Quantum aspect of the classical dynamics

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“The laws of nature are written as differential equations.”

Induced dynamics. Let $\mathcal{A}_N$ denote a phase space of a dynamical system of N free particles

$$\dot{q}_i = h'(p_i), \qquad \dot{p}_i = 0, \quad i = 1, \ldots, N,$$

i.e. the Hamiltonian system with respect to the canonical Poisson bracket

$$\{q_i, p_j\} = \delta_{ij}, \quad \dot{q}_i = \{q_i, H\}, \quad \dot{p}_i = \{p_i, H\}, \quad H = \sum_{i=1}^{N} h(p_i),$$

where h is a function of one variable. We assume that q_i are either real or pairwise complex conjugate and the same are properties of the corresponding p_i.

Let $f(q_1, \ldots, q_N, p_1, \ldots, p_N)$ be a function on $\mathcal{A}_N$ such that equation

$$f(q_1 - x, \ldots, q_N - x, p_1, \ldots, p_N) = 0$$

has M simple real zeros $x_1, \ldots, x_M$, where $0 < M \leq N$. Moreover, let there exists such open subset $\mathcal{A}'_N \subset \mathcal{A}_N$, that $M = N$ for any $\{q_1, \ldots, q_N, p_1, \ldots, p_N\} \in \mathcal{A}'_N$.

The **induced system** is a system with configuration space given by **real** zeros of equation

$$f(q_1(t) - x, \dots, q_N(t) - x, p_1, \dots, p_N) = 0.$$

This system is dynamical as all roots

$$x_i = x_i(q_1(t), \dots, q_N(t), p_1, \dots, p_N)$$

are functions on $\mathcal{A}_N$ and depend on t via q_i only. Evolution of this system is given by the same Hamiltonian H, $\dot{x}_i = \{x_i, H\}$, under the same Poisson bracket. Moreover, the induced system is not only Hamiltonian but also integrable, as by construction it has N independent integrals of motion in involution.

Below I use notation $\mathbf{q} = (q_1, \dots, q_N)$, $\mathbf{p} = (p_1, \dots, p_N)$, and $\mathbf{e} = (\underbrace{1, \dots, 1}_{N})$, so the equation above takes the form

$$f(\mathbf{q}(t) - x\mathbf{e}, \mathbf{p}) = 0.$$

Differential equations. Assume, that equation $f(\mathbf{q}(t) - x\mathbf{e}, \mathbf{p}) = 0$ has N (different) solutions $x_i(t)$:

$$f(q_1(t) - x_i(t), \dots, q_N(t) - x_i(t), p_1, \dots, p_N) = 0, \qquad i = 1, \dots, N, \tag{1}$$

Taking $\dot{q}_i = h'(p_i)$, $\dot{p}_i = 0$ into account we have:

$$\dot{x}_i = \frac{\sum_{j=1}^{N} h'(p_j) f_{q_j}(\mathbf{q} - x_i\mathbf{e}, \mathbf{p})}{\sum_{j=1}^{N} f_{q_j}(\mathbf{q} - x_i\mathbf{e}, \mathbf{p})}, \tag{2}$$

$$\ddot{x}_i \sum_{j=1}^{N} f_{q_j}(\mathbf{q} - x_i\mathbf{e}, \mathbf{p}) = \sum_{j,k=1}^{N} (h'(p_j) - \dot{x}_i)(h'(p_k) - \dot{x}_k) f_{q_j q_k}(\mathbf{q} - x_i\mathbf{e}, \mathbf{p}). \tag{3}$$

(1)+(2)$=2N$ equations on $2N$ unknowns q_i and p_i. $\Rightarrow$
$\mathbf{q} = \mathbf{q}(x_1, \dots, x_N; \dot{x}_1, \dots, \dot{x}_N)$,
$\mathbf{p} = \mathbf{p}(x_1, \dots, x_N; \dot{x}_1, \dots, \dot{x}_N)$ (under condition of unique solvability).
Inserting these functions in (3) we prove existence of the Newton-type equations of **the induced dynamical system**:

$$\ddot{x}_i = F_i(x_1, \dots, x_N, \dot{x}_1, \dots, \dot{x}_N), \quad i = 1, \dots, N.$$

Cauchy problem. These results leed to the solution of the Cauchy problem for the induced system. Let we have $2N$ initial data: $x_i(0)$ and $\dot{x}_j(0)$, say at $t = 0$, where $i, j = 1, \ldots, N$. Equations (1) and (2) define values $(\mathbf{q}(0), \mathbf{p}(0))$ that belong to $\mathcal{A}'_N$ by construction. Then $q_i(t) = q_i(0) + th'(p_i)$, $p_i(t) = p_i$, that after substitution in

$$f(\mathbf{q}(t) - x\mathbf{e}, \mathbf{p}) = 0$$

gives M real roots $x_1(t), \ldots, x_M(t)$ for any $t \in \mathbb{R}$. We will see that M is not obliged to be equal to N at any moment of time.

Thus the scheme of solution of the Cauchy problem for the induced system is close to the one for integrable nonlinear PDE's.

Singular solutions of the KdV equation.

N-soliton solution of the KdV equation $4u_t - 6uu_x + u_{xxx} = 0$ is given by

$$u(t,x) = -2\partial_x^2 \log\det A(t,x),$$

where A is $N \times N$-matrix

$$A_{ij}(t,x) = \epsilon_i e^{2p_i(x-q_i)}\delta_{ij} + \frac{2p_i}{p_i + p_j}, \quad \mathrm{Re} p_i > 0,$$

and $q_i(t) = q_{0,i} + p_i^2 t$. Here

$$\text{either } \mathrm{Im} p_i = 0 \text{ and then } \epsilon_i = \pm 1, \quad \mathrm{Im} q_i = 0$$
$$\text{or } \mathrm{Im} p_i \neq 0, \text{ then exists } p_l = \overline{p}_i \text{ and } \epsilon_i = \epsilon_i = \pm 1, \quad q_= \overline{q}_j.$$

In the case where $\mathrm{Im} p_i = 0$ sign $\epsilon_i = +1$ gives regular soliton, sign $\epsilon_i = -1$ gives the singular one. Every pair of $p_i = \overline{p}_l$ with $\mathrm{Im} p_i \neq 0$ gives one line of singularity. In order to make regular solitons more observable we introduce "charge conjugation": $A(t,x) \to \widetilde{A}(t,x)|_{\epsilon_i \to -\epsilon_i}$ for all i. Thus we look for the zeros $x_i(t)$ of the function

$$f(\mathbf{q}(t) - x, \mathbf{p}) \equiv \det\big(A(t,x)\widetilde{A}(t,x)\big) = 0,$$

where again $\mathbf{q} - x\mathbf{e} = (q_1 - x, \ldots, q_N - x)$, $\mathbf{p} = (p_1, \ldots, p_N)$, $\dot{q}_i = p_i^2$, $\dot{p}_i = 0$.

Sinh–Gordon equation.

N-soliton solution of the Sinh–Gordon equation $u_{xt} = \frac{1}{16}\sinh u$, where x and t are cone variables, equals

$$e^{u(x,t)} = \frac{\det(A(x,t)+v)}{\det(A(x,t)-v)}, \qquad \text{where } v_{ij} = \frac{p_i}{p_i+p_j},$$

$$A(x,t) = \operatorname{diag}\{\epsilon_i e^{2p_i(x-q_i)}\}_{i=1}^{N}, \qquad q_i(\eta) = q_{0,i} - \frac{t}{p_i^2}.$$

$\mathrm{Re}p_i > 0$, and either $p_i = \overline{p}_i$ and $\epsilon_i = \pm 1$, $q_i = \overline{q}_i$
or $p_i = \overline{p}_k$ for some $k \neq i$ and $p_k \neq p_i$, $\epsilon_i = \epsilon_k$, $q_i = \overline{q}_k$.

Singularities $x_1(t), \ldots, x_N(t)$ of these solutions are given by zeros of both determinants, so they are given as zeros of the product

$$f(\mathbf{q}(t) - x, \mathbf{p}) \equiv \det(A(x,t)+v)\det(A(x,t)-v) = 0.$$

These zeros form N smooth time-like curves $x_i(t)$. The first factor gives $u = -\infty$ and the second one $u = +\infty$. Lines corresponding to singularities of the different signs can intersect and in these points and only in this points they are light-like.

This system is Hamiltonian: $x_i' = \{H, x_i\}$, where $H = \sum_{i=1}^{N} \frac{1}{p_i}$.

Equation of motion, $N = 2$, $x_1(t) + x_2(t) = 0$:

$$\frac{\ddot{x}_{12}\,\operatorname{sgn} x_{12}}{\sqrt{4-\dot{x}_{12}^2}} = \frac{4\varepsilon}{\cosh\left(\frac{4x_{12}}{\sqrt{4-\dot{x}_{12}^2}}\sqrt{1+\frac{\ddot{x}_{12}\,\operatorname{sgn} x_{12}}{\sqrt{4-\dot{x}_{12}^2}}}\right)-\varepsilon},$$

where $x_{12}(t) = x_1(t) - x_2(t)$ and where $\varepsilon = 1$ for the case of repulsion and $\varepsilon = -1$ for the both, soliton-antisoliton and breather, cases of attraction.

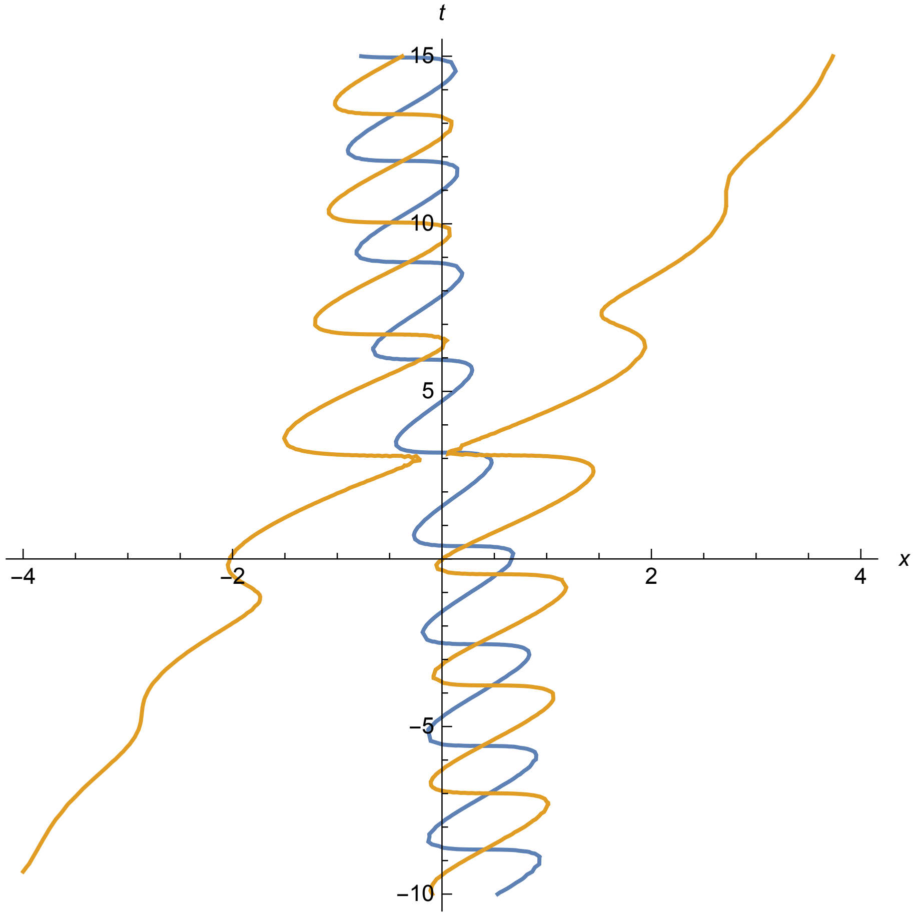

Figure 1: Soliton–breather collision.

Calogero–Moser and Ruijsenaars–Schneider models. Rational case.

$$\text{CM:} \qquad \ddot{x}_j = \sum_{\substack{k=1,\\ k\neq j}}^{N} \frac{2\gamma^2}{(x_k - x_j)^3}, \qquad \overline{\gamma} = -\gamma,$$

$$\text{RS:} \qquad \ddot{x}_j = \sum_{\substack{k=1,\\ k\neq j}}^{N} \frac{2\gamma^2 \dot{x}_j \dot{x}_k}{(x_j - x_k)\big(\gamma^2 - (x_j - x_k)^2\big)}.$$

Both systems are completely integrable, their L-operators can be written as

$$L(t) = \operatorname{diag}\{\dot{x}_1(t), \ldots, \dot{x}_N(t)\} + V(t),$$

where for CM and RS models

$$V_{\mathrm{CM}}(t) = \left(\frac{\gamma}{x_k(t) - x_j(t)}\right)_{\substack{j,k=1,\\ k\neq j}}^{N}, \quad V_{\mathrm{RS}}(t) = \left(\frac{\gamma \dot{x}_k(t)}{x_k(t) - x_j(t) + \gamma}\right)_{\substack{j,k=1,\\ k\neq j}}^{N}.$$

Solutions $x_i(t)$ of the equations above obey asymptotic behavior $x_i(t) = a_i + tp_i + O(t^{-1})$, say, at $t \to -\infty$, a_i and p_i are constants. They are given (Olshanetski–Perelomov, Ruijsenaars) as eigenvalues of the matrix $X(0) + tL(0)$, where $X(t) = \operatorname{diag}\{x_1(t), \ldots, x_N(t)\}$.

CM and RS models as induced systems Thanks to the translation invariance it was proved (AP2021) that the rational versions of CM and RS models are given by the roots of the equation

$$f(\mathbf{q}-x\mathbf{e},\mathbf{p})\equiv\det\bigl(Q(t)+W-xI\bigr)=0,$$

where $Q(t)=\operatorname{diag}\{q_1(t),\ldots,q_N(t)\}$, $q_i(t)=a_i+tp_i$ and

$$W_{\text{CM}}=\left(\frac{\gamma}{p_j-p_k}\right)_{\substack{j,k=1,\\ k\neq j}}^{N},\qquad W_{\text{RS}}=\left(\frac{\gamma p_j}{p_j-p_k}\right)_{\substack{j,k=1,\\ k\neq j}}^{N}.$$

Here again (q_i,p_j) are canonical variables, and Hamiltonian $H=\sum_i p_i^2/2$ for **both these models**. This gives another proof of the Liouville integrability for CM and RS systems in the rational case. Moreover, integrability takes place for any choice of the matrix $W(\mathbf{p})$, that obeys conditions of solvability of the systems formulated above. Specific property of the CM and RS models is possibility to write down equations of motion, Lax pairs and Hamiltonian explicitly.

Calogero–Moser and Ruijsenaars–Schneider models.
Hyperbolic case.

Notation: $X = \mathrm{diag}\{x_1, \dots, x_N\}$, $\dot{X} = \mathrm{diag}\{\dot{x}_1, \dots, \dot{x}_N\}$, $Q = \mathrm{diag}\{q_1, \dots, q_N\}$.
Lax operators: $L = \dot{X} + V$, where

$$V_{\mathrm{CM}} = \left(\frac{\gamma}{\sinh(x_j - x_k)}\right)_{\substack{j,k=1,\\ k\neq j}}^{N}, \qquad V_{\mathrm{RS}} = \left(\frac{(\sinh\gamma)\dot{x}_k}{\sinh(x_j - x_k + \gamma)}\right)_{\substack{j,k=1,\\ k\neq j}}^{N}.$$

In analogy to the rational case, $x_i(t)$ are given as roots of the characteristic equation

$$\det\left(e^{2Q(t)}K^\dagger K - e^{2x}I\right) = 0,$$

where K is upper ($p_1 < \cdots < p_N$) triangular matrix

$$\text{CM:} \qquad K_{j,k} = \delta_{j,k} - \frac{\gamma}{p_j - p_k}\prod_{i=j+1}^{k-1}\left(1 - \frac{\gamma}{p_i - p_k}\right), \quad k \geq j.$$

Again, integrability is preserved if we substitute $K^\dagger K$ by and arbitrary matrix depending on $p_1, \dots, p_N$ only.

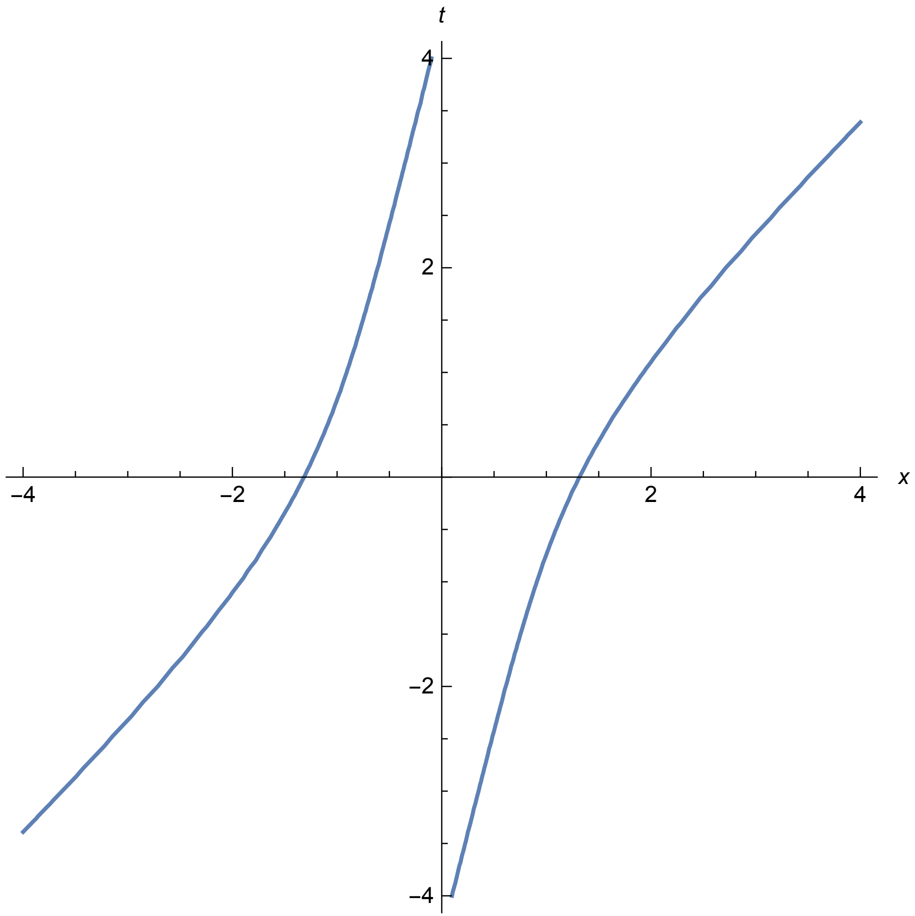

Figure 2: Ruijsenaars–Schneider model, 2 particles

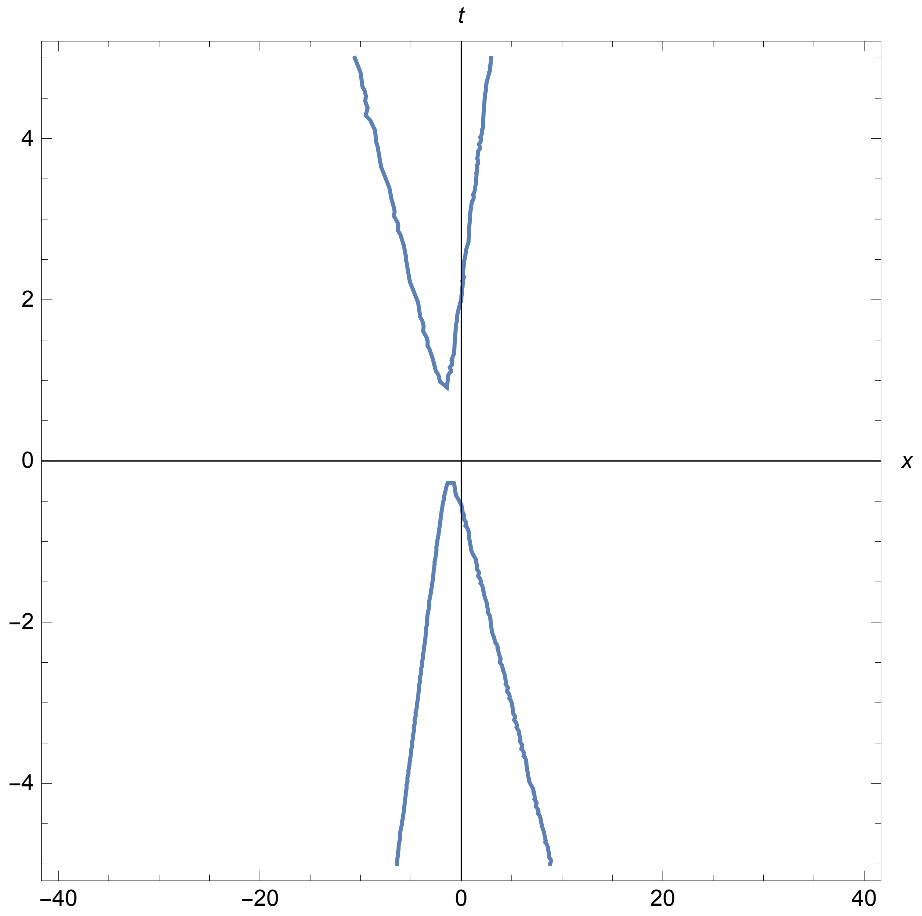

Figure 3: Ruijsenaars–Schneider model, 2 particles

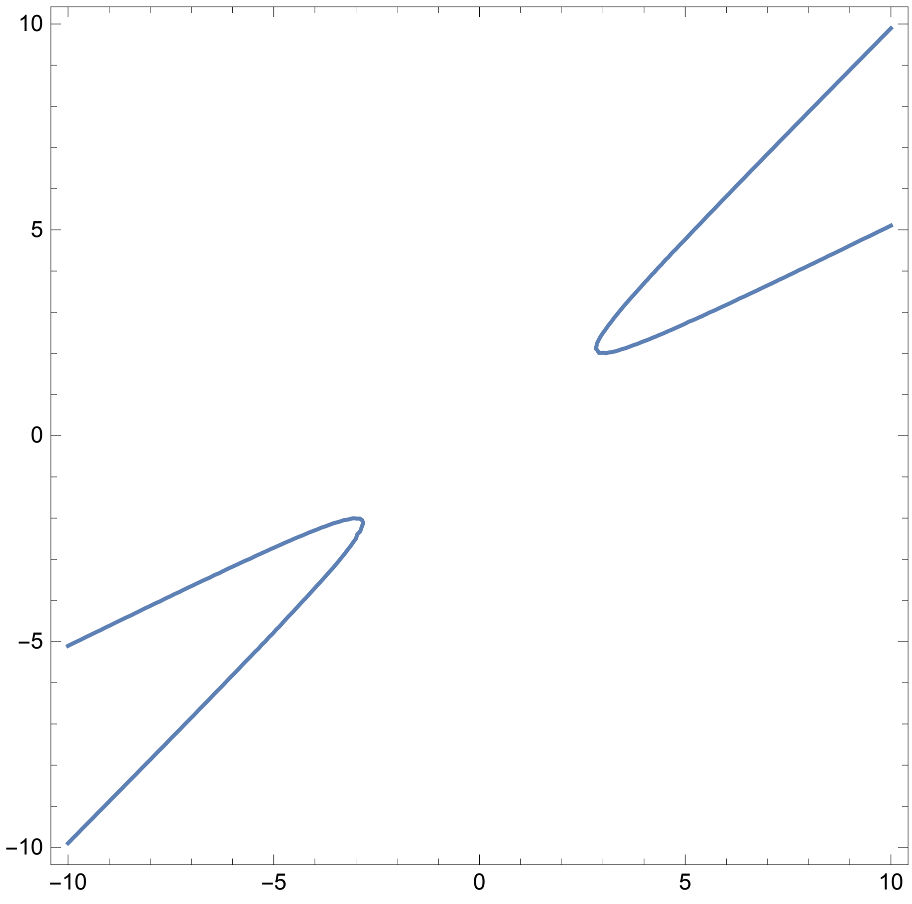

Figure 4: Calogero–Moser model, 2 particles

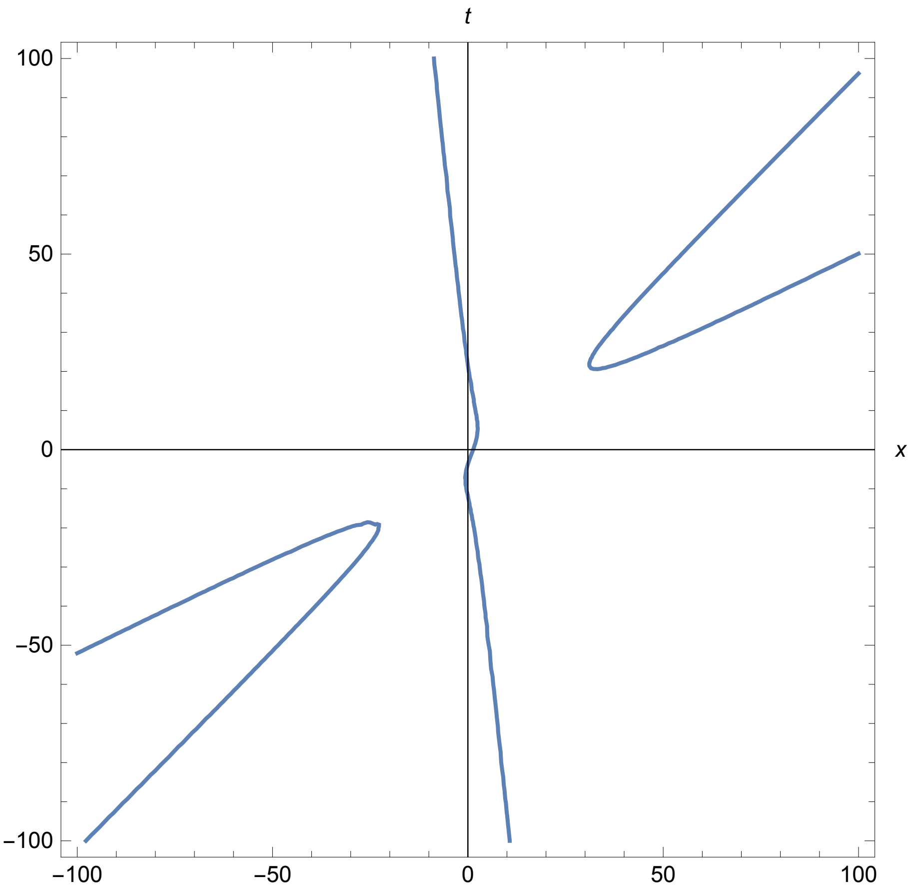

Figure 5: Calogero–Moser model, 3 particles

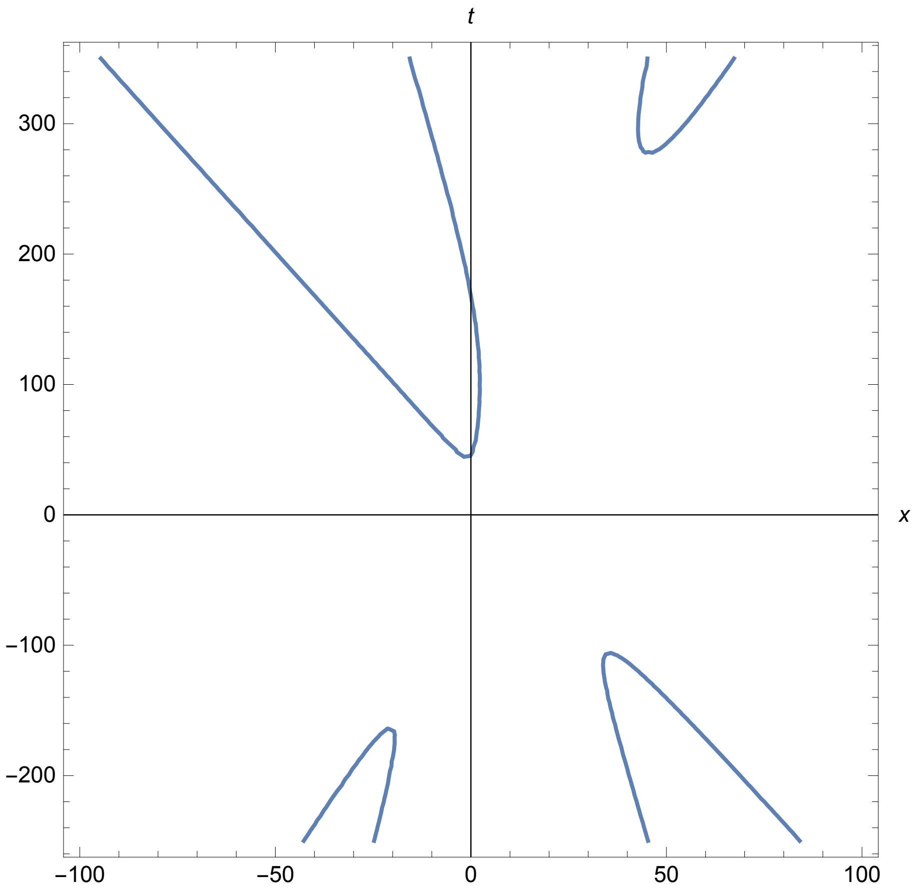

Figure 6: Calogero–Moser model, 4 particles

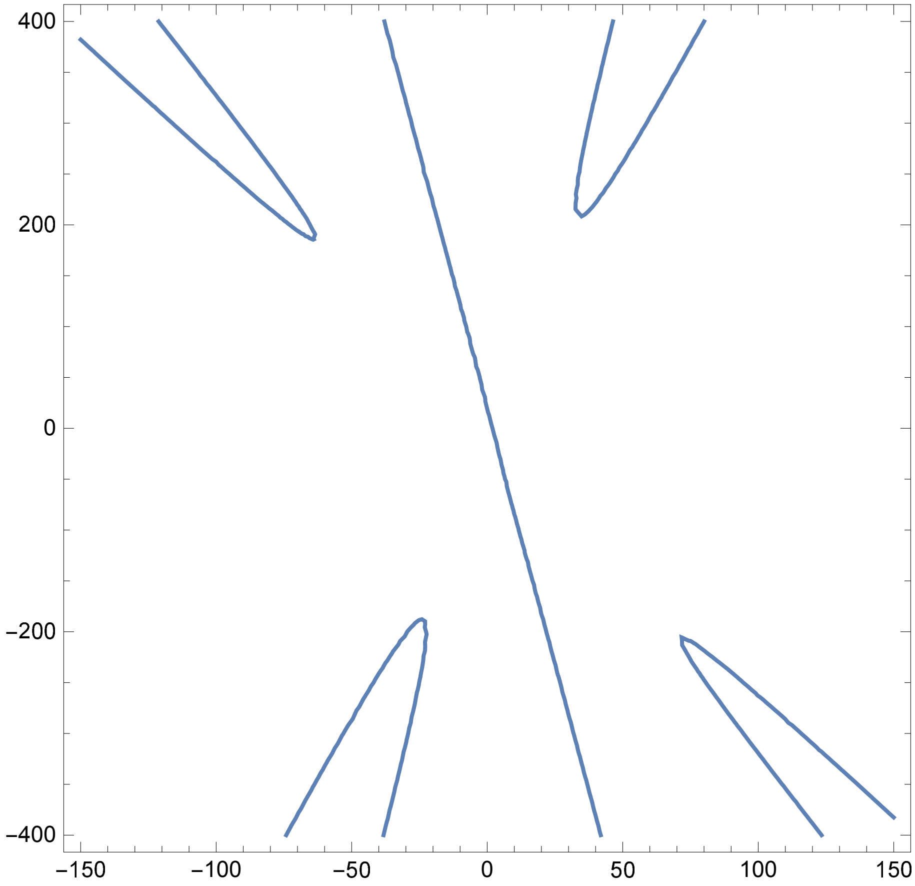

Figure 7: Calogero–Moser model, 5 particles

The "Goldfish" model.

Ruijsenaars–Schneider system:

$$\ddot{x}_j = \sum_{\substack{k=1,\\ k\neq j}}^{N} \frac{2\gamma^2 \dot{x}_j \dot{x}_k}{(x_j - x_k)\big(\gamma^2 - (x_j - x_k)^2\big)}$$

is completely integrable. L-operator:

$$L(t) = \mathrm{diag}\{\dot{x}_1(t), \ldots, \dot{x}_N(t)\} + \left(\frac{\gamma \dot{x}_k(t)}{x_k(t) - x_j(t) + \gamma}\right)_{\substack{j,k=1,\\ k\neq j}}^{N}.$$

Solutions $x_i(t)$:

obey asymptotic behavior $x_i(t) = a_i + tp_i + O(t^{-1})$, $t \to \infty$,

are given (Olshanetski–Perelomov) as eigenvalues of the matrix $X(0) + tL(0)$, where $X(t) = \mathrm{diag}\{x_1(t), \ldots, x_N(t)\}$.

In the limit $\gamma \to \infty$ we get nice system that Calogero called Goldfish model:

$$\ddot{x}_j = 2 \sum_{\substack{k=1,\\ k\neq j}}^{N} \frac{\dot{x}_j \dot{x}_k}{x_j - x_k}, \quad \text{with } L\text{-operator} \quad L = (1, \ldots, 1)^{\mathrm{T}} \otimes (\dot{x}_1, \ldots, \dot{x}_N).$$

This L-operator is senseless as $L^n = \left(\sum_{j=1}^N \dot{x}_j\right)^{n-1} L$, nevertheless it obeys extremely rich symmetry: $x \to \alpha + \lambda x$, $t \to \beta + \mu t$, where $\alpha, \beta, \lambda, \mu$ are arbitrary constants. In particular this model is Lorentz invariant.

Any particle initially at rest maintains this state of rest forever—since $\dot{x}_n = 0$ implies $\ddot{x}_n = 0$ and so on. Calogero proved that the solution of the initial-value problem for this N-body model is given by the following simple prescription: the values of the N coordinates $x_n(t)$ are given by the N roots of the following equation in x:

$$\sum_{j=1}^{N} \frac{\dot{x}_j(0)}{x - x_j(0)} = \frac{1}{t}.$$

It was proved (Gorsky, Vasilyev and Zotov) that this system is completely integrable. It has N integrals of motion

$$J_k = \sum_{I_k} \dot{x}_{i_1} \cdots \dot{x}_{i_k} \prod_{l<j} (x_{i_l} - x_{i_j})^2,$$

where J_k is a k-tuple $(i_1, \ldots, i_k)$ such that $1 \le i_1 < \ldots < i_k \le N$.

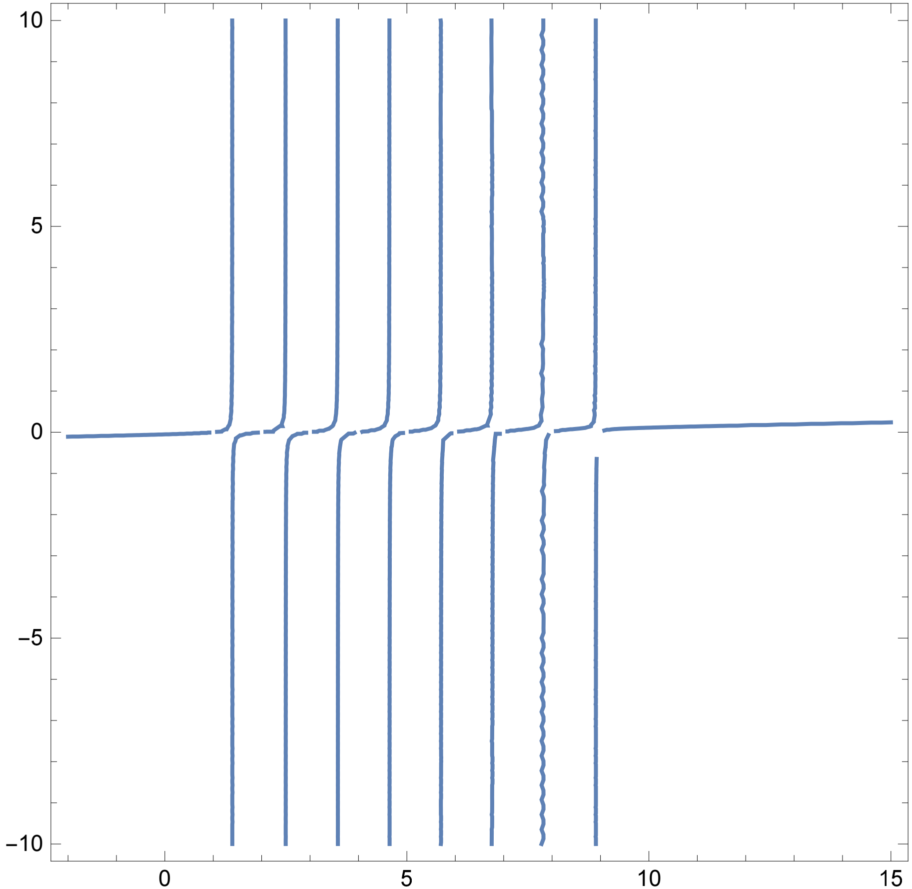

Figure 8: “Goldfish” model, 9 particles

Rutherford planetary model of the atom (uncertainty principle)
Creation/annihilation of the particles
The Big Bang
The expanding universe
The dark matter