \cup \Lambda_\Gamma$

– Iterate the procedure until $\Lambda_\Gamma^1 = \emptyset$.

Assumption: *We have* $\Lambda_\Gamma = \mathcal{G}$.



Theorem. *Under the assumptions above,*

$$q(t) \to q_c^{\pm} \quad as \quad t \to \pm\infty,$$

where $q_c^{\pm}$ are critical points of the effective potential V^{eff}. Moreover,

$$\frac{d^k}{dt^k} q(t) \to 0 \quad as \quad t \to \pm\infty$$

for any $k \geq 1$.

• In various (but different) infinite dimensional Hamiltonian systems similar effects where found by Al. Komech and al. via different methods.
• D. Treschev'10: one thermostat + one oscillator (linear or nonlinear).
• A.D.'12: one thermostat + n linear oscillators
• S. Saulin'17: n thermostats + n linear of nonlinear oscillators

Our assumption on Λ coincides with that from the paper by Cuneo, Eckmann, Hairer, Rey-Bellet on networks of oscillators coupled to thermal baths.

Thermostats absorb the energy to fast Fourier modes of the functions $\xi_m(\cdot, t)$ as $t \to \infty$. The coupling therms $\int_{\infty}^{\infty} \kappa_m(\nu)\xi(\nu, t)\, d\nu$ average out.

In the Fourier space: radiation to infinity.

• Assume for the moment that $q_m(t) = \sin(\lambda t)$ or $\cos(\lambda t)$ for some $m \in \Lambda$ and $\lambda \neq 0$. Since $\kappa(\lambda) \neq 0$, in eq.

$$\ddot{\xi}_m(\nu) = -\nu^2 \xi_m(\nu) + \kappa_m(\nu) q_m \qquad (1)$$

with $\nu = \lambda$ we observe a parametric resonance – *contradicts to the energy conservation!*

$\Rightarrow$ the functions $q_m(t)$ with $m \in \Lambda$ cannot have "components, oscillating with any frequency $\lambda \neq 0$". This rough idea lead to

$$\frac{d^k}{dt^k} q_m(t) \to 0 \quad \text{as} \quad t \to \pm\infty$$

for any $k \geq 1$ and any $m \in \Lambda$.

- Expressing $\xi(\nu, t)$ from (1) via the Duhamel formula we prove that

$$\int_{-\infty}^{\infty} \kappa_m(\nu)\xi_m(\nu)\,d\nu = K_m q_m(t) + \theta_m(t),$$

where $\frac{d^k}{dt^k}\theta_m(t) \to 0$ for any $k \geq 0$.

- Inserting this formula into equation for q_m, we find

$$\ddot{q}_m = -U'_m(q_m) + \sum_{i \in \mathcal{G}:\, i \sim m} V'_{im}(q_i - q_m) + K_m q_m + \theta_m.$$

- In the case when $\Lambda = \mathcal{G}$ we conclude the proof by sending $t \to \infty$.
- In general case we "thermalize"oscillators one by one:

$$\dddot{q}_m = -U''_m(q_m)\dot{q}_m + \sum_{i \in \mathcal{G}:\, i \sim m} V''_{im}(q_i - q_m)(\dot{q}_i - \dot{q}_m) + K_m \dot{q}_m + \dot{\theta}_m.$$

There is a unique $i \sim m$ which does not belong to Λ. Then $V''_{im}(q_i - q_m)\dot{q}_i \to 0 \Rightarrow \dot{q}_i \to 0 \Rightarrow \dot{q}_i^{(k)} \to 0$ for any $k \geq 1$.

Duhamel formula $\Rightarrow$

$$\xi_m(\nu, t) = \xi_m^0(\nu, t) + \xi_m^1(\nu, t),$$

where

$$\xi_m^0(\nu, t) := \Re\Big(\frac{1}{i\nu}\boldsymbol{\xi}_{0m}e^{i\nu t}\Big), \quad \boldsymbol{\xi}_{0m} := \dot{\xi}_{0m} + i\nu\xi_{0m},$$

and

$$\xi_m^1(\nu, t) := \Re\Big(\frac{\kappa_m}{i\nu}\hat{q}_m^t e^{i\nu t}\Big),$$

where $q_m^t := q_m \mathbb{I}_{[0,t]}$ if $t \geq 0$, $q_m^t := -q_m \mathbb{I}_{[t,0]}$ if $t < 0$, and $\hat{q}_m^t := \mathcal{F}(q_m^t)$.
Direct computation $\Rightarrow$

$$\mathcal{E}_m(t) = \|\kappa_m \hat{q}_m^t + \boldsymbol{\xi}_{0m}\|_{L^2}^2 < const$$

since $\mathcal{E}_m(t)$ is bounded. Then

$$\kappa_m \hat{q}_m \in L_2.$$