Discrete quasi-infinitely divisible distributions

I.A. Alexeev

Saint-Petersburg State University
Department of probability theory and mathematical statistics

December 23, 2020

Summary

Introduction

Lévy-Khintchine representation

$$f(t) = \exp\left\{it\gamma - \frac{a^2t^2}{2} + \int\limits_{\mathbb{R}\setminus\{0\}} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\nu(dx)\right\}.$$

Infinity divisible: $\displaystyle\int\limits_{\mathbb{R}\setminus\{0\}} \min(x^2, 1)\nu(dx) < \infty.$

Quasi-infinity divisible: $\nu|_{\mathbb{R}\setminus(-r,r)}$ is a finite signed measure,

$$\int\limits_{\mathbb{R}\setminus\{0\}} \min(x^2, 1)|\nu|(dx) < \infty, \quad |\nu|(A) = \sup \sum_{j=1}^{k} \nu(A_j), \quad A \subset \bigcup_{j=1}^{k} A_j.$$

Introduction

Lévy-Khintchine representation

$$f(t) = \exp\left\{it\gamma - \frac{a^2t^2}{2} + \int\limits_{\mathbb{R}\setminus\{0\}} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\nu(dx)\right\}.$$

Infinity divisible: $$\int\limits_{\mathbb{R}\setminus\{0\}} \min(x^2, 1)\nu(dx) < \infty.$$

Quasi-infinity divisible: $\nu|_{\mathbb{R}\setminus(-r,r)}$ is a finite signed measure,

$$\int\limits_{\mathbb{R}\setminus\{0\}} \min(x^2, 1)|\nu|(dx) < \infty, \quad |\nu|(A) = \sup \sum_{j=1}^{k} \nu(A_j), \quad A \subset \bigcup_{j=1}^{k} A_j.$$

Lévy-Khintchine representation

$$f(t) = \exp\left\{it\gamma - \frac{a^2t^2}{2} + \int\limits_{\mathbb{R}\setminus\{0\}} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\nu(dx)\right\}.$$

Infinity divisible: $\int\limits_{\mathbb{R}\setminus\{0\}} \min(x^2, 1)\nu(dx) < \infty.$

Quasi-infinity divisible: $\nu|_{\mathbb{R}\setminus(-r,r)}$ is a finite signed measure,

$$\int\limits_{\mathbb{R}\setminus\{0\}} \min(x^2, 1)|\nu|(dx) < \infty, \quad |\nu|(A) = \sup \sum_{j=1}^{k} \nu(A_j), \quad A \subset \bigcup_{j=1}^{k} A_j.$$

Example

$$f(t) = p + (1-p)e^{it} = p\left(1 + \frac{1-p}{p}e^{it}\right), \quad p > \frac{1}{2} \Rightarrow \frac{1-p}{p} < 1.$$

$$\begin{aligned} \ln f(t) &= \ln p + \ln\left(1 + \frac{p}{1-p}e^{it}\right) \\ &= \ln p + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{1-p}{p}\right)^k e^{itk} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{1-p}{p}\right)^k \left(e^{itk} - 1\right). \end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1-p}{p}\right)^k = -\ln\left(1 - \frac{1-p}{p}\right) < \infty.$$

Example

$$f(t) = p + (1-p)e^{it} = p\left(1 + \frac{1-p}{p}e^{it}\right), \quad p > \frac{1}{2} \Rightarrow \frac{1-p}{p} < 1.$$

$$\begin{aligned}
\ln f(t) &= \ln p + \ln\left(1 + \frac{p}{1-p}e^{it}\right) \\
&= \ln p + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{1-p}{p}\right)^k e^{itk} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{1-p}{p}\right)^k \left(e^{itk} - 1\right).
\end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1-p}{p}\right)^k = -\ln\left(1 - \frac{1-p}{p}\right) < \infty.$$

Example

$$f(t) = p + (1-p)e^{it} = p\left(1 + \frac{1-p}{p}e^{it}\right), \quad p > \frac{1}{2} \Rightarrow \frac{1-p}{p} < 1.$$

$$\begin{aligned} \ln f(t) &= \ln p + \ln\left(1 + \frac{p}{1-p}e^{it}\right) \\ &= \ln p + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\frac{1-p}{p})^k e^{itk} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{1-p}{p}\right)^k (e^{itk} - 1). \end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1-p}{p}\right)^k = -\ln\left(1 - \frac{1-p}{p}\right) < \infty.$$

Distributions concentrated on integers

Theorem(Lindner, Pan, Sato), 2018

Let f be a c.f. of distribution concentrated on integers. Then f is quasi-infinitely divisible iff $f(t)$ does not have zeros and, moreover,

$$\operatorname{Ln} f(t) = itk + \sum_{n \in \mathbb{Z} \setminus \{0\}} \lambda_n (e^{itn} - 1), \quad k \in \mathbb{Z}. \tag{1}$$

$$\operatorname{Ln} f_m(t) = itk_m + \sum_{n\in\mathbb{Z}\setminus\{0\}} \lambda_{m,n}\big(e^{itn} - 1\big), \quad k \in \mathbb{Z},$$

$$\operatorname{Ln} f(t) = itk + \sum_{n\in\mathbb{Z}\setminus\{0\}} \lambda_{n}\big(e^{itn} - 1\big), \quad k \in \mathbb{Z},$$

Theorem(Lindner, Pan, Sato), 2018

Let $f_m(t)$, $f(t)$ do not have zeros. Then f_m converges weakly to f iff

(i) $k_m \to k$ as $m \to \infty$;

(ii) $\lim\limits_{m\to\infty} \sum\limits_{n\in\mathbb{Z}} |\lambda_{n,m} - \lambda_n| = 0.$

Sufficient condition

$$f(t)=\sum_{n=1}^{\infty}p_n e^{itx_n},\quad x_n\in\mathbb{R},\quad p_n\geqslant 0,\quad \sum_{n=1}^{\infty}p_n=1.$$

Theorem(Khartov and A.)

If $\inf\limits_{t\in\mathbb{R}}|f(t)|>0$ then f is quasi-infinitely divisible and, moreover,

$$f(t)=\exp\Big\{it\gamma+\sum_{x\in X\setminus\{0\}}\lambda_x(e^{itx}-1)\Big\},$$

where $X=\mathcal{L}(x_1,x_2,\ldots)$, $\gamma\in X$, $\lambda_x\in\mathbb{R}$, $\sum\limits_{x\in X}|\lambda_x|<\infty$.

Necessary condition

Theorem(Khartov and A.)

Consider the following c.f.

$$f(t) = \sum_{n=1}^{N} p_n e^{itx_n}, \quad p_n \geqslant 0, \quad \sum_{n=1}^{N} p_n = 1.$$

If f is quasi-infinitely divisible then $\inf_{t\in\mathbb{R}} |f(t)| > 0$.

Corollary

Consider the c.f. from previous theorem. Then f is quasi-infinitely divisible iff $\inf_{t\in\mathbb{R}} |f(t)| > 0$.

Definitions

$$f_n(t) = \sum_{m=1}^{\infty} p_{n,m} e^{itx_m} = \exp\left\{ it\gamma_n + \sum_{x\in X} \lambda_{n,x}\left(e^{itx} - 1\right)\right\},$$

Convergence in variation

$f_n \xrightarrow{var} f$ if $\sum_{m=1}^{\infty} |p_{n,m} - p_m| \to 0.$

Convergence with shift in variation

$f_n \xrightarrow{s.var} f$ if $f_n e^{-ita_n} \xrightarrow{var} f$, $a_n \to 0$.

Example

$f_n(t) = e^{ita_n}$, $f(t) = 1$, $a_n \to 0$. Then $f_n \not\xrightarrow{var} f$, but $f_n \xrightarrow{s.var} f$.

Convergence

We assume that

$$\sup_{n\in\mathbb{N}} \sum_{x\in X} |\lambda_{n,x}| < \infty. \qquad (2)$$

Theorem(Khartov and A.)

If (2) holds then the following statements are equivalent:

(i) $f_n \xrightarrow{s.var} f,\ n \to \infty.$

(ii) $\gamma_n \to \gamma$ and $\hat{f}_n \xrightarrow{var} f,\ n \to \infty,\ \hat{f}_n(t) = f_n(t)e^{-it(\gamma_n-\gamma)},\ t \in \mathbb{R}.$

(iii) $\gamma_n \to \gamma$ and $\sum_{x\in X} |\lambda_{n,x} - \lambda_x| \to 0,\ n \to \infty.$