Correlation Functions of Heisenberg XX Chain and Enumeration of Constrained Plane Partitions

Nikolay M. Bogoliubov, Cyril Malyshev

St.-Petersburg Department of Steklov Mathematical Institute
St.-Petersburg, RUSSIA

Mathematical and Theoretical Physics

dedicated to Ludwig Faddeev

28-31 May 2024 PDMI RAS

ABSTRACT

Relations between the mean values of distributions of flipped spins on periodic Heisenberg XX chain and some aspects of enumerative combinatorics are discussed. The Bethe vectors, which are the state-vectors of the model, are considered both as on- and off-shell. It is this approach that makes it possible to represent and to study the correlation functions in the form of nests of non-intersecting lattice walks and related plane partitions. The determinantal representation for the norm-trace generating function of plane partitions with fixed height of diagonal parts is obtained as the expectation of the generating exponential over off-shell N-particle Bethe states. The asymptotics of the mean value of the generating exponential is calculated provided that the evolution parameter is large. It is shown that the amplitudes of the leading asymptotics depend on the number of diagonally constrained plane partitions.

TABLE OF CONTENT

References

/ N. Bogoliubov, C. Malyshev /

[1] Spin correlation functions, Ramus-like identities, and enumeration of constrained lattice walks and plane partitions // J. Phys. A: Math. Theor., vol 55 (2022) 225002 (37 pp)

[2] How to draw a correlation function // Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), vol 17 (2021) 106

[3] The partition function of the four-vertex model in inhomogeneous external field and trace statistics // J. Phys. A: Math. Theor., vol 52 (2019) 495002

[4] The phase model and the norm-trace generating function of plane partitions // Journal of Statistical Mechanics: Theory and Experiment, vol 2018 (2018) 083101

[5] The integrable models and combinatorics //Russ. Math. Surveys, vol 70 (2015) 789

****

I. OUTLINE OF THE PROBLEM

••• Let us consider the quantum system of $\frac{1}{2}$-spins on a chain consisting of M sites. Spin "up", $|\uparrow\rangle_n$, and spin "down", $|\downarrow\rangle_n$, states are defined on n^{th} site, $n \in \{1, 2, \ldots, M\}$. Let two operators, q_n and $\bar{\mathsf{q}}_n$, be local projectors (i.e., spin "up" and "down" on-site densities) which respect

$$\mathsf{q}_n\,|\downarrow\rangle_n = |\downarrow\rangle_n\,, \quad \mathsf{q}_n\,|\uparrow\rangle_n = 0\,, \qquad \bar{\mathsf{q}}_n\,|\uparrow\rangle_n = |\uparrow\rangle_n\,, \quad \bar{\mathsf{q}}_n\,|\downarrow\rangle_n = 0\,.$$

Let us introduce the sum of q_n taken with inhomogeneous weights α_n:

$$\mathcal{Q} \equiv \sum_{n=1}^{M} \alpha_n \mathsf{q}_n\,.$$

The state $|\Uparrow\rangle \equiv \bigotimes_{n=1}^{M} |\uparrow\rangle_n$ (spins "up" on all sites) is chosen as the *reference state*. Reversed spin on n^{th} site $|\downarrow\rangle_n$ is called *flipped* spin.

The sum $Q(m) \equiv \sum_{k=1}^{m} \mathsf{q}_k$ is the operator of number of flipped spins on first m sites, and $\mathcal{N} \equiv \mathcal{Q}(\mathcal{M})$ is the total number of flipped spins.

Let us consider the mean value of *generating exponential operator* $e^{\mathcal{Q}}$:

$$\prec e^{\mathcal{Q}} \succ \equiv \text{trace}\,(e^{\mathcal{Q}}\boldsymbol{\rho})\,, \qquad \boldsymbol{\rho} \equiv \frac{e^{-\beta H}}{\text{trace}\,(e^{-\beta H})}\,,$$

where β is a real positive, H is the Hamiltonian, and $\boldsymbol{\rho}$ is the density matrix. The parameter β might be treated either as an "evolution" parameter or inverse absolute temperature. In what follows trace implies summation over N-particle states, trace $\equiv \mathrm{tr}_{\,N}(\cdot)$.

Our approach is to demonstrate the enumerative combinatorial implications of the averages generated by $\prec e^{\mathcal{Q}} \succ$. Multiple differentiation of $\prec e^{\mathcal{Q}} \succ$ with respect to α_n leads to the correlation functions of flipped spins, which demonstrate combinatorial implications provided that the Bethe state-vectors are expressed in terms of symmetric functions.

♦ Mean values $\prec \mathit{\Pi}_{\mathbf{k}} \succ$ of product $\mathit{\Pi}_{\mathbf{k}} \equiv \prod_{j=1}^{l} \mathsf{q}_{k_j}$ of flipped spin operators q_n, not necessarily at consecutive sites arise provided that the average $G(\mathbf{a}_M) \equiv \prec \exp\big(\mathcal{Q}(\mathbf{a}_{\mathcal{M}})\big) \succ$ parameterized by the elements of M-tuple $\mathbf{a}_M \equiv (\alpha_1, \alpha_2, \ldots, \alpha_M)$ is used as the generating function:

$$\begin{aligned} \prec \mathit{\Pi}_{\mathbf{k}} \succ &= \text{trace}\left(\mathit{\Pi}_{\mathbf{k}}\, \boldsymbol{\rho}\right) \\ &= \lim_{\mathbf{a}_M \to 0} \frac{\partial^l\, G(\mathbf{a}_M)}{\partial \alpha_{k_1} \partial \alpha_{k_2}\, \ldots\, \partial \alpha_{k_l}} \equiv \lim_{\mathbf{a}_M \to 0} \partial^l_{\alpha_{k_1} \alpha_{k_2} \ldots \alpha_{k_l}} G(\mathbf{a}_M)\,, \end{aligned}$$

where $M \geq k_1 > k_2 > \cdots > k_l \geq 1$.

♦ Assume a linear dependence of the elements of $\mathbf{a}_M$ on the site coordinates, $\mathbf{a}_M = \frac{\alpha}{N} \times (1, 2, \ldots, M)$, $\alpha \in \mathbb{R}$. The operator $\mathcal{Q}$ is reduced to $\mathcal{Q} = \alpha \mathsf{M}$, where the operator M would be considered as the operator of first moment of the distribution $\frac{n_i}{N}$ of N flipped spins (n_i is an eigen-value of q_i):

$$\mathsf{M} \equiv \frac{1}{N} \sum_{n=1}^{M} n\, \mathsf{q}_n \, . \quad \text{Then,} \quad \prec \mathsf{M}^l \succ = \mathsf{trace}\left(\mathsf{M}^l \boldsymbol{\rho}\right) = \mathcal{D}_\alpha^l \prec e^{\alpha \mathsf{M}} \succ ,$$

where $\mathcal{D}_\alpha^l$ denotes differentiation of l^{th} order with respect to α at $\alpha = 0$. Therefore, the mean value $\prec e^{\alpha \mathsf{M}} \succ$ is the generating function of the mean values of powers of M.

Temporal evolution of $e^{\alpha \mathsf{M}}$ has been studied for the Quantum Phase Model, where M is given above, q_n is on-site boson number operator, and M is the operator of first moment of the distribution $\frac{n_i}{N}$ of N bosons with n_i be the occupation number.

N. M. Bogoliubov, C. Malyshev // The phase model and the norm-trace generating function of plane partitions — Journal of Statistical Mechanics: Theory and Experiment, vol 2018 issue 8 (2018) 083101

Therefore we shall study $\prec e^{\mathcal{Q}} \succ \equiv \mathsf{trace}\,(e^{\mathcal{Q}}\boldsymbol{\rho})$ and its enumerative combinatorial implications using, as a well-developed model, the Heisenberg XX spin chain. The combinatorial implications to be elaborated below are hopefully of interest from the viewpoint of the models mentioned.

• • • Let us introduce the local spin operators $\sigma_n^{\pm} = \frac{1}{2}(\sigma_n^x \pm i\sigma_n^y)$ and σ_n^z dependent on the lattice argument $n \in \{1, 2, \dots, M\}$ and acting on the state space $\mathfrak{H}_M \equiv (\mathbb{C}^2)^{\otimes M}$:

$$\sigma_n^{\#} = \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \underbrace{\sigma^{\#}}_{n} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}\,,$$

where $\mathbb{I}$ is unit 2×2 matrix, and $\sigma^{\#}$ at n-th place is the Pauli matrix, $\sigma^{\#} \in \mathfrak{su}(2)$ ($\#$ means x, y, z, or $\pm$):

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spin operators satisfy the commutation relations:

$$[\sigma_k^+, \sigma_l^-] = \delta_{kl}\, \sigma_l^z\,, \qquad [\sigma_k^z, \sigma_l^{\pm}] = \pm 2\delta_{kl}\, \sigma_l^{\pm}\,.$$

The spin "up" $|\uparrow\rangle_n \equiv \begin{pmatrix}1\\0\end{pmatrix}_n$ and spin "down" $|\downarrow\rangle_n \equiv \begin{pmatrix}0\\1\end{pmatrix}_n$ states constitute a natural base in $\mathbb{C}^2$ and thus enable that $\sigma_n^\pm$ act on them as the rising/lowering operators:

$$\sigma_n^+ \, |\downarrow\rangle_n = |\uparrow\rangle_n \,, \qquad \sigma_n^- \, |\uparrow\rangle_n = |\downarrow\rangle_n \,, \qquad \sigma_n^- \, |\downarrow\rangle_n = \sigma_n^+ \, |\uparrow\rangle_n = 0 \,.$$

The operators ensure the definition of q_n, $\bar{\mathsf{q}}_n$ presented above:

$$\mathsf{q}_n \equiv \sigma_n^- \sigma_n^+ = \frac{1}{2}(1 - \sigma_n^z) \,, \qquad \bar{\mathsf{q}}_n \equiv \sigma_n^+ \sigma_n^- = \frac{1}{2}(1 + \sigma_n^z) \,.$$

• • • The Hamiltonian of the XX Heisenberg model is chosen:

$$H = H_{\mathrm{xx}} - hS^z \,, \qquad H_{\mathrm{xx}} \equiv -\frac{1}{2} \sum_{n,m=1}^{M} \Delta_{nm} \sigma_n^+ \sigma_m^- \,,$$

$$S^z = \frac{1}{2} \sum_{n=1}^{M} \sigma_n^z \,,$$

where S^z is the third component of total spin, and $h \geq 0$ is homogeneous magnetic field. The number of sites is $M = 0 \pmod 2$, the periodic boundary conditions $\sigma_{n+M}^{\pm,z} = \sigma_n^{\pm,z}$, $\forall n \in \{1, 2, \ldots, M\}$, are imposed, and H commutes with S^z.

Besides, Δ_{nm} are the entries of *exchange matrix* $\boldsymbol{\Delta}$ given by

$$\Delta_{nm} \equiv \delta_{|n-m|,1} + \delta_{|n-m|,M-1} \, ,$$

$\delta_{n,l} (\equiv \delta_{nl})$ is the Kronecker symbol, or

$$\boldsymbol{\Delta} \equiv (\Delta_{nm})_{1 \le n,m \le M} = \begin{pmatrix} 0 & 1 & & & & 1 \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & & \cdots & & \\ & & & 1 & 0 & 1 \\ 1 & & & & 1 & 0 \end{pmatrix} .$$

The exchange matrix $\boldsymbol{\Delta}$ is expressed in terms of the *circulant matrix*, which is a square matrix of order $M \in \mathbb{N}$ of the form

$$\mathsf{C}_M = \begin{pmatrix} c_0 & c_1 & \dots & c_{M-2} & c_{M-1} \\ c_{M-1} & c_0 & c_1 & \dots & c_{M-2} \\ \vdots & c_{M-1} & c_0 & \ddots & \vdots \\ c_2 & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \dots & c_{M-1} & c_0 \end{pmatrix}. \qquad (3)$$

The first row of C_M (3), $(c_0, c_1, \dots, c_{M-1})$, is called the *generator* of C_M. The exchange matrix $\boldsymbol{\Delta}$ is $\boldsymbol{\Delta} = \mathsf{S}_M + \mathsf{S}_M^T$, where

$$\mathsf{S}_M \equiv \mathrm{Circ}_M(0, 0, \dots, 0, 1) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & 1 & 0 & \ddots & \vdots \\ 0 & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

is known as the *basic circulant permutation matrix*.

Depending on the definition of trace, $\prec e^{\mathcal{Q}(\mathbf{a}_{\mathcal{M}})} \succ$ implies the mean value denoted, in what follows, either $G_{N,\beta}(\mathbf{a}_M) \equiv \langle\langle e^{\mathcal{Q}(\mathbf{a}_{\mathcal{M}})} \rangle\rangle_{N,\beta}$ (N flipped spins and β are fixed) or $G_{\beta}(\mathbf{a}_M) \equiv \langle\langle e^{\mathcal{Q}(\mathbf{a}_{\mathcal{M}})} \rangle\rangle_{\beta}$ (β is fixed). It will be shown that the combinatorial implications of $G_{N,\beta}(\mathbf{a}_M)$ are related to enumeration of nests of closed lattice trajectories with initial/final positions constrained, as well as with enumeration of boxed plane partitions with constrained diagonal parts. The implications of $G_{\beta}(\mathbf{a}_M)$ are related to enumeration of random turns walks with constrained initial/final positions.

II. The state-vectors, the Schur polynomials and non-intersecting lattice walks

••• The Bethe state-vectors

Let us introduce a *strict partition* $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_N)$ consisting of elements μ_k, $1 \le k \le N$, called *parts* of $\boldsymbol{\mu}$, which respect

$$M \ge \mu_1 > \mu_2 > \dots > \mu_N \ge 1 \,.$$

We also introduce the "staircase" partition

$$\boldsymbol{\delta}_N \equiv (N, N-1, \dots, 2, 1) \,,$$

and introduce non-strict partition $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)$ consisting of weakly decreasing non-negative integers:

$$\mathcal{M} \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N \ge 0 \,, \qquad \mathcal{M} \equiv M - N \,.$$

The relationship between the parts of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ is expressed as

$$\lambda_j = \mu_j + j - N - 1 \,, \qquad 1 \le j \le N \,,$$

or $\boldsymbol{\lambda} = \boldsymbol{\mu} - \boldsymbol{\delta}_N$. The *volume* of partition, for instance, $\boldsymbol{\lambda}$ is the sum of its parts: $|\boldsymbol{\lambda}| \equiv \sum_{i=1}^N \lambda_i$. The volumes of $\boldsymbol{\mu}$, $\boldsymbol{\lambda}$, and $\boldsymbol{\delta}$ are related: $|\boldsymbol{\mu}| = |\boldsymbol{\lambda}| + \frac{N}{2}(N+1)$.

We define an arbitrary state $|\boldsymbol{\mu}\rangle$ labelled by parts of $\boldsymbol{\mu}$ corresponding to N flipped spins and its conjugate $\langle\boldsymbol{\nu}|$:

$$|\boldsymbol{\mu}\rangle \equiv \left(\prod_{k=1}^{N} \sigma^{-}_{\mu_k}\right) |\Uparrow\rangle \,, \qquad \langle\boldsymbol{\nu}| \equiv \langle\Uparrow| \left(\prod_{k=1}^{N} \sigma^{+}_{\nu_k}\right) ,$$

where $|\Uparrow\rangle \equiv \bigotimes_{n=1}^{M} |\uparrow\rangle_n$, and $\sigma^{\pm}_n$ act on $|\uparrow\rangle_n$ and $|\downarrow\rangle_n$ as rising/lowering operators. The orthogonality is valid: $\langle\boldsymbol{\nu}|\boldsymbol{\mu}\rangle = \delta_{\boldsymbol{\nu}\boldsymbol{\mu}} \equiv \prod_{n=1}^{N} \delta_{\nu_n \mu_n}$.

Define N-particle state-vectors as the linear combinations of $|\boldsymbol{\mu}\rangle$:

$$|\,\Psi(\mathbf{u}_N)\rangle \equiv \sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2_N)\,|\boldsymbol{\lambda} + \boldsymbol{\delta}_N\rangle \,,$$

where the coefficients are the *Schur polynomials*:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}_N) \equiv S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_N) \equiv \frac{\det(x_j^{\lambda_k+N-k})_{1\le j,k\le N}}{\mathcal{V}(\mathbf{x}_N)} \,,$$

where $\mathcal{V}(\mathbf{x}_N)$ is the *Vandermonde determinant*

$$\mathcal{V}(\mathbf{x}_N) \equiv \det(x_j^{N-k})_{1\le j,k\le N} = \prod (x_l - x_m) \,.$$

The scalar product of the states takes the form:

$$\langle \Psi(\mathbf{v}_N) \mid \Psi(\mathbf{u}_N) \rangle = \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{v}_N^{-2}) S_{\boldsymbol{\lambda}}(\mathbf{u}_N^2)\,,$$

and further due to the Cauchy–Binet formula:

$$\langle \Psi(\mathbf{v}_N) \mid \Psi(\mathbf{u}_N) \rangle = \frac{1}{\mathcal{V}(\mathbf{v}_N^{-2})\mathcal{V}(\mathbf{u}_N^2)} \det\left(\frac{1-(u_i/v_j)^{2M}}{1-(u_i/v_j)^2}\right)_{1\le i,j\le N}.$$

The following identity is valid:

$$\sum_{k=1}^{N} S_{\boldsymbol{\lambda}+\mathbf{e}_k}(\mathbf{x}_N) = \left(\sum_{k=1}^{N} x_k\right) S_{\boldsymbol{\lambda}}(\mathbf{x}_N)\,,$$

where $\mathbf{e}_k$, $1 \le k \le N$, are N-tuples consisting of zeros except of a unity at $k^{\rm th}$ place, say, from left.

Let the exponential parametrization $\mathbf{x}_N = e^{i\boldsymbol{\theta}_N}$ be adopted, where $e^{i\boldsymbol{\theta}_N}$ is N-tuple $(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N})$. Assume that $e^{iM\theta_j} = (-1)^{N-1}$, $1 \leq j \leq N$ holds. As well, $|\boldsymbol{\mu}\rangle$, $\langle\boldsymbol{\nu}|$ are given by $\sigma_n^{\pm}$ subjected to the periodicity $\sigma_{n+M}^{\pm} = \sigma_n^{\pm}$, $\forall n \in \{1, 2, \dots, M\}$. Then the state-vector $|\,\Psi(e^{i\boldsymbol{\theta}_N/2})\rangle$ with the coefficients $S_{\boldsymbol{\lambda}}(e^{i\boldsymbol{\theta}_N})$ and its conjugate are N-particle Bethe state-vectors of the XX model.

Statement: *The Bethe state-vectors of XX model $|\,\Psi(e^{i\boldsymbol{\theta}_N/2})\rangle$ are the eigen-states of H (1) and S^z (2) on periodic chain:*

$$\Big(H_{\rm xx} - hS^z\Big)\,|\,\Psi(e^{i\boldsymbol{\theta}_N/2})\rangle \;=\; E_N(\boldsymbol{\theta}_N)\,|\,\Psi(e^{i\boldsymbol{\theta}_N/2})\rangle\,,$$

$$S^z\,|\,\Psi(e^{i\boldsymbol{\theta}_N/2})\rangle \;=\Big(\frac{M}{2} - N\Big)|\,\Psi(e^{i\boldsymbol{\theta}_N/2})\rangle\,,$$

where

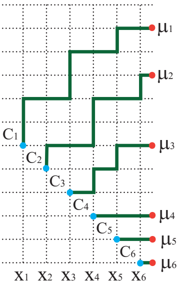
$$E_N(\boldsymbol{\theta}) \;=\; -\frac{hM}{2} + \sum_{j=1}^{N} \varepsilon(\theta_j)\,, \qquad \varepsilon(\theta_j) \equiv h - \cos\theta_j\,.$$

• • • The Schur polynomials, non-intersecting lattice paths, and plane partitions

• The Schur polynomials $S_{\boldsymbol{\lambda}}(\mathbf{x}_N)$ admit a combinatorial interpretation since are related to the *semi-standard Young tableaux*, which are in one-to-one correspondence with the *nests of non-intersecting lattice paths*.

A semi-standard Young tableau T of shape $\boldsymbol{\lambda}$ is a diagram possessing λ_i cells in i^{th} row ($i = 1, 2, \dots, N$) such that the cells are filled with positive integers $n \in \mathbb{N}^+$ weakly increasing along rows and strictly increasing downwards along columns (right-hand side of Figure below).

Definition *Star $\mathcal{C}$, corresponding to semi-standard Young tableau* T *of shape $\boldsymbol{\lambda}$, is a nest of N non-intersecting lattice paths (left-hand side of Figure) counted from the top of* T *and going from points $C_i = (i, N+1-i)$ to points $(N, \mu_i = \lambda_i + N + 1 - i), 1 \le i \le N$. An i^{th} path makes λ_i upward steps along vertical lines encoded by the integers in i^{th} row of* T.



1	1	3	3	5
2	4	4	6	
4	5			

Рис.: A star $\mathcal{C}$ of $N = 6$ lattice paths and semi-standard tableau T of shape $\boldsymbol{\lambda} = (5, 4, 2, 0, 0, 0)$.

The number l_j of upward steps along the line x_j coincides with the number of occurrences of j in T. Then, $S_{\boldsymbol{\lambda}}(\mathbf{x}_N)$ corresponding to T of shape $\boldsymbol{\lambda}$ takes the form:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}_N) = \sum_{\{\mathcal{C}\}} \prod_{j=1}^{N} x_j^{l_j},$$

where summation is over all admissible stars $\mathcal{C}$.
The number of nests of non-intersecting lattice paths is given by $S_{\boldsymbol{\lambda}}(\mathbf{1}_N) \equiv S_{\boldsymbol{\lambda}}(1, 1, \ldots, 1)$ equal to

$$S_{\boldsymbol{\lambda}}(\mathbf{1}_N) = \prod_{1\le j<k\le N} \frac{\lambda_j - j - \lambda_k + k}{k-j} = \prod_{1\le j<k\le N} \frac{\mu_j - \mu_k}{k-j}\,.$$

- Let us describe the nest of lattice paths called *watermelon*. Let us consider the nest of N non-intersecting lattice paths with equidistantly arranged start and end points, C_l and B_l, respectively ($1 \leq l \leq N$). Only upward and rightward steps are allowed for the path in the nest so that an l^{th} one is contained within the rectangle whose lower left and upper right vertices are C_l and B_l, respectively. Moreover, the total number $\mathcal{M} = M - N$ of upward steps and the total number N of rightward steps are the same for each path in the nest.

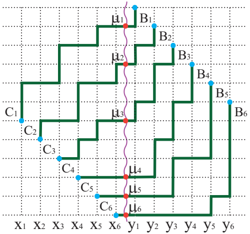


Рис.: *Watermelon* as the nest of lattice paths at $\mathcal{M} = 6$, $N = 6$.

- A *plane partition* $\boldsymbol{\pi}$ contained in $N \times N \times \mathcal{M}$ box $\mathcal{B}(N, N, \mathcal{M})$ is an array $(\pi_{ij})_{i,j \geq 1}$ of non-negative integers that satisfy $\pi_{ij} \leq \mathcal{M}$, $\pi_{ij} \geq \pi_{i+1,j}$ and $\pi_{ij} \geq \pi_{i,j+1}$ for all $i, j \geq 1$. Furthermore, $\pi_{ij} = 0$ whenever i or j exceed N.

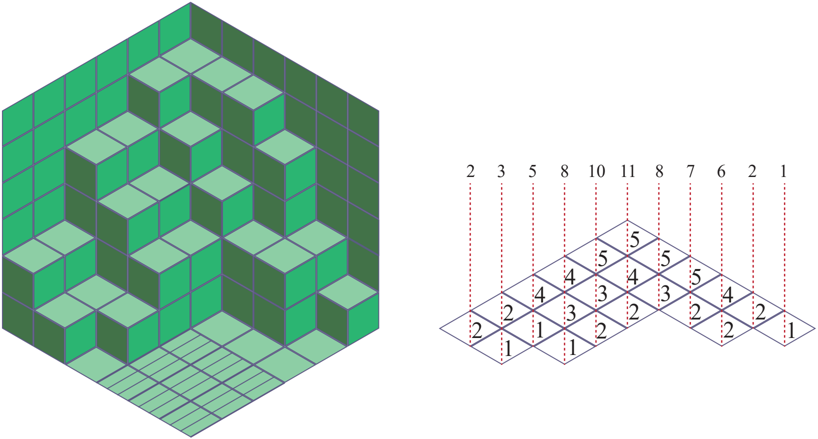


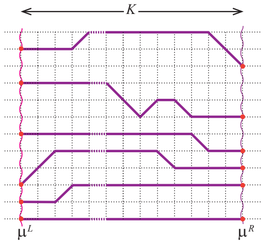
Рис.: Plane partition with $|\boldsymbol{\lambda}| = 11$ equivalent to watermelon in Figure 6.

There exists bijection between the watermelon configuration of non-intersecting lattice paths (Figure 6) and the plane partition in $\mathcal{B}(N, N, \mathcal{M})$ (Figure 7). The trace of s^{th} diagonal of plane partition counted from lower left corner is $\mathrm{tr}_s \boldsymbol{\pi} \equiv \sum_{N+j-i=s} \pi_{ij}$, $1 \leq s \leq 2N-1$. The *volume* of $\boldsymbol{\pi}$ is $|\boldsymbol{\pi}| = \sum_{s=1}^{2N-1} \mathrm{tr}_s \boldsymbol{\pi}$. The bijection is such that the heights of diagonal columns are given by parts of $\boldsymbol{\lambda}$, and thus $\mathrm{tr}_N \boldsymbol{\pi} = |\boldsymbol{\lambda}|$ (all traces are depicted above; notice that 3×3 square hatched in Figure 7 is commented further).

III. The transition amplitude and random turns walks of vicious walkers

••• Multi-particle transition amplitude

Let us turn to one-dimensional random walks of *vicious walkers*, who annihilate one another whenever they meet at the same site. Suppose that there are N walkers on a one-dimensional lattice. In the *random turns model* only a single randomly chosen walker jumps at each tick of a clock to either of adjacent sites while the remaining walkers are staying. It has been proposed, *N.M. Bogoliubov // XX0 Heisenberg chain and random walks, J. Math. Sci.* **138** *(2006) 5636*, to use XX chain to interpret random movements in the random turns model as transitions between spin "up" and "down" states.



The generating function of the lattice trajectories of N random turns vicious walkers is given by N-particle *transition amplitude* between the states $\langle \boldsymbol{\mu}^L |$ and $| \boldsymbol{\mu}^R \rangle$ parameterized by parts of $\boldsymbol{\mu}^L \equiv (\mu_1^L, \mu_2^L, \dots, \mu_N^L)$ and $\boldsymbol{\mu}^R \equiv (\mu_1^R, \mu_2^R, \dots, \mu_N^R)$ interpreted as initial and final positions:

$$G_{\boldsymbol{\mu}^L ; \boldsymbol{\mu}^R}(\beta) \equiv \langle \boldsymbol{\mu}^L | \, e^{-\beta H_{\rm xx} + \beta h S^z} \, | \boldsymbol{\mu}^R \rangle \,,$$

where XX Hamiltonian is used. Equivalently,

$$G_{\boldsymbol{\mu}^L ; \boldsymbol{\mu}^R}(\beta) = \, e^{\beta h (\frac{M}{2} - N)} G^0_{\boldsymbol{\mu}^L ; \boldsymbol{\mu}^R}(\beta) \,,$$

$$G^0_{\boldsymbol{\mu}^L ; \boldsymbol{\mu}^R}(\beta) \equiv \langle \boldsymbol{\mu}^L | \, e^{-\beta H_{\rm xx}} \, | \boldsymbol{\mu}^R \rangle \,,$$

where the exponential factor is due to coupling of the spin chain to homogeneous magnetic field, and the corresponding exponent is proportional to the eigen-value of the total spin: $S^z \; |\Uparrow\rangle = \frac{M}{2} \; |\Uparrow\rangle$.

The transition amplitude $G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta)$ respects

$$\frac{d\,G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta)}{d(\beta/2)} = \sum_{k=1}^{N}\left(G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R+\mathbf{e}_k}(\beta) + G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R-\mathbf{e}_k}(\beta)\right) + h(M-2N)\,G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta)\,,$$

Proposition: *The transition amplitude respecting the initial condition* $G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(0) = \delta_{\boldsymbol{\lambda}^L\boldsymbol{\lambda}^R}$, *as well as the periodicity condition and non-intersection property takes the form*:

$$G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta) = e^{\beta h(\frac{M}{2}-N)} G^0_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta)\,, \quad G^0_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta) = \det\left(G^0_{\mu^L_n;\mu^R_k}(\beta)\right)_{1\le n,k\le N}\,,$$

where:

$$G^0_{j;\,m}(\beta) = \frac{1}{M}\sum_{n=1}^{M} e^{\beta\cos\phi_n}\, e^{i\phi_n(m-j)}\,,$$

and the sum is over $\phi_n = \frac{2\pi}{M}\left(n - \frac{M}{2}\right)$, and $1 \le n \le M$.

The function $G^0_{j;\,m}(\beta)$ is approximated by the modified Bessel function $G^0_{j;\,m}(\beta) \approx I_{|j-m|}(\beta)$ provided that $\frac{1}{M}\sum_{n=1}^{M}$ is approximately replaced by $\frac{1}{2\pi}\int_{-\pi}^{\pi} dp$ at large enough $M \gg 1$. Let $\mathcal{D}^K_s$ be differentiation of $K^{\rm th}$ order with respect to s at $s = 0$. Applying $\mathcal{D}^K_{\beta/2}$ to $G^0_{j;\,m}(\beta)$, one obtains the number $|P^0_K(m \to j)|$ of K-step paths between $m^{\rm th}$ and $j^{\rm th}$ sites (*N.M. Bogoliubov // XX0 Heisenberg chain and random walks, J. Math. Sci.* **138** *(2006) 5636*):

$$|P^0_K(m \to j)| = \binom{2L + |m-j|}{L},$$

where L is one-half of the total number of turns: $L \equiv (K - |m-j|)/2$.

• • • The random turns walks and the circulant matrix

Acting by $\mathcal{D}^{K}_{\beta/2}$ on $G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta)$ one obtains the matrix element:

$$\mathfrak{G}(\boldsymbol{\mu}^L;\boldsymbol{\mu}^R\,|K) \equiv \mathcal{D}^{K}_{\beta/2}\,G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta) = \langle\boldsymbol{\mu}^L|\,(-2H)^K\,|\,\boldsymbol{\mu}^R\rangle\,.$$

Due to the orthogonality, it follows that $\mathfrak{G}(\boldsymbol{\mu}^L;\boldsymbol{\mu}^R\,|0) = \delta_{\boldsymbol{\mu}^L\boldsymbol{\mu}^R}$. Let us consider the power series:

$$G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta) \,=\, \sum_{K=0}^{\infty}\frac{(\beta/2)^K}{K!}\,\mathfrak{G}(\boldsymbol{\mu}^L;\boldsymbol{\mu}^R\,|K)\,,$$

where the coefficients $\mathfrak{G}(\boldsymbol{\mu}^L;\boldsymbol{\mu}^R\,|K)$ respect the equation:

$$\begin{aligned}\mathfrak{G}(\boldsymbol{\mu}^L;\boldsymbol{\mu}^R\,|K+1) &= h(M-2N)\,\mathfrak{G}(\boldsymbol{\mu}^L;\boldsymbol{\mu}^R\,|K)\\ &\quad+\sum_{k=1}^{N}\big(\mathfrak{G}(\boldsymbol{\mu}^L;\boldsymbol{\mu}^R+\mathbf{e}_k\,|K)+\mathfrak{G}(\boldsymbol{\mu}^L;\boldsymbol{\mu}^R-\mathbf{e}_k\,|K)\big)\,,\end{aligned}$$

which is supplied with the initial condition $\mathfrak{G}(\boldsymbol{\mu}^L; \boldsymbol{\mu}^R \,|0) = \delta_{\boldsymbol{\mu}^L \boldsymbol{\mu}^R}$, as well as with appropriate periodicity and non-intersection requirements. The corresponding coefficients $\mathfrak{G}^{\,0}(\boldsymbol{\mu}^L; \boldsymbol{\mu}^R \,|K)$ defined as follows,

$$\mathfrak{G}^{\,0}(\boldsymbol{\mu}^L; \boldsymbol{\mu}^R \,|K) \equiv \mathcal{D}^K_{\beta/2}\, G^{\,0}_{\boldsymbol{\mu}^L; \boldsymbol{\mu}^R}(\beta) = \langle \boldsymbol{\mu}^L |\, (-2H_{\mathrm{xx}})^K \,|\, \boldsymbol{\mu}^R \rangle \,,$$

respect the equation above at $h = 0$. Expanding the exponential $e^{\beta h(\frac{M}{2} - N)}$, one obtains the identity:

$$\mathfrak{G}(\boldsymbol{\mu}^L; \boldsymbol{\mu}^R \,|K) \;=\; \sum_{p=0}^{K} \binom{K}{p} \big(h(M - 2N)\big)^p \,\mathfrak{G}^{\,0}(\boldsymbol{\mu}^L; \boldsymbol{\mu}^R \,|K - p) \,,$$

where $\binom{K}{p}$ is the binomial coefficient.

The circulant matrix $\mathbf{\Delta}$ gives $N=1$ solution at $h=0$:

$$\mathfrak{G}^0(j,m|K) = \langle \Uparrow | \, \sigma_j^+ (-2H_{\rm xx})^K \sigma_m^- \, | \Uparrow \rangle = \left(\mathbf{\Delta}^K\right)_{jm}, \tag{4}$$

where $\left(\mathbf{\Delta}^K\right)_{jm}$ is the entry of $K^{\rm th}$ power of $\mathbf{\Delta}$, which obeys

$$\left(\mathbf{\Delta}^{K+1}\right)_{jm} = \left(\mathbf{\Delta}^K\right)_{j,m+1} + \left(\mathbf{\Delta}^K\right)_{j,m-1}.$$

The initial condition is fulfilled since $\mathfrak{G}^0(j,m|0)$ is the Kronecker symbol δ_{jm}. The periodicity is also consistent with the circulant matrix.
Position of the walker on the chain is labelled by spin "down", the empty sites correspond to spin "up" states. Let $|P_K^0(j \to m)|$ to denote the number of K-step paths between $j^{\rm th}$ and $m^{\rm th}$ sites ($h=0$). At $N=1$ one gets $|P_K^0(j \to m)| = \left(\mathbf{\Delta}^K\right)_{jm}$.
In the case of N random turns vicious walkers at $h=0$ with initial and final positions arranged as $\boldsymbol{\mu}^L$ and $\boldsymbol{\mu}^R$, the numbers of nests of K-step paths $|P_K^0(\boldsymbol{\mu}^L \to \boldsymbol{\mu}^R)|$ are given by

Proposition: [Malyshev, Bogoliubov (2022)] *The number of nests of non-intersecting lattice paths of N random turns vicious walkers with K steps is equal to the amplitude $\mathfrak{G}^{\,0}(\boldsymbol{\mu}^L;\boldsymbol{\mu}^R\,|K)$ at $h=0$*:

$$|P_K^0(\boldsymbol{\mu}^L \to \boldsymbol{\mu}^R)| = \mathfrak{G}^{\,0}(\boldsymbol{\mu}^L;\boldsymbol{\mu}^R|K)$$
$$= \sum_{|\mathbf{n}|=K} P(\mathbf{n}) \det\big((\boldsymbol{\Delta}^{n_j})_{\mu_i^L;\mu_j^R}\big)_{1\le i,j\le N}\,,$$

where $\mathbf{n}=(n_1,n_2,\dots,n_N)$, $|\mathbf{n}|\equiv n_1+n_2+\ldots+n_N$, $P(\mathbf{n})$ is the multinomial coefficient,

$$P(\mathbf{n}) \equiv \frac{(n_1+n_2+\ldots+n_N)!}{n_1!\,n_2!\,\ldots n_N!}\,,$$

the entry $(\boldsymbol{\Delta}^n)_{jm}$ *is defined by* (4), *and* $(\boldsymbol{\Delta}^0)_{jm}=\delta_{jm}$.

C. Malyshev, N. Bogoliubov // *Spin correlation functions, Ramus-like identities, and enumeration of constrained lattice walks and plane partitions*, — J. Phys. A: Math. Theor. **55** (2022), 225002

Expression for $\mathfrak{G}(\boldsymbol{\mu}^L; \boldsymbol{\mu}^R|K)$ demonstrates that either a single walker chosen randomly jumps to one of adjacent sites with equal probabilities or all walkers are staying stationary. Right-hand side of

$$\mathfrak{G}(\boldsymbol{\mu}^L; \boldsymbol{\mu}^R\,|K) = \sum_{p=0}^{K} \binom{K}{p} \big(h(M-2N)\big)^p\, \mathfrak{G}^{\,0}(\boldsymbol{\mu}^L; \boldsymbol{\mu}^R\,|K-p)\,,$$

is the polynomial of a single variable $h(M-2N)$. The coefficients $\mathfrak{G}^{\,0}(\boldsymbol{\mu}^L; \boldsymbol{\mu}^R|K-p)$ enumerate, due to Proposition, $(K-p)$-step nests of paths of N walkers. In turn, the number $\binom{K}{p}$ of p-element combinations of the set of K steps enumerates all the possibilities for N walkers to stay stationary p times.

A typical nest of $N = 6$ paths is shown in Figure ($K = 13$, $p = 1$) where dashed lines imply that walkers are staying. As far as $|P^0_{K-p}(\boldsymbol{\mu}^L \to \boldsymbol{\mu}^R)|$ is concerned, the nest in Figure corresponds to $n_1 = 0$, $n_2 = 1$, $n_3 = 3$, $n_4 = 1$, $n_5 = 4$, $n_6 = 3$.

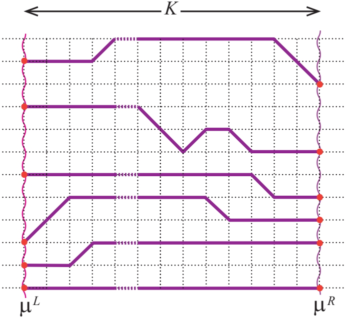


Рис.: Random turns vicious walkers.

• • • Transition amplitude as the generating function of random turns walks

Connection between the solution $G^0_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta)$ in the determinantal form and the series form (with coefficients given by Proposition) is expressed by

Proposition: *The determinantal solution $G^0_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta)$ is the generating function of the numbers $|P^0_K(\boldsymbol{\mu}^L_N \to \boldsymbol{\mu}^R_N)|$ given by Proposition (a kind of generalized Ramus's identity):*

$$\mathcal{D}^K_{\beta/2} \det\big(G^0_{\mu^L_n;\mu^R_k}(\beta)\big)_{1\le n,k\le N} = \mathfrak{G}^0(\boldsymbol{\mu}^L;\boldsymbol{\mu}^R|K)$$

$$= |P^0_K(\boldsymbol{\mu}^L_N \to \boldsymbol{\mu}^R_N)| \left(= \sum_{|\mathbf{n}|=K} P(\mathbf{n}) \det\big((\boldsymbol{\Delta}^{n_j})_{\mu^L_i;\mu^R_j}\big)_{1\le i,j\le N} \right),$$

♦ Ramus's identity Vanishing $\left(\boldsymbol{\Delta}^K\right)_{jm} = 0$ occurs for the circulant matrix $\boldsymbol{\Delta}$ in the case $K - |j-m| = 1 (\bmod 2)$. In the case $K - |j-m| = 0 (\bmod 2)$, Ramus's identity allows us to formulate

Proposition: [Malyshev, Bogoliubov (2022)] *The row-column indices j, m of $M \times M$ matrix respect $|j-m| \leq M-1$. Let us consider $L \equiv \frac{K-|j-m|+pM}{2}$, where $p \in \mathbb{Z}$ is chosen so that $0 \leq L \leq \frac{M}{2}$. Then,*

$$\left(\boldsymbol{\Delta}^K\right)_{jm} = \binom{K}{L\bar{\delta}_{L,\frac{M}{2}}}_{M/2},$$

where $\bar{\delta}_{L,\frac{M}{2}} \equiv 1 - \delta_{L,\frac{M}{2}}$, and the notation for the lacunary sum of binomial coefficients is used:

$$\binom{K}{L}_{M/2} \equiv \sum_{n=0,1,2,\ldots} \binom{K}{L + \frac{M}{2} \cdot n}.$$

****

Ramus's identity (*Christian Ramus (1806–1856) Danmark*)

$$\frac{2^n}{R}\sum_{j=0}^{R-1}\cos^n\frac{\pi j}{R}\cos\frac{\pi j(n-2t)}{R} = \sum_{l=0,1,2,\ldots}\binom{n}{t+R\cdot l}, \quad 0 \le t < R.$$

Ramus C., Solution Generale d'un Probleme d'Analyse Combinatoire, J. reine angew. Math. ***11*** *(1834) 353-355*

The entries in question in terms of the binomial coefficients thus stressing the connection with enumeration of the lattice walks.

Appendix

Useful identities arise provided that $\left(\boldsymbol{\Delta}^K\right)_{jm}$, on one hand, and by *Rimas J. // On computing of arbitrary positive integer powers for one type of even order symmetric circulant matrices — Applied Mathematics and Computation* **172** *(2005) 86;* **174** *(2006) 511*, on another, are equated each to other.

We specify the matrix $\left(\boldsymbol{\Delta}^K\right)_{jm} \equiv \left(\boldsymbol{\Delta}^K\right)_{j-m}$ of the size 6×6 to $K = 14$:

$$\left(\boldsymbol{\Delta}^{14}\right)_0 = \binom{14}{1}_3 = 5462\,,$$
$$\left(\boldsymbol{\Delta}^{14}\right)_2 = \binom{14}{0}_3 = \left(\boldsymbol{\Delta}^{14}\right)_4 = \binom{14}{2}_3 = 5461\,,$$
$$\left(\boldsymbol{\Delta}^{14}\right)_1 = \left(\boldsymbol{\Delta}^{14}\right)_3 = \left(\boldsymbol{\Delta}^{14}\right)_5 = 0\,.$$

We obtain in notations of

$$\left(\boldsymbol{\Delta}^{14}\right)_0 = \frac{a_1}{6} = \frac{2(2^{14}+2)}{6}\,,$$

$$\left(\boldsymbol{\Delta}^{14}\right)_2 = \left(\boldsymbol{\Delta}^{14}\right)_4 = \frac{a_3}{6} = \frac{2(2^{14}-1)}{6} = \frac{a_1}{6} - 1\,.$$

♦ Generalized Ramus's identity

Proposition 5 (generalized Ramus's identity): *The identity is valid*:

$$\sum_{|\mathbf{n}|=K} P(\mathbf{n})\, \boldsymbol{\Delta}^{\mathbf{n}}_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R} = \frac{2^{K+N}}{M^N} \sum_{\mathbf{l}_N \in \mathsf{P}^N} \left(\sum_{k=1}^{N} \cos\left(\frac{2\pi}{M} l_k\right)\right)^K \times \prod_{s=1}^{N} \cos\left(\frac{2\pi}{M} l_s (\mu_s^L - \mu_s^R)\right), \qquad (5)$$

where

$$\boldsymbol{\Delta}^{\mathbf{n}}_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R} \equiv \prod_{j=1}^{N} (\boldsymbol{\Delta}^{n_j})_{\mu_j^L;\mu_j^R},$$

$(\boldsymbol{\Delta}^n)_{jm}$ *is defined by above, and* $(\boldsymbol{\Delta}^0)_{jm} = \delta_{jm}$. *Summation indices* n_j *in left-hand side of* (5) *are of the same parity as* $|\mu_j^L - \mu_j^R|$, $1 \le j \le N$. *Summation in right-hand side of* (5) *is over* N*-tuples* $\mathbf{l}_N = (l_1, l_2, \ldots, l_N)$, $l_k \in \mathsf{P} \equiv \{0, 1, \ldots, \frac{M}{2} - 1\}$.

- **Corollary:** *Determinantal generalization*:

$$\sum_{|\mathbf{n}|=K} P(\mathbf{n}) \det\big((\boldsymbol{\Delta}^{n_j})_{\mu_i^L;\mu_j^R}\big)_{1\le i,j\le N} = \frac{1}{M^N} \sum_{\{\boldsymbol{\phi}_N\}} \Big(2\sum_{m=1}^{N} \cos\phi_m\Big)^K \times |\mathcal{V}(e^{i\boldsymbol{\phi}_N})|^2 S_{\boldsymbol{\lambda}^L}(e^{i\boldsymbol{\phi}_N}) S_{\boldsymbol{\lambda}^R}(e^{-i\boldsymbol{\phi}_N}),$$

where the entries $(\boldsymbol{\Delta}^{n_j})_{\mu_i^L;\mu_j^R}$, $1 \le i,j \le N$, *are given above.*
Summation is over N*-tuples* $\boldsymbol{\phi}_N = (\phi_{k_1}, \phi_{k_2}, \dots, \phi_{k_N})$, *where* $\phi_n = \frac{2\pi}{M}\left(n - \frac{M}{2}\right)$ *and* $M \ge k_1 > k_2 > \dots > k_N \ge 1$.

IV. Norm-trace generating function of plane partitions

••• N-Particle averages and plane partitions with constrained columns

Let us obtain the average of $e^{\mathcal{Q}}$ over 'off-shell' N-particle states:

$$\langle \Psi(\mathbf{v}_N) \mid e^{\mathcal{Q}} \mid \Psi(\mathbf{u}_N) \rangle = \mathcal{P}(\mathbf{v}_N^{-2}, \mathbf{u}_N^2, \mathbf{a}_M)\,,$$

where $$\mathcal{P}(\mathbf{v}_N^{-2}, \mathbf{u}_N^2, \mathbf{a}_M) \equiv \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{v}_N^{-2}) S_{\boldsymbol{\lambda}}(\mathbf{u}_N^2) \prod_{i=1}^{N} e^{\alpha_{\mu_i}}$$

is parameterized by M-tuple $\mathbf{a}_M \equiv (\alpha_1, \alpha_2, \ldots, \alpha_M)$, while $\boldsymbol{\lambda} = \boldsymbol{\mu} - \boldsymbol{\delta}$ connects the parts. The generic *Cauchy–Binet formula* leads to

Proposition: *The sum* $\mathcal{P}(\mathbf{v}_N^{-2}, \mathbf{u}_N^2, \mathbf{a}_M)$ *parameterized by* M*-tuple* $\mathbf{a}_M$ *admits the determinantal representation*:

$$\mathcal{P}(\mathbf{v}_N^{-2}, \mathbf{u}_N^2, \mathbf{a}_M) = \frac{1}{\mathcal{V}(\mathbf{u}_N^2)\mathcal{V}(\mathbf{v}_N^{-2})} \det \left(\sum_{n=1}^{M} e^{\alpha_n} \left(\frac{u_i^2}{v_j^2} \right)^{n-1} \right)_{1 \le i,j \le N},$$

where the Vandermonde determinant is used.

Let us introduce the 'tilded' notations to be used below.

Definition: *Let us fix a strict partition* $\mathbf{k}_l \equiv (k_1, k_2, \dots, k_l)$ *of length* l, $M \geq k_1 > k_2 > \dots > k_l \geq 1$ (*clearly,* $k_j \geq l - j + 1$, $1 \leq j \leq l$). *Among all strict partitions* $\boldsymbol{\mu}_N$, *there exists a subset of such partitions* $\widetilde{\boldsymbol{\mu}}_N$ *that* l *their parts are given by* $\mathbf{k}_l$, $l \leq N$. *We introduce strict partitions* $\mathbf{m}_l \subset \boldsymbol{\delta}_N$ *such that parts of* $\mathbf{m}_l$ *label positions of the parts of* $\mathbf{k}_l$ *in* $\widetilde{\boldsymbol{\mu}}_N$. *Non-strict partitions* $\widetilde{\boldsymbol{\lambda}} \equiv \widetilde{\boldsymbol{\lambda}}_N = \widetilde{\boldsymbol{\mu}}_N - \boldsymbol{\delta}_N$ *are characterized by* l *parts occupying the positions* $\mathbf{m}_l$ *and given by non-strict partition* $\mathbf{k}_l - \mathbf{m}_l$.

Then, off-shell N-particle matrix element of $\mathit{\Pi}_{\mathbf{k}}$ arises:

$$\langle \Psi(\mathbf{v}_N) \mid \mathit{\Pi}_{\mathbf{k}} \mid \Psi(\mathbf{u}_N) \rangle = \lim_{\mathsf{a}_M \to 0} \partial^l_{\alpha_{k_1} \alpha_{k_2} \dots \alpha_{k_l}} \times \langle \Psi(\mathbf{v}_N) \mid e^{\mathcal{Q}} \mid \Psi(\mathbf{u}_N) \rangle = \widetilde{\mathcal{P}}(\mathsf{v}_N^{-2}, \mathsf{u}_N^2, \mathbf{k}_l)\,,$$

where the tilded notation $\widetilde{\mathcal{P}}(\mathsf{v}_N^{-2}, \mathsf{u}_N^2, \mathbf{k}_l)$ implies the sum

$$\widetilde{\mathcal{P}}(\mathsf{v}_N^{-2}, \mathsf{u}_N^2, \mathbf{k}_l) \equiv \sum_{\{\widetilde{\boldsymbol{\lambda}}\}} S_{\widetilde{\boldsymbol{\lambda}}}(\mathsf{v}_N^{-2}) S_{\widetilde{\boldsymbol{\lambda}}}(\mathsf{u}_N^2)\,,$$

and summation goes over $\widetilde{\boldsymbol{\lambda}}$ with respect to fixed l-tuple $\mathbf{k}_l$.

The matrix element of $\mathit{\Pi}_{\mathbf{k}}$ under q-parametrization,

$$\mathbf{v}^{-2} = \mathbf{q}_N \equiv (q, q^2, \ldots, q^N)\,, \qquad \mathbf{u}^2 = \mathbf{q}_N / q\,,$$

arises:

$$< \mathit{\Pi}_{\mathbf{k}} >_{N,q} \equiv \langle \Psi(\mathbf{q}_N^{-1/2}) \mid \mathit{\Pi}_{\mathbf{k}} \mid \Psi((\mathbf{q}_N/q)^{1/2}) \rangle = \widetilde{\mathcal{P}}\Big(\mathbf{q}_N, \frac{\mathbf{q}_N}{q}, \mathbf{k}_l\Big)\,.$$

Equation in the case $\mathbf{k}_l = \boldsymbol{\delta}_l$ reads:

$$< \prod_{i=1}^{l} \mathsf{q}_i >_{N,q} = \widetilde{\mathcal{P}}\Big(\mathbf{q}_N, \frac{\mathbf{q}_N}{q}, \boldsymbol{\delta}_l\Big)\,,$$

so that $\widetilde{\boldsymbol{\mu}}_N$ and $\widetilde{\boldsymbol{\lambda}}_N = \widetilde{\boldsymbol{\mu}}_N - \boldsymbol{\delta}_N$ are concretized:

$$\widetilde{\boldsymbol{\mu}}_N = (\mu_1, \mu_2, \ldots, \mu_{N-l}, l, l-1, \ldots, 1)\,,$$
$$\widetilde{\boldsymbol{\lambda}}_N = (\lambda_1, \lambda_2, \ldots, \lambda_{N-l}, 0, 0, \ldots, 0)\,,$$

and summation in $\widetilde{\mathcal{P}}_{\mathcal{M}}$ is over $\mathcal{M} \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-l} \geq 0$.

Right-hand side provides the generating function of watermelons:

$$\lim_{q\to 1} < \prod_{i=1}^{l} \mathsf{q}_i >_{N,q} = \widetilde{\mathcal{P}}(\mathbf{1}_N, \mathbf{1}_N, \boldsymbol{\delta}_l) = \sum_{\{\widetilde{\boldsymbol{\lambda}}\}} S_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{1}_N) S_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{1}_N)\,.$$

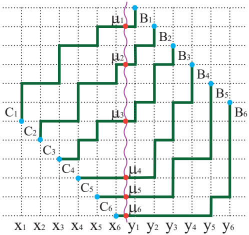


Рис.: *Watermelon* as the nest of lattice paths at $\mathcal{M} = 6$, $N = 6$.

Indeed, $S_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{1}_N)$ corresponds to paths connecting the points $C_i = (i, N + 1 - i)$ and $(N, \widetilde{\mu}_i)$. The nest in Figure is just depicted for $\widetilde{\boldsymbol{\mu}}$ at $l = 3$. An $i^{\rm th}$ path in Figure makes $\lambda_i \in \boldsymbol{\lambda}_{N-l} \equiv (\lambda_1, \lambda_2, \ldots, \lambda_{N-l})$ steps upwards at $1 \le i \le N - l$, while only rightward steps are allowed at $N - l + 1 \le i \le N$. Notice that $\mu_4 = 3$, $\mu_5 = 2$, $\mu_6 = 1$ on Figure.

As far as the bijection between the watermelons and the plane partitions is concerned, the watermelons characterized by $\widetilde{\boldsymbol{\mu}}_N$ and $\widetilde{\boldsymbol{\lambda}}_N$ are mapped to such stacks of cubes that $l \times l$ square remains empty on the bottom of $N \times N \times \mathcal{M}$ box. Indeed, the watermelon in Figure is characterized by $\mu_4 = 3$, $\mu_5 = 2$, $\mu_6 = 1$, and $\lambda_4 = \lambda_5 = \lambda_6 = 0$. Three columns of zero height on the diagonal of the plane partition imply the empty 3×3 square dashed in Figure. Therefore, $\widetilde{\mathcal{P}}_{\mathcal{M}}(\mathbf{1}_N, \mathbf{1}_N, \boldsymbol{\delta}_l)$ enumerates the plane partitions constrained by presence of the empty square: stacks are forbidden.

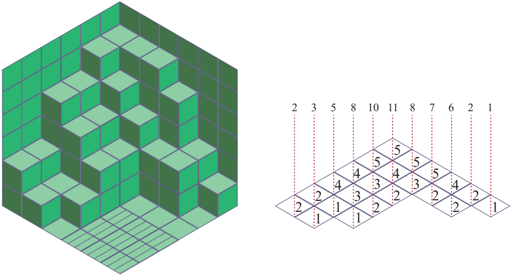


Рис.: Plane partition with $|\boldsymbol{\lambda}| = 11$ equivalent to watermelon in Figure 6.

In the case of $l = 0$, the sum $\mathcal{P}_{\mathcal{M}}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{0}_M)$ provides the number of such watermelons that upward steps are allowed for all paths from 1^{st} to N^{th}.
The number $\mathcal{P}_{\mathcal{M}}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{0}_M)$ gives the number $A(N, N, M-N)$ of plane partitions in $N \times N \times (M-N)$ box (MacMahon formula):

$$\mathcal{P}_{\mathcal{M}}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{0}_M) = A(N, N, M-N) = \prod_{k=1}^{N} \prod_{j=1}^{N} \frac{M-N+k+j-1}{k+j-1} \,.$$

Generally, $\widetilde{\mathcal{P}}_{\mathcal{M}}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{k}_l)$,

$$\lim_{q\to 1} < \mathit{\Pi}_{\mathbf{k}} >_{N,q} = \lim_{q\to 1} < \prod_{i=1}^{l} \mathsf{q}_i >_{N,q} = \widetilde{\mathcal{P}}_{\mathcal{M}}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{k}_l)\,,$$

where

$$< \mathit{\Pi}_{\mathbf{k}} >_{N,q} \equiv \langle \Psi(\mathbf{q}_N^{-1/2}) \mid \mathit{\Pi}_{\mathbf{k}} \mid \Psi((\mathbf{q}_N/q)^{1/2}) \rangle = \widetilde{\mathcal{P}}\Big(\mathbf{q}_N, \frac{\mathbf{q}_N}{q}, \mathbf{k}_l\Big)\,.$$

enumerates the plane partitions with l columns of heights given by $\mathbf{k}_l - \mathbf{m}_l$ at the positions labelled by parts of $\mathbf{m}_l$ (cf. Definition of non-strict partitions $\widetilde{\boldsymbol{\lambda}} \equiv \widetilde{\boldsymbol{\lambda}}_N = \widetilde{\boldsymbol{\mu}}_N - \boldsymbol{\delta}_N$ characterized by l parts at the positions $\mathbf{m}_l$ and given by non-strict partition $\mathbf{k}_l - \mathbf{m}_l$).

• • • The N-particle mean values and plane partitions with constrained diagonal volume

Norm-trace generating function $G(N, N, \mathcal{M}|\, q, \gamma)$ is the generating function of plane partitions with fixed total volume of the parts on principal diagonal in box of height $\mathcal{M}$ and bottom of size $N \times N$.

Derived in *Stanley R.P. // The conjugate trace and trace of a plane partition — J. Comb. Theor. A, vol 14 (1973) 53* and generalized in *Gansner E. // The enumeration of plane partitions via the Burge correspondence — Illinois J. Math. 25 (1981) 533*.
The determinantal representation for $G(N, N, \mathcal{M}|\, q, \gamma)$ has been derived for the model of strongly correlated bosons, *N. Bogoliubov, C. Malyshev // The phase model and the norm-trace generating function of plane partitions — Journal of Statistical Mechanics: Theory and Experiment, vol 2018 (2018) 083101*.

The norm-trace generating function $G(N,N,\mathcal{M}|\, q,\gamma)$ arises from – (see -42-) – under the q-parametrization

$$\mathbf{v}^{-2} = \mathbf{q}_N \equiv (q, q^2, \ldots, q^N)\,, \qquad \mathbf{u}^2 = \mathbf{q}_N/q\,.$$

Indeed, let us consider the linear parametrization of $\mathbf{a}_M$ and specify α_n so that $e^{\alpha_n} = \gamma^n$, $0 < \gamma \le 1$. Using $< e^{\mathcal{Q}(\gamma)} >_{N,q}$ to denote the q-parameterized average,

$$< e^{\mathcal{Q}(\gamma)} >_{N,q} \equiv \langle \Psi(\mathbf{q}_N^{-1/2}) \,|\, e^{\mathcal{Q}(\gamma)} \,|\, \Psi((\mathbf{q}_N/q)^{1/2}) \rangle\,,$$

one formulates

Proposition: *The determinantal representation for the norm-trace generating function of plane partitions with fixed total volume of the parts on principal diagonal in a box of height $\mathcal{M}$ and $N \times N$ bottom is given:*

$$G(N,N,\mathcal{M}|\, q,\gamma) = \gamma^{\frac{-N}{2}(N+1)} < e^{\mathcal{Q}(\gamma)} >_{N,q}$$

$$= \sum_{\boldsymbol{\lambda} \subseteq \{^N\}} \gamma^{|\boldsymbol{\lambda}|} S_{\boldsymbol{\lambda}}\Big(\frac{\mathbf{q}}{q}\Big) S_{\boldsymbol{\lambda}}(\mathbf{q}) = \frac{\det\big(\mathsf{h}_M(\gamma\, q^{i+j-1})\big)_{1\le i,j\le N}}{\mathcal{V}(\mathbf{q}_N/q)\mathcal{V}(\gamma\,\mathbf{q}_N)}\,,$$

where $\mathsf{h}_M(x) \equiv \frac{1-x^M}{1-x}$.

The series above is the norm-trace generating function of plane partitions with fixed height of their diagonal parts in $N \times N \times \mathcal{M}$ box.
Equation at $\gamma = 1$ gives the determinantal formula for the generating function of boxed plane partitions in $N \times N \times \mathcal{M}$ box:

$$\lim_{q\to 1} G(N, N, \mathcal{M}|\, q, 1) = A(N, N, \mathcal{M})\,,$$

where $A(N, N, \mathcal{M})$ is the number of plane partitions.
Assume that the approximation $\mathsf{h}_M(x) \simeq (1-x)^{-1}$ is valid at $|x| < 1$ and large enough M. Then:

$$\lim_{M\to\infty} G(N, N, \mathcal{M}|\, q, \gamma) = \frac{\det\Big(\big(1-\gamma q^{i+j-1}\big)^{-1}\Big)_{1\le i,j\le N}}{\mathcal{V}(\mathbf{q}_N/q)\mathcal{V}(\gamma\,\mathbf{q}_N)}$$

$$= \prod_{i=1}^{N}\prod_{j=1}^{N} \frac{1}{1-\gamma q^{i+j-1}}\,.$$

Evaluation of the Cauchy-type determinant leads to the double product, which is nothing but the norm-trace generating function of plane partitions with unbounded height.

Further,

$$\lim_{N/M \ll 1, N \to \infty} G(N, N, \mathcal{M}|\, q, \gamma) \;=\; \prod_{n=1}^{\infty} \frac{1}{(1-\gamma q^n)^n}\,.$$

V. Matrix elements of the generating exponential on the Bethe states and the lattice paths

Consider the normalized matrix element on the on-shell Bethe states:

$$\langle e^{\mathcal{Q}}\, e^{-\beta H} \rangle_N \equiv \frac{\langle \Psi(e^{i\boldsymbol{\theta}_N/2}) | e^{\mathcal{Q}}\, e^{-\beta H} | \Psi(e^{i\boldsymbol{\theta}_N/2}) \rangle}{\mathcal{N}^2(e^{i\boldsymbol{\theta}_N/2})} \,.$$

We express $\langle e^{\mathcal{Q}}\, e^{-\beta H} \rangle_N$ in the integral form at $M \gg 1$:

$$\langle e^{\mathcal{Q}}\, e^{-\beta H} \rangle_N \simeq \frac{e^{\beta h M/2}}{\mathcal{N}^2(e^{i\boldsymbol{\theta}_N/2}) N!} \int\limits_{\mathsf{S}^N} \mathcal{P}(e^{-i\mathbf{p}_N}, e^{i\boldsymbol{\theta}_N}, \mathbf{0}_M)$$

$$\times\, \mathcal{P}(e^{-i\boldsymbol{\theta}_N}, e^{i\mathbf{p}_N}, \mathbf{a}_M)\, |\mathcal{V}(e^{i\mathbf{p}_N})|^2\, e^{\beta \sum_{l=1}^{N} (\cos p_l - h)} \frac{d^N p}{(2\pi)^N} \,,$$

where $\mathbf{p}_N \equiv (p_1, p_2, \ldots, p_N)$, $d^N p = dp_1 dp_2 \cdots dp_N$, and the integration domain S^N is N-fold product of $\mathsf{S} \equiv [-\pi, \pi]$.

Provided that the transition amplitude $G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta)$ is used, the representation takes equivalent form:

$$\langle e^{\mathcal{Q}}\, e^{-\beta H}\rangle_N \;\simeq\; \mathcal{N}^{-2}(e^{i\boldsymbol{\theta}_N/2}) \sum_{\boldsymbol{\lambda}^{L,R}\subseteq\{^N\}} S_{\boldsymbol{\lambda}^L}(e^{-i\boldsymbol{\theta}_N})\, S_{\boldsymbol{\lambda}^R}(e^{i\boldsymbol{\theta}_N}) \times \exp\Big(\sum_{k=1}^{N} \alpha_{\mu_k^L}\Big)\, G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta)\,,$$

where $G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta)$ is expressed at $M \gg 1$:

$$G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta) \;\simeq\; e^{\beta h(\frac{M}{2}-N)} \det\big(I_{|\mu_i^L-\mu_j^R|}(\beta)\big)_{1\le i,j\le N}\,.$$

Applying $\lim_{\mathsf{a}_M \to 0} \partial^l_{\alpha_1 \alpha_2 \ldots \alpha_l}$ to the numerator taken over the ground state solution, one obtains:

$$\mathcal{D}^K_{\beta/2} \langle \Psi(e^{i\boldsymbol{\theta}^{\mathrm{g}}_N/2}) \Big| \prod_{i=1}^{l} \mathsf{q}_i \, e^{-\beta H} \Big| \Psi(e^{i\boldsymbol{\theta}^{\mathrm{g}}_N/2}) \rangle = \mathfrak{P}(e^{i\boldsymbol{\theta}^{\mathrm{g}}_N/2}; e^{i\boldsymbol{\theta}^{\mathrm{g}}_N/2} \,|K)\,,$$

where $\mathfrak{P}(e^{i\boldsymbol{\theta}^{\mathrm{g}}_N/2}; e^{i\boldsymbol{\theta}^{\mathrm{g}}_N/2} \,|K)$ is the value of the polynomial

$$\mathfrak{P}(\mathbf{v}_N; \mathbf{u}_N \,|K) \equiv \sum_{\widetilde{\boldsymbol{\lambda}}^L, \boldsymbol{\lambda}^R} S_{\widetilde{\boldsymbol{\lambda}}^L}(\mathbf{v}_N^{-2}) S_{\boldsymbol{\lambda}^R}(\mathbf{u}_N^2)\, \mathfrak{G}(\widetilde{\boldsymbol{\mu}}^L; \boldsymbol{\mu}^R \,|K)$$

at $\mathbf{u}_N = \mathbf{v}_N = e^{i\boldsymbol{\theta}^{\mathrm{g}}_N/2}$. Summation is over $\boldsymbol{\lambda}^R$ and $\widetilde{\boldsymbol{\lambda}}^L$, while $\mathfrak{G}(\widetilde{\boldsymbol{\mu}}^L; \boldsymbol{\mu}^R \,|K)$ is given above.

The replacement $e^{i\boldsymbol{\theta}_N^{\mathrm{g}}} \longmapsto 1$ is appropriate at $M \gg N$, and one obtains:

$$\lim_{q\to 1} \mathcal{D}^K_{\beta/2} < \prod\nolimits_{i=1}^{l} \mathsf{q}_i \, e^{-\beta H} >_{N,q} = \mathfrak{P}(\mathbf{1}_N; \mathbf{1}_N | K)\,,$$

where $< \cdot >_{N,q}$ is defined above. Right-hand side is expressed:

$$\mathfrak{P}(\mathbf{1}_N; \mathbf{1}_N | K) = \sum_{p=0}^{K} \binom{K}{p} \big(h(M-2N)\big)^p \, \mathfrak{P}^0(\mathbf{1}_N; \mathbf{1}_N | K-p)\,,$$

where $\mathfrak{P}^0(\mathbf{1}_N; \mathbf{1}_N | K-p)$ corresponds to $\mathfrak{G}^0(\widetilde{\boldsymbol{\mu}}^L; \boldsymbol{\mu}^R \,|\cdot)$,

$$\mathfrak{P}^0(\mathbf{1}_N; \mathbf{1}_N | K-p) = \sum_{\widetilde{\boldsymbol{\lambda}}^L, \boldsymbol{\lambda}^R} S_{\widetilde{\boldsymbol{\lambda}}^L}(\mathbf{1}_N) S_{\boldsymbol{\lambda}^R}(\mathbf{1}_N) \, |P^0_{K-p}(\widetilde{\boldsymbol{\mu}}^L \to \, \boldsymbol{\mu}^R)|\,, \quad (6)$$

and $|P^0_{K-p}(\widetilde{\boldsymbol{\mu}}^L \to \, \boldsymbol{\mu}^R)|$ is given by Proposition 3.

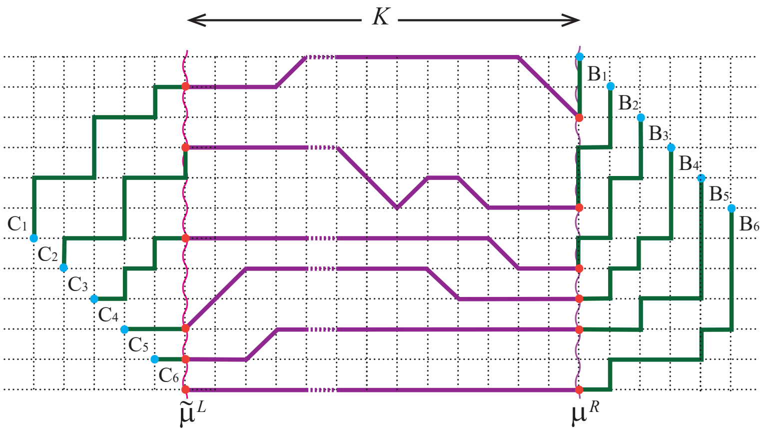


Рис.: Nest of paths contributing to $\mathfrak{P}(\mathbf{1}_N; \mathbf{1}_N|K)$ at $N = 6$, $K = 13$, and $p = 1$.

The coefficient $\mathfrak{P}^0(\mathbf{1}_N; \mathbf{1}_N|K-p)$ (56) enumerates, according to (6), the compound paths. Indeed, the factor $S_{\widetilde{\boldsymbol{\lambda}}^L}(\mathbf{1}_N)$ in (6) corresponds to walks by the lock steps rules from C_i, $1 \le i \le N$, to the sites $\widetilde{\boldsymbol{\mu}}^L$ (Figure 1). The contribution $|P^0_{K-p}(\widetilde{\boldsymbol{\mu}}^L \to \boldsymbol{\mu}^R)|$ in (6) corresponds to the random turns walks from $\widetilde{\boldsymbol{\mu}}^L$ to $\boldsymbol{\mu}^R$. The factor $S_{\boldsymbol{\lambda}^R}(\mathbf{1}_N)$ accounts for the lock steps walks from $\boldsymbol{\mu}^R$ to B_i, $1 \le i \le N$. The coefficients at powers of $h(M-2N)$ in $\mathfrak{P}(\mathbf{1}_N; \mathbf{1}_N|K)$ (56) are responsible for enumeration of non-intersecting lattice walks corresponding to (6) but with stays inserted. Therefore (55) and (56) demonstrate that the matrix element of $e^{\mathcal{Q}}\, e^{-\beta H}$ over on-shell Bethe states is the generating function of numbers of nests of non-intersecting paths of the type presented in Figure 8. A typical nest is depicted in Figure 8 for $K=13$ and $p=1$, so that the paths are characterized by $\mathbf{n}=(0,1,3,1,4,3)$, $|\mathbf{n}|=12$.

VI. Constrained lattice paths and plane partitions

• • • The N-particle mean values and constrained lattice paths

Let trace to imply summation over all N-particle Bethe solutions, and let us consider the N-particle trace of the Boltzmann-weighted generating exponential:

$$\mathrm{tr}_N(e^{\mathcal{Q}} e^{-\beta H}) \equiv \sum_{\{\boldsymbol{\theta}_N\}} \langle e^{\mathcal{Q}}\, e^{-\beta H} \rangle_N \,,$$

$$\langle e^{\mathcal{Q}}\, e^{-\beta H} \rangle_N \equiv \frac{\langle \Psi(e^{i\boldsymbol{\theta}_N/2}) | e^{\mathcal{Q}}\, e^{-\beta H} | \Psi(e^{i\boldsymbol{\theta}_N/2}) \rangle}{\mathcal{N}^2(e^{i\boldsymbol{\theta}_N/2})} \,.$$

The definition leads to N-particle mean value dependent on β:

$$G_{N,\beta} \equiv \langle\langle e^{\mathcal{Q}} \rangle\rangle_{N,\beta} \equiv \mathrm{tr}_N(e^{\mathcal{Q}} \boldsymbol{\rho}_N) \,, \qquad \boldsymbol{\rho}_N \equiv \frac{e^{-\beta H}}{\mathrm{tr}_N(e^{-\beta H})} \,.$$

In order to investigate $G_{N,\beta}$ at $M \gg 1$, let us use the representation:

$$\mathrm{tr}_N(e^{\mathcal{Q}} e^{-\beta H}) = \sum_{\{\boldsymbol{\mu}_N\}} \exp\Big(\sum_{k=1}^{N} \alpha_{\mu_k}\Big)\, G_{\boldsymbol{\mu};\boldsymbol{\mu}}(\beta) \,.$$

The transition amplitude $G_{\boldsymbol{\mu}^L;\boldsymbol{\mu}^R}(\beta)$ is related with enumeration of random walks of N vicious walkers with initial/final positions given by partitions $\boldsymbol{\mu}^L$ and $\boldsymbol{\mu}^R$. Only closed walks contribute to $\mathrm{tr}\,_N(e^{\mathcal{Q}}e^{-\beta H})$, since the summands in right-hand side contain the diagonal entries $G_{\boldsymbol{\mu};\boldsymbol{\mu}}(\beta)$.

- With regard at the definitions, we obtain:

$$\langle\langle \mathit{\Pi}_{\mathbf{k}} \rangle\rangle_{N,\beta}\Big|_{M\gg 1} = \frac{\mathrm{tr}\,_N(\mathit{\Pi}_{\mathbf{k}}\, e^{-\beta H})}{\mathrm{tr}\,_N(e^{-\beta H})}\Big|_{M\gg 1} \simeq \frac{\widetilde{\mathcal{I}}_N(\beta|\mathbf{k}_l)}{\mathcal{I}_N(\beta, \mathbf{0}_M)}\,,$$

where

$$\widetilde{\mathcal{I}}_N(\beta|\mathbf{k}_l) \equiv \sum_{\{\widetilde{\boldsymbol{\mu}}_N\}} \det\Big(I_{|\widetilde{\mu}_i-\widetilde{\mu}_j|}(\beta)\Big)_{1\le i,j\le N}\,,$$

and summation is over strict partitions with marked parts for a fixed l-tuple $\mathbf{k}_l$.

The Ramus's identity tells us that $\widetilde{\mathcal{I}}_N(\beta|\mathbf{k}_l)$ is the generating function of numbers $\widetilde{\mathcal{P}}_{K,N}(\mathbf{k}_l)$ of nests of trajectories (Proposition 6):

$$\mathcal{D}^K_{\beta/2}\,\widetilde{\mathcal{I}}_N(\beta|\mathbf{k}_l) = \widetilde{\mathcal{P}}_{K,N}(\mathbf{k}_l)\,,$$

where

$$\widetilde{\mathcal{P}}_{K,N}(\mathbf{k}_l) \equiv \sum_{\{\widetilde{\boldsymbol{\mu}}_N\}} \left| P_K^0(\widetilde{\boldsymbol{\mu}}_N \to \widetilde{\boldsymbol{\mu}}_N) \right|,$$

$$\left| P_K^0(\widetilde{\boldsymbol{\mu}}_N \to \widetilde{\boldsymbol{\mu}}_N) \right| = \sum_{|\mathbf{n}|=K} P(\mathbf{n}) \det \left(\binom{n_j}{\frac{n_j+\widetilde{\mu}_j-\widetilde{\mu}_i}{2}} \right)_{1\le i,j \le N}.$$

Equation defines the number of nests of trajectories of N random turns vicious walkers initially located at all admissible $\widetilde{\boldsymbol{\mu}}_N$ and returning after K steps to their initial positions. The generating function $e^{\beta h(\frac{M}{2}-N)}\widetilde{\mathcal{I}}_N(\beta|\mathbf{k}_l)$ enables enumeration of nests of closed paths with stays allowed and with a part of initial/final positions pinned.

••• Diagonally constrained plane partitions

The representation for $\mathrm{tr}_N(e^{\mathcal{Q}} e^{-\beta H})$ is estimated at $M \gg 1$:

$$\frac{\mathrm{tr}_N(e^{\mathcal{Q}} e^{-\beta H})}{e^{\beta h M/2}} = \frac{1}{N!} \int_{\mathsf{S}^N} \mathcal{P}(e^{-i\mathbf{p}_N}, e^{i\mathbf{p}_N}, \mathbf{a}_M) \, |\mathcal{V}(e^{i\mathbf{p}_N})|^2 \times e^{\beta \sum_{l=1}^{N} (\cos p_l - h)} \frac{d^N p}{(2\pi)^N},$$

where the integration domain S^N is N-fold product of $\mathsf{S} \equiv [-\pi, \pi]$, and $\mathcal{P}(e^{-i\mathbf{p}_N}, e^{i\mathbf{p}_N}, \mathbf{a}_M)$ is given by

$$\mathcal{P}(\mathbf{v}_N^{-2}, \mathbf{u}_N^2, \mathbf{a}_M) \equiv \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{v}_N^{-2}) S_{\boldsymbol{\lambda}}(\mathbf{u}_N^2) \prod_{i=1}^{N} e^{\alpha_{\mu_i}}$$

is the sum parameterized by M-tuple $\mathbf{a}_M \equiv (\alpha_1, \alpha_2, \ldots, \alpha_M)$, while the parts of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are related.

Let us introduce the notation $\mathbf{a}_M^\gamma \equiv \log\gamma \cdot (1, 2, \ldots, M)$, which implies that the parametrization $\alpha_n = n \log\gamma$, $0 < \gamma \leq 1$, is used in $\mathcal{P}$. Then we obtain the estimate at $1 \ll M \ll \beta$:

$$\frac{\mathrm{tr}_N(e^{\mathcal{Q}(\gamma)} e^{-\beta H})}{e^{\beta h M/2}} \simeq \mathcal{P}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{a}_M^\gamma)\, V_N(\beta, h)\,,$$

$$V_N(\beta, h) \equiv \frac{e^{\beta N(1-h)}\, \mathfrak{I}_N}{\beta^{N^2/2}} = e^{\beta N(1-h) - \frac{N^2}{2}\log\beta + \varphi_N}\,, \quad \varphi_N \equiv \log \mathfrak{I}_N\,,$$

where $\mathfrak{I}_N$ is Mehta integral expressed in terms of the Barnes G-function:

$$\mathfrak{I}_N = \frac{G(N+1)}{(2\pi)^{N/2}}\,, \qquad G(N+1) \equiv \frac{(N!)^N}{1^1\, 2^2\, \ldots N^N} = \prod_{k=1}^{N} \Gamma(k)\,.$$

The behaviour of $\mathfrak{I}_N = e^{\varphi_N}$ is governed at $1 \ll N \ll M$ by the estimate:

$$\varphi_N = \frac{N^2}{2}\log N - \frac{3N^2}{4} + \mathcal{O}(\log N)\,, \qquad N \gg 1\,.$$

It is seen that $V_N(\beta, h)$ only depends on β, N via the ratio $\frac{\beta}{N}$ at $\beta > N \gg 1$.

The coefficient $\mathcal{P}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{a}_M^\gamma)$ arises at $q \to 1$ from the q-parametrized sum:

$$< e^{\mathcal{Q}(\gamma)} >_{N,q} \equiv \mathcal{P}\Big(\mathbf{q}_N, \frac{\mathbf{q}_N}{q}, \mathbf{a}_M^\gamma\Big) \underset{q\to 1}{\longrightarrow} \mathcal{P}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{a}_M^\gamma)\,,$$

where $< e^{\mathcal{Q}(\gamma)} >_{N,q}$ is defined by

$$\begin{aligned} < e^{\mathcal{Q}(\gamma)} >_{N,q} &= \gamma^{\frac{N}{2}(N+1)} \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^N\}} \gamma^{|\boldsymbol{\lambda}|} S_{\boldsymbol{\lambda}}\Big(\frac{\mathbf{q}}{q}\Big) S_{\boldsymbol{\lambda}}(\mathbf{q}) \\ &= \frac{\gamma^{\frac{N}{2}(N+1)}}{\mathcal{V}(\mathbf{q}_N/q)\mathcal{V}(\gamma\,\mathbf{q}_N)} \det\left(\sum_{n=0}^{M-1} \gamma^n q^{n(i+j-1)}\right)_{1\le i,j\le N}, \end{aligned}$$

where $|\boldsymbol{\lambda}|$ is the volume of $\boldsymbol{\lambda}$, and the homogeneity property $\gamma^{|\boldsymbol{\lambda}|} S_{\boldsymbol{\lambda}}(\mathbf{q}) = S_{\boldsymbol{\lambda}}(\gamma\,\mathbf{q})$ is used.

The limiting expression leads from the generating function $G(N, N, \mathcal{M}|\, 1, \gamma)$ of plane partitions with fixed sum of their diagonal parts confined in $N \times N \times \mathcal{M}$ box to the case of box with infinite height:

$$\mathcal{P}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{a}^{\gamma}_M))\Big|_{M\gg 1} = \gamma^{\frac{N}{2}(N+1)}\, G(N, N, \mathcal{M}|\, 1, \gamma)\Big|_{M\gg 1}$$

$$\underset{M\to\infty}{\longrightarrow} \gamma^{\frac{N}{2}(N+1)} \lim_{q\to 1} \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{1}{1-\gamma q^{i+j-1}}\,,$$

where

$$G(N, N, \mathcal{M}|\, 1, \gamma) = \sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^N\}} \gamma^{|\boldsymbol{\lambda}|} S_{\boldsymbol{\lambda}}(\mathbf{1}) S_{\boldsymbol{\lambda}}(\mathbf{1})$$

- The mean value of the generating exponential is estimated:

$$G_{N,\beta}(\mathbf{a}_M^{\gamma})\Big|_{1\ll M\ll\beta} \simeq \gamma^{\frac{N}{2}(N+1)} \frac{G(N,N,\mathcal{M}|\,1,\gamma)\Big|_{M\gg1}}{A(N,N,\mathcal{M})\Big|_{M\gg1}},$$

where $A(N,N,\mathcal{M})$ is the number of unconstrained plane partitions in $N\times N\times\mathcal{M}$ box ($\mathcal{M}\equiv M-N$):

$$\mathcal{P}(\mathbf{1}_N,\mathbf{1}_N,\mathbf{0}_M) = A(N,N,\mathcal{M}) = \prod_{k=1}^{N}\prod_{j=1}^{N}\frac{M-N+k+j-1}{k+j-1}.$$

The numerator is the generating function $G(N,N,\mathcal{M}|\,1,\gamma)$ which is of a polynomial form. Indeed, application of $\mathcal{D}_\alpha^l$ at $\gamma=e^{\alpha/N}$ gives the mean value of power l of the first momentum M:

$$\langle\langle \mathsf{M}^l\rangle\rangle_{N,\beta}\Big|_{1\ll M\ll\beta} \simeq \frac{N^{-l}\sum_{\{\boldsymbol{\mu}_N\}}|\boldsymbol{\mu}_N|^l S_{\boldsymbol{\lambda}_N}(\mathbf{1}_N)S_{\boldsymbol{\lambda}_N}(\mathbf{1}_N)\Big|_{M\gg1}}{A(N,N,\mathcal{M})\Big|_{M\gg1}},$$

where $\boldsymbol{\lambda}_N=\boldsymbol{\mu}_N-\boldsymbol{\delta}_N$. The numerator may be viewed as a sum of the terms $(\frac{m}{N})^l A(N,N,\mathcal{M}|\, m)$, where

$$A(N,N,\mathcal{M}|\, m) \equiv \sum_{\boldsymbol{\mu}_N \vdash m} S_{\boldsymbol{\lambda}_N}(\mathbf{1}_N) S_{\boldsymbol{\lambda}_N}(\mathbf{1}_N)$$

is the number of plane partitions $\boldsymbol{\pi}$ confined in $N \times N \times \mathcal{M}$ box which are *diagonally constrained* since their trace is $\mathrm{tr}\,_N \boldsymbol{\pi} = m - \frac{N}{2}(N+1)$.

♦ The mean value of product $\Pi_{\mathbf{k}}$ of flipped spins on the sites $\mathbf{k}_l \equiv (k_1, k_2, \dots, k_l)$ of length l, $M \geq k_1 > k_2 > \dots > k_l \geq 1$ is estimated:

$$\langle\langle \Pi_{\mathbf{k}} \rangle\rangle_{N,\beta}\Big|_{1 \ll M \ll \beta} \simeq \frac{\widetilde{\mathcal{P}}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{k}_l)}{A(N,N,\mathcal{M})}\,,$$

where $\widetilde{\mathcal{P}}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{k}_l)$ is the number of the plane partitions at $q \to 1$:

$$\lim_{q \to 1} < \Pi_{\mathbf{k}} >_{N,q} = \widetilde{\mathcal{P}}_{\mathcal{M}}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{k}_l) = \sum_{\{\widetilde{\boldsymbol{\lambda}}\}} S_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{1}_N) S_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{1}_N)\,.$$

The numbers $\widetilde{\mathcal{P}}(\mathbf{1}_N, \mathbf{1}_N, \mathbf{k}_l)$ (67) enumerate the plane partitions diagonally constrained in the sense that l columns on the principal diagonal are of prescribed heights and positions (see Definition 3). In the case of $\mathbf{k}_l = \boldsymbol{\delta}_l$, the estimate is given by $\widetilde{\mathcal{P}}(\mathbf{1}_N, \mathbf{1}_N, \boldsymbol{\delta}_l)$.

THANKS !

nmnmnm