On Adiabatic Normal Modes in a Wedge-Shaped Shallow Sea

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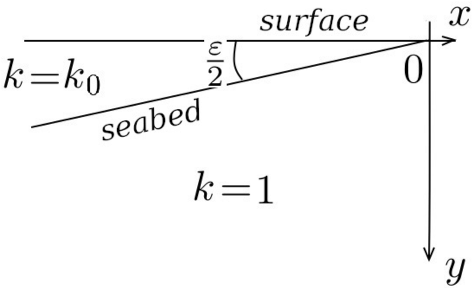
Problem formulation

Consider $\Delta U(x,y) + k^2 U(x,y) = 0$ in $P = \{(x,y) \in \mathbb{R}^2 : y > 0\}$.

Here $k = k(\varepsilon_1 x, y)$, $\varepsilon_1 = \mathrm{tg}\,(\varepsilon/2)$, $0 < \varepsilon \ll 1$,

$$k(\xi, y) = \begin{cases} k_0, & 0 \leqslant y \leqslant -\xi, \\ 1 & \text{otherwise}, \end{cases} \qquad k_0 > 1.$$



Boundary conditions:

- $U|_{y=0} = 0$ (at the "surface");
- U and $\partial U/\partial n$ are continuous at $y = -\varepsilon_1 x$, $x < 0$ (at the "seabed").

Background

- $\varepsilon \sim 1$ — a difficult problem;
- in ocean acoustics, $\varepsilon \sim 0.01$.

- Heuristic approach: physicists A. D. Pierce, L. B. Felsen et al. (1980's); reflected waves were not described!
- The first step: A. A. Fedotov (2018).

Operator $H(\xi)$

If $\varepsilon \ll 1$, one "separates" the variables asymptotically.

Consider the operator

$$H(\xi) = \frac{\partial^2}{\partial y^2} + k^2(\xi, y), \quad \xi = \varepsilon_1 x,$$

which depends on ξ as on a parameter and acts in $L_2(\mathbb{R}_+)$ with the Dirichlet boundary condition at $y = 0$.

Let

$$\xi_m = -\frac{\pi(m - 1/2)}{\sqrt{k_0^2 - 1}}, \quad m \in \mathbb{N}.$$

The m-th eigenvalue $E_m(\xi)$ of $H(\xi)$ exists if $\xi < \xi_m$, approaches the edge of the continuous spectrum as $\xi \to \xi_m$, and disappears for $\xi \geqslant \xi_m$.

Adiabatic normal modes

A. Fedotov constructed explicitly a set of solutions (adiabatic normal modes) $\{U_m\}_{m=1}^{\infty}$ in $Q_L = \{(x, y) \in \mathbb{R}^2 : y > 0,\ x < 0\}$ for $\varepsilon \ll 1$.

As $\varepsilon \to 0$,

$$U_m(x, y) \sim e^{\frac{i}{\varepsilon_1} \int\limits_{\xi_0}^{\xi} \sqrt{E_m(\xi')}\, \mathrm{d}\xi'} \sum_{n=0}^{\infty} \varepsilon_1^n \psi_{m,n}(\xi, y), \quad \xi = \varepsilon_1 x,$$

$\xi \leqslant \xi_0 < \xi_m$, ξ_0 is a fixed number, and $\psi_{m,0}(\xi, \cdot\,)$ is the m-th eigenfunction of $H(\xi)$.

U_m is constructed using ideas similar to those of the Zommerfield–Malyuzhinets method.

Asymptotics of U_m far away from the wedge tip

Theorem 1 (Fedotov, 2018)

Let $N \in \mathbb{N}$, $\Xi_1 < \Xi_2 < \xi_m$, $\xi = \varepsilon_1 x$. *If* $\Xi_1 \leqslant \xi \leqslant \Xi_2$ *and* $0 \leqslant y \leqslant -\xi$,

$$U_m(x, y) = \sqrt{-\frac{\mathrm{d}\sqrt{E_m}}{\mathrm{d}\xi}}\, e^{\frac{2i}{\varepsilon}\int\limits_{\xi_m}^{\xi} \sqrt{E_m}\,\mathrm{d}\xi' + \frac{i\theta_m}{\varepsilon} + \frac{3\pi i}{4}} \left(\sum_{n=0}^{N-1} \varepsilon^n \psi_{m,n}(\xi, y) + O(\varepsilon^N) \right).$$

Here $E_m = E_m(\xi)$, *and* $\psi_{m,0}$ *is the* m*-th eigenfunction of* $H(\xi)$.

Change in asymptotic behaviour

As $\xi = \varepsilon_1 x \to \xi_m$, $E_m(\xi)$ approaches the edge of the continuous spectrum of $H(\xi)$.

Modifying A. Fedotov's technique, we obtain asymptotics of U_m for $\xi_m - \delta \leqslant \xi \leqslant \xi_m$.

Proof.

By analyzing the integral representation of U_m. □

The asymptotics is expressed in terms of Airy functions.

Main results

Let

$$F(z) = \sqrt{\pi} e^{-\frac{2}{3} z^3 - \frac{\pi i}{12}} (z \operatorname{Ai}(z^2) - \operatorname{Ai}'(z^2)),$$

where Ai is the Airy function, and

$$Z_m = \left(-\frac{3}{2\varepsilon} \int_{\xi_m}^{\xi} (\sqrt{E_m} - 1) \, \mathrm{d}\xi' \right)^{\frac{1}{3}}.$$

Main results

Theorem 2

Fix a sufficiently small $\delta > 0$. *Let* $\xi_m - \delta \leqslant \xi \leqslant \xi_m$ *and* $0 \leqslant y \leqslant -\xi$, *where* $\xi = \varepsilon_1 x$. *Then*

$$U_m(x,y) = \sqrt{-\frac{1}{Z_m}\frac{\mathrm{d}\sqrt{E_m}}{\mathrm{d}\xi}}\, e^{\frac{2i}{\varepsilon}(\xi-\xi_m)+\frac{i\theta_m}{\varepsilon}+\frac{3\pi i}{4}} F(e^{\frac{i\pi}{6}} Z_m)\psi_{m,0} + O(\varepsilon^{\frac{2}{3}}(1+|Z_m|^{\frac{1}{2}})),$$

where $E_m = E_m(\xi), \ \ \psi_{m,0} = \psi_{m,0}(\xi, y)$.

If $\xi_m - \xi \sim 1$, then, as $\varepsilon \to 0$, the leading term in this formula turns into the leading term from theorem 1.