Lie algebras of multidimensional Schrödinger operators

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The talk is devoted to applications to the theory of the Korteweg–de Vries hierarchy of the following results:

V.M. Buchstaber, V.Z. Enolskii, D.V. Leykin:

1. Construction of hyperelliptic analogs of Weierstrass elliptic functions;

2. Description of all algebraic relations in the field of hyperelliptic functions;

V.M. Buchstaber, D.V. Leykin:

3. Construction of the polynomial Lie algebras theory;

4. Construction for each $g > 0$ of a system of $2g$ multidimensional Schrödinger equations, which determines the sigma function of a hyperelliptic curve of genus g in the model $V_\lambda = \left\{ (x, y) \in \mathbb{C}^2 : y^2 = x^{2g+1} + \sum_{k=2}^{2g+1} \lambda_{2k} x^{2g-k+1} \right\}$;

5. Construction of a polynomial Lie algebra $Sch\Lambda_g$, the generators of which are $2g$ Schrödinger operators $Q_0, Q_2, \ldots, Q_{4g-2}$ over $\Lambda = \mathbb{C}[\lambda_4, \ldots, \lambda_{4g+2}]$;

V.M. Buchstaber, E.Yu. Bunkova:

6. Explicit description of structure polynomials of the Lie algebras $Sch\Lambda_g$;

7. Explicit description of operators Q_0, Q_2, Q_4;

8. Recurrence formulas for Q_{2k}, $k > 2$, in terms of Lie brackets of operators Q_0, Q_2, Q_4.

The stationary Korteweg–de Vries equation (KdV for short)

$$u''' = 6u'u$$

defines a dynamical system in $\mathbb{C}^3$ with coordinates $x_{1,1}$, $x_{2,1}$ and $x_{3,1}$:

$$x'_{1,1} = x_{2,1}, \quad x'_{2,1} = x_{3,1}, \quad x'_{3,1} = 12x_{2,1}x_{1,1}.$$

The first integral of this system is the first integral of the Newton equation

$$u'' = 3u^2 + \alpha_4, \text{ where } \alpha_4 \text{ is a constant parameter.}$$

Solution of this system can be given in the form

$$u = 2x_{1,1}; \quad x_{1,1} = \wp,\ x_{2,1} = \wp',\ x_{3,1} = \wp''; \quad \wp = -(\log \sigma)''$$

where σ and $\wp$ are the Weierstrass functions of a non-singular elliptic curve

$$y^2 = 4x^3 - g_2 x - g_3, \quad g_2^3 - 27g_3^2 \neq 0,$$

with constant parameters g_2, g_3. In the coordinates of the space $\mathbb{C}^3$, this curve is the intersection of two hypersurfaces given by the equations

$$2(6x_{1,1}^2 - x_{3,1}) = g_2, \qquad 2x_{3,1}x_{1,1} - 8x_{1,1}^3 - x_{2,1}^2 = g_3.$$

We will describe the KdV-hierarchy with an infinite set of parameters $\alpha_4, \alpha_6, \ldots$ and for each $g = 1, 2, \ldots$ introduce the stationary parametric g-equation.

This equation corresponds to an ordinary differential equation of order $2g$, which is called the universal Novikov g-equation.

Each such g-equation defines a polinomial dynamical system in $\mathbb{C}^{3g}$ with coordinates

$$x_{1,2k-1}, \quad x_{2,2k-1}, \quad x_{3,2k-1}, \; k = 1, \ldots, g.$$

Let $\wp_{2k}(z, \lambda) = -\frac{\partial^2}{\partial z_1 \partial z_{2k-1}} \ln \sigma(z, \lambda)$ be the hyperelliptic function of non-singular hyperelliptic curves

$$V_\lambda = \left\{(x, y) \in \mathbb{C}^2 \; : y^2 = x^{2g+1} + \lambda_4 x^{2g-1} + \ldots + \lambda_{4g} x + \lambda_{4g+2}\right\}$$

where $z = (z_1, \ldots, z_{2g-1}), \; \lambda = (\lambda_4, \ldots, \lambda_{4g+2})$.

It will be shown that our dynamical system has the solution of the form

$$x_{1,2k-1} = \wp_{2k}, \quad x_{2,2k-1} = \wp_{2k}', \quad x_{3,2k-1} = \wp_{2k}'', \; k = 1, \ldots, g,$$

if the parameters $\{\alpha_{2k}\}$ are given by the recursion

$$2\alpha_{2k+2} = \lambda_{2k+2} - \sum_{i=1}^{k-2} \alpha_{2i+2}\alpha_{2k-2i}, \; k \geqslant 3, \; 2\alpha_4 = \lambda_4, \; 2\alpha_6 = \lambda_6.$$

The resulting dynamic system has $2g$ polynomial integrals.

Thus we realize the Jacobian of a nonsingular hyperelliptic curve of genus g as a g-dimensional submanifold in $\mathbb{C}^{3g}$ given by the system of $2g$ polynomial equations

$$\lambda_{2k}(x_{1,1}, \dots, x_{s,2q-1}) = \lambda_{2k},\ s + 2q - 1 \leqslant 2k,\ k = 2, \dots, 2g + 1,\ s = 1, 2, 3.$$

We obtain that for any $g \geqslant 2$ the function

$$u = 2\wp_{1,1}(z, \lambda)$$

of a hyperelliptic curve of genus g satisfies the parametric hierarchy.

This hierarchy begins with KdV-equation

$$4\dot{u} = u''' - 6uu'$$

where $\dot{u} = \frac{\partial u}{\partial z_3},\ u' = \frac{\partial u}{\partial z_1}$.

KdV-equation in the Lax form

Set $\frac{\partial}{\partial x} = \partial,\ u' = \partial(u)$ and $L = \partial^2 - u$.

In 1968 P.Lax showed that for the operator $A_3 = \partial^3 - \frac{3}{2}u\partial - \frac{3}{4}u'$,
the commutator $[A_3, L]$ is the operator of multiplication
by the differential polynomial

$$-\frac{1}{4}(u''' - 6uu').$$

Using that $\partial_t(L) = [\partial_t, L] = -[\partial_t, u] = -u_t$, he obtained that the KdV-equation

$$4\dot{u} = u''' - 6uu',$$

where $u = u(x,t),\ \dot{u} = \frac{\partial u}{\partial t},\ u' = \frac{\partial u}{\partial x}$

is equivalent to the equation

$$\partial_t(L) = [A_3, L].$$

The graded commutative differential polynomial algebra

Consider a graded commutative differential polynomial algebra

$$\mathfrak{A}_0 = (\mathbb{C}[u_0, u_1, \ldots],\ D),$$

where D is a derivation of $\mathbb{C}[u_0, u_1, \ldots]$ such that $D(u_k) = u_{k+1},\ k = 0, 1, \ldots$
Set $|u_0| = 2,\ |D| = 1$ and $|u_k| = k + 2,\ k \in \mathbb{N}$. Thus,

$$\mathfrak{A}_0 = \bigoplus_{k \geqslant 0} \mathfrak{A}_{0,k}$$

where $\mathfrak{A}_{0,k}$ is a graded linear space generated by homogeneous polynomials of weight k,

$$\mathfrak{A}_{0,0} = \mathbb{C}\{1\},\ \mathfrak{A}_{0,1} = \emptyset,\ \mathfrak{A}_{0,2} = \mathbb{C}\{u_0\},\ \mathfrak{A}_{0,3} = \mathbb{C}\{u_1\},\ \mathfrak{A}_{0,4} = \mathbb{C}\{u_2, u_0^2\}, \ldots$$

For further constructions, it is important that

the operator D defines monomorphisms $D\colon \mathfrak{A}_{0,k} \to \mathfrak{A}_{0,k+1}$ for all $k > 1$.

A-KdV-equation

Let us consider a homogeneous differential operator of order n

$$A = \sum_{i=0}^{n} a_i D^i, \ a_i \in \mathfrak{A}_{0,|a_n|+n-i}, \ a_n \neq 0,$$

such that $[L, A]$ is an operator of multiplication by a polynomial $P_A([u]) \in \mathfrak{A}_{0,|a_n|+n+2}$. We will call such operators the KdV-operators.

Denote by $\mathcal{V}_n$ the set of all KdV-operators of order at most n.
It is clear that $\mathcal{V}_n$ is a linear space over $\mathbb{C}$.

Let $A \in \mathcal{V}_n$ and $[L, A] = P_A([u])$.
We will call by A-KdV-equation the equation

$$\partial_t(u) = P_A([u])$$

which is equivalent to the equation $\partial_t(L) = [A, L]$.

Algebra of homogeneous pseudodifferential operators

The set of homogeneous pseudodifferential operators (in short, *pd*-operators)

$$\mathfrak{A}_0^D = \Big\{ A = \sum_{i \leqslant m} a_i D^i \mid a_i \in \mathfrak{A}_{0,|a_m|+m-i},\ a_m \neq 0,\ m \in \mathbb{Z} \Big\} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{A}_{0,k}^D,$$

where $|A| = |a_m| + m$, $\deg A = m$, is a non-commutative associative graded algebra over $\mathbb{C}$ with an additive homogeneous basis $\{aD^n,\ |aD^n| = |a| + n\}$.
The homogeneous polynomial $a \in \mathfrak{A}_0$ is considered as the multiplication operator

$$a \colon \mathfrak{A}_0 \to \mathfrak{A}_0 \ : \ a(b) = ab,\ b \in \mathfrak{A}_0.$$

The multiplication rule in $\mathfrak{A}_0^D$ is given by commutation relations

$$[D, u_k] = u_{k+1}, \qquad [D^{-1}, u_k] = \sum_{i \geqslant 0} (-1)^{i-1} u_{k+i} D^{-(i+1)}.$$

The set of homogeneous differential operators

$$\mathfrak{A}_0[D] = \Big\{ A = \sum_{i=0}^{m} a_i D^i \mid a_i \in \mathfrak{A}_{0,|a_m|+m-i},\ a_m \neq 0,\ m \geqslant 0 \Big\}$$

is a subalgebra in $\mathfrak{A}_0^D$.

Multiplication rule and residues

$$bD^k a\, D^l = \sum_{i\geqslant 0} \binom{k}{i} ba^{(i)} D^{k+l-i} \qquad (1)$$

Here $a^{(i)} = D^i(a) \in \mathfrak{A}_0$ and $\binom{0}{0} = 1$, $\binom{k}{0} = 1$,

$$\binom{k}{i} = \frac{k(k-1)\cdots(k-i+1)}{i!} = (-1)^i \binom{-k+i-1}{i}, \; k \in \mathbb{Z}, \; i > 0.$$

For negative k the series (1) is infinite.

For any $A \in \mathfrak{A}_0^D$ the coefficient a_{-1} of the term $a_{-1}D^{-1}$ is called the residues of A and denoted $\mathrm{res}\, A$.

For any $A \in \mathfrak{A}_0^D$ we have $[D, A] = \sum\limits_{i\leqslant m} D(a_i)D^i = D(A)$.

Corollary.

For any $A \in \mathfrak{A}_0^D$, $\mathrm{res}\,[D, A] = D(\mathrm{res}\, A)$.

The skew-symmetric bilinear form

Lemma 1.
On the space of *pd*-operators, a homogeneous skew-symmetric bilinear over $\mathbb{C}$ form

$$\sigma(\cdot,\cdot)\colon \mathfrak{A}_0^D \otimes \mathfrak{A}_0^D \to \mathfrak{A}_0, \quad |\sigma(A,B)| = |A| + |B|,$$

is defined, such that for $n, m \in \mathbb{Z}$

$$\sigma(aD^n, bD^m) = \begin{cases} 0 \ \text{ if } n+m<0, \\ \binom{n}{n+m+1} \sum_{s=0}^{n+m} (-1)^s a^{(s)} b^{(n+m-s)} \ \text{ if } n+m \geqslant 0. \end{cases} \tag{2}$$

The formula (2) is extended to *pd*-operators since

$$\sigma(aD^n, bD^m) = 0 \ \text{ if } nm \geqslant 0 \ \text{ or } \ n+m<0.$$

For example: $\sigma(D^n, bD^m) = \binom{n}{n+m+1} b^{(n+m)}$.

Therefore $\sigma(D, D^{-1}) = 1$ and $\sigma(D, A) = \operatorname{res} \mathrm{A}$, $\mathrm{A} \in \mathfrak{A}_0^{\mathrm{D}}$.

Cocycle equation

Corollary.

For any $A, B \in \mathfrak{A}_0^D$

$$\mathrm{res}\,[\mathrm{A},\mathrm{B}] = \mathrm{D}\big(\sigma(\mathrm{A},\mathrm{B})\big).$$

Corollary.

For any $A, B, C \in \mathfrak{A}_0^D$

$$\sigma([A,B],C) + \sigma([B,C],A) + \sigma([C,A],B) = 0.$$

For $A = \sum_{i \leqslant m} a_i D^i$, $a_m \neq 0$, we put $A = A_+ + A_-$ where $A_+ = 0$ if $m < 0$ and $A_+ = \sum_{i=0}^{m} a_i D^i$ if $m \geqslant 0$.

For any $A, B \in \mathfrak{A}_0^D$ we have $\sigma(A,B) = \sigma(A_+, B_-) + \sigma(A_-, B_+)$.

The square root of the Schrödinger operator

Let us consider a homogeneous operator $L = D^2 - u$, $|L| = 2$.

The equation $\mathcal{L}^2 = L$ uniquely defines homogeneous *pd*-operator

$$\mathcal{L} = D + \sum_{n\geqslant 1} l_{1,n} D^{-n},\ |\mathcal{L}| = 1,\ l_{1,n} \in \mathfrak{A}_{0,n+1}.$$

The polynomials $l_{1,n}$ can be calculated by the recursion

$$2l_{1,n} + l'_{1,n-1} + \sum_{k=1}^{n-2} l_{1,k} \sum_{i=0}^{n-k-2} \binom{-n}{i} l^{(i)}_{1,n-k-i-1} = 0,\ n \geqslant 3,$$

with initial conditions $l_{1,1} = -\frac{1}{2}u$, $l_{1,2} = \frac{1}{4}u_1$. Thus,

$$\mathcal{L} = D - \frac{1}{2}uD^{-1} + \frac{1}{4}u_1 D^{-2} - \frac{1}{8}(u_2 + u^2)D^{-3} + \frac{1}{16}(u_3 + 6uu_1)D^{-4} - \\ - \frac{1}{32}(u_4 + 14u_2 u + 11u_1^2 + 2u^3)D^{-5} + \dots$$

Let us define a sequence of homogeneous differential operators

$$A_{2k} = \mathcal{L}_+^{2k} = L^k, \qquad A_{2k-1} = \mathcal{L}_+^{2k-1}$$

and homogeneous differential polynomials $\rho_{2k} \in \mathfrak{A}_{0,2k}$,

$$\rho_0 = 1, \quad \rho_{2k} = \operatorname{res} \mathcal{L}^{2\mathrm{k}-1}, \ \mathrm{k} = 1, 2, \ldots$$

Thus,

$$\mathcal{L}^{2k-1} = A_{2k-1} + \rho_{2k} D^{-1} + \ldots, \ k > 0.$$

We have $A_1 = D$ and

$$A_{2k-1} = D^{2k-1} - \frac{1}{2}(2k-1)uD^{2k-3} + \cdots + a_{2k-1}, \ k = 2, \ldots$$

where $a_{2k-1} = A_{2k-1}(1) \in \mathfrak{A}_{0,2k-1}$.

Canonical KdV-operators

We have $[A_{2k}, L] = [L^k, L] = 0$. It is easy to show that $[\mathcal{L}^{2k-1}, L] = 0$ and therefore the commutator

$$[A_{2k-1}, L] = [\mathcal{L}^{2k-1} - \mathcal{L}_-^{2k-1}, L] = 2D(\rho_{2k}) \in \mathfrak{A}_0$$

is the operator of multiplication on the function $2D(\rho_{2k})$.
Thus, we have obtained a sequence of homogeneous KdV-operators

$$A_0 = 1, \quad A_1 = D, \quad A_n = D^n + \sum_{k=1}^{n} a_{n,k} D^{n-k}, \; n > 1.$$

Lemma 2.

Any KdV-operator $A \in \mathfrak{A}_0 \otimes \mathfrak{F}$ of order n, where $\mathfrak{F}$ is a field of constants, can be uniquely written in the form

$$A = \sum_{k=0}^{n} c_k A_k, \text{ where } A_k = \mathcal{L}_+^k \text{ and } c_k \in \mathfrak{F}.$$

Since $[A_{2k}, L] = [L^k, L] = 0$, then under considering A-KdV-equations it is sufficient to deal with operators A such that $c_{2m} = 0$ for all m.

The operators A_{2k-1}, $k \geqslant 1$, will be called canonical KdV-operators.

Canonical KdV-derivations ∂_{2k-1}

Let's define

$$\partial_{t_{2k-1}}(u) = \partial_{2k-1}(u) = [L, A_{2k-1}] = -2D(\rho_{2k}) \in \mathfrak{A}_{0,2k+1}, \; k \in \mathbb{N}.$$

The operator ∂_{2k-1} on $\mathfrak{A}_0$ extends to the differentiation operator on $\mathfrak{A}_0^D$ which we will also denote by ∂_{2k-1}.

Let $A = \sum_{i \leqslant m} a_i D^i$. Then $\partial_{2k-1}(A) = [\partial_{2k-1}, A] = \sum_{i \leqslant m} \partial_{2k-1}(a_i) D^i$.

Lemma 3.

$$\partial_{2k-1}(\mathcal{L}) = [A_{2k-1}, \mathcal{L}],$$

and therefore,

$$\partial_{2k-1}(\mathcal{L}^{2n-1}) = [A_{2k-1}, \mathcal{L}^{2n-1}], \; k, n \in \mathbb{N}.$$

Lemma 4.

For any $A, B \in \mathfrak{A}_0^D$

$$\partial_{2k-1}\big(\sigma(A, B)\big) = \sigma\big(\partial_{2k-1}(A), B\big) + \sigma\big(A, \partial_{2k-1}(B)\big).$$

Canonical evolutionary derivations ∂_{2k-1} commute

A derivation ∂_t of the ring $\mathfrak{A}_0$ is said to be evolutionary if $[\partial_t, D] = 0$.

To set a homogeneous evolutionary derivation ∂_t, it is necessary and sufficient to choose a polynomial $P([u]) \in \mathfrak{A}_{0,k}$, $k \geqslant 0$, and put $\partial_t(u) = P([u])$.

Let $k, n \in \mathbb{N}$. Let's put

$$\sigma_{2k-1,2n-1} = \sigma(A_{2k-1}, \mathcal{L}^{2n-1}) \in \mathfrak{A}_{0,2k+2n-2}.$$

We obtain

$$\sigma_{1,2n-1} = \rho_{2n}, \qquad \sigma_{2k-1,2n-1} = \sigma_{2n-1,2k-1}.$$

Lemma 5.

$$\partial_{2k-1}(\rho_{2n}) = \partial_{2n-1}(\rho_{2k}) = D(\sigma_{2k-1,2n-1}).$$

Corollary.

$$[\partial_{2k-1}, \partial_{2n-1}] = 0.$$

Canonical KdV-hierarchy

The infinite system of compatible homogeneous differential equations

$$\partial_{2k-1}(u) = -2D(\rho_{2k}),\ k \in \mathbb{N},$$

is called the canonical KdV-hierarchy.

This system is written in the form of conservation laws for the KdV-equation and is equivalent to the system of compatible homogeneous operator equations in the Lax form

$$\partial_{2k-1}(L) = [A_{2k-1}, L].$$

Namely,

$$\begin{aligned} \partial_1(u) &= D(u) = u_1, \\ 4\,\partial_3(u) &= D(u_2 - 3u^2) = u_3 - 6uu_1, \\ 16\,\partial_5(u) &= D(u_4 - 10u_2u - 5u_1^2 + 10u^3) = u_5 - 10uu_3 - 20u_1u_2 + 30u^2u_1, \end{aligned}$$

and so on.

The universal Novikov's N-equation

For fixed $N \in \mathbb{N}$, let $\mathcal{A} = \mathbb{C}[\alpha_2, \alpha_4, \ldots, \alpha_{2N+2}]$, $|\alpha_{2n}| = 2n$, $n \geqslant 1$, be a graded algebra of parameters and $\mathfrak{A} = \mathcal{A}[u_0, u_1, \ldots] = \oplus \mathfrak{A}_{2k}$, $k \geqslant 0$. By definition $D(\alpha_{2n}) = 0$.

Let us define the universal N-symmetry of the KdV-equation

$$\partial_\tau(u) = \partial_{2N+1}(u) + \sum_{k=1}^{N} \alpha_{2(N-k+1)} \partial_{2k-1}(u).$$

We have $\partial_\tau(u) = -2D(F_{2N+2})$ where

$$F_{2N+2} = \rho_{2N+2} + \sum_{k=0}^{N} \alpha_{2(N-k+1)} \rho_{2k} \in \mathfrak{A}_{2N+2}, \; \rho_0 = 1.$$

Equation $\partial_\tau(u) = 0$ is called the universal Novikov's N-equation.

Let us restrict ourselves with the solution u of the KdV-hierarchy such that $\partial_\tau(u) = 0$. It implies that

$$\rho_{2N+2} + \sum_{k=0}^{N} \alpha_{2(N-k+1)} \rho_{2k} = 0, \; \rho_0 = 1.$$

Since $2^{2k-1}\rho_{2k} = -u_{2k-2} + \dots,\ k \in \mathbb{N}$, the equation $F_{2N+2} = 0$ can be written in the form

$$u_{2N} = f_{2N+2}(u_0, u_1, \dots, u_{2N-2};\, \alpha_2, \dots, \alpha_{2N+2}),\ f_{2N+2} \in \mathfrak{A}_{2N+2}. \qquad (3)$$

Equation (3) with $\alpha_{2k} \in \mathbb{C}$ is called Novikov's N-equation.
These equations depend linearly on $\alpha_2, \alpha_4, \dots, \alpha_{2N+2}$.

Let us assume $\alpha_2 = 0$, then:

$$N = 1 :\ u_2 = 3u^2 + 8\alpha_4,$$

$$N = 2 :\ u_4 = 10(u_2 - u^2)u + 5u_1^2 - 16\alpha_4 u + 32\alpha_6,$$

$$N = 3 :\ u_6 = 14(u_4 - 5u_2 u + 5u_1^2)u + 28u_1 u_3 + 21u_2^2 + 35u^4 - \\ - 16\alpha_4(u_2 - 3u^2) - 64\alpha_6 u + 128\alpha_8.$$

Since $u_k = D^k(u)$, then for the function $u = u(x)$

the Novikov's N-equation is an homogeneous ordinary nonlinear differential equation of the 2N-th order with graded parameters $\alpha_4, \dots, \alpha_{2N+2}$.

Let $\mathfrak{I}_N = (F_{2N+2}) \subset \mathfrak{A}$ be a differential ideal generated by the polynomial F_{2N+2} and its derivatives $D^k(F_{2N+2})$. For any element of $\mathfrak{A}$ the canonical projection

$$\pi_N : \mathfrak{A} \mapsto \mathfrak{A} / \mathfrak{I}_N$$

is the result of the elimination of variables $u_k, \ k \geqslant 2N$, using equation (3) and equation $u_{2N+k} = D^k(u_{2N}) = D^k(f_{2N+2})$ recursively.

Thus $\mathfrak{A} / \mathfrak{I}_N = \mathcal{A}[u_0, \ldots, u_{2N-1}]$.

Proposition 1.

The ideal $\mathfrak{I}_N$ is invariant with respect to evolutionary derivations ∂_{2k-1}, $k \in \mathbb{N}$.

The canonical operators $\partial_{2k-1}, \ k = 1, 2, \ldots$, on $\mathfrak{A}_0$ induce operators on $\mathfrak{A}_0 / \mathfrak{I}_N$. These operators, which we will also denote by ∂_{2k-1}, one can write in the form

$$\partial_1 = \mathcal{D} = \sum_{i=0}^{2N-2} u_{i+1} \frac{\partial}{\partial u_i} + f_{2N+2} \frac{\partial}{\partial u_{2N-1}},$$

$$\partial_{2k-1} = -2 \sum_{i=0}^{2N-1} \mathcal{D}^{i+1}(\rho_{2k}) \frac{\partial}{\partial u_i}, \ 2 \leqslant k \leqslant N,$$

The universal Novikov's N-hierarchy

Corollary.

In $\mathbb{C}^{3N}$ with coordinates $u_0, u_1, \ldots, u_{2N-1}; \alpha_4, \ldots, \alpha_{2N+2}$,
the operators $\partial_1 = \mathcal{D}$ and $\partial_{2k-1},\ k = 2, \ldots, N$, (see above) define N compatible dynamical systems with parameters $\alpha_4, \ldots, \alpha_{2N+2}$

$$\partial_{2k-1}(u_s) = -2\mathcal{D}^{s+1}(\rho_{2k}), \quad s = 0, \ldots, 2N-1,\ k = 1, \ldots, N,$$

which we will call universal Novikov's N-hierarchy.

The system given by the operator ∂_1 is the dynamic system corresponding to the Novikov's N-equation

$$\partial_1(u_s) = u_{s+1},\ s = 0, \ldots, 2N-1,$$
$$\partial_1(u_{2N-1}) = f_{2N+2}.$$

Theorem 1.

The Novikov's N-equation possesses N first integrals

$$H_{2n+1,2N+1} = \sigma_{2n+1,2N+1} + \sum_{k=1}^{N-1} \alpha_{2N-2k+2}\sigma_{2n+1,2k-1}, \quad n = 1, \dots, N.$$

The polynomials $H_{2n+1,2N+1} \in \mathfrak{A}_{2N+2n+2}$ are algebraically independent.

Proof. Since $\partial_{2n+1}(\rho_{2k}) = D(\sigma_{2n+1,2k-1})$ (see Lemma 5), then for any $n \in \mathbb{N}$ we have:

$$H_{2n+1,2N+1} \notin \mathfrak{I}_N, \qquad D(H_{2n+1,2N+1}) = \partial_{2n+1}(F_{2N+2}) \in \mathfrak{I}_N.$$

Thus, the polynomials $H_{2n+1,2N+1}$ give the first integrals for all $n \geqslant 1$.

A direct verification shows that in the ring $\mathfrak{A}/\mathfrak{I}_N$ only the polynomials $H_{2n+1,2N+1}$, $1 \leqslant n \leqslant N$, are algebraically independent.

Liouville-integrable polynomial dynamical systems

Let us consider $\mathbb{C}^{3N}$ with coordinates

$$[u] = (u_0, u_1, \dots, u_{2N-1}), \quad [\alpha] = (\alpha_4, \alpha_6, \dots, \alpha_{2N+2})$$

and $\mathbb{C}^N$ with coordinates

$$[H] = (H_{3,2N+1}, \dots, H_{2N+1,2N+1}).$$

Corollary.

The N first integrals of the Novikov's N-equation define a polynomial mapping

$$\pi_N \colon \mathbb{C}^{3N} \to \mathbb{C}^N \;\; : \pi_N([u], [\alpha]) = [H].$$

For fixed values of $[\alpha]$, we obtain a Liouville-integrable polynomial system.

Example for $N = 1$

The Novikov's 1-equation gives a dynamical system in $\mathbb{C}^3$ with coordinates u, u_1, α_4

$$\partial_1 u = u_1, \quad \partial_1 u_1 = 3u^2 + 8\alpha_4.$$

According Theorem 1 we get one first integral

$$H_{3,3} = -\frac{3}{16}\left(\frac{1}{2}u_1^2 - u^3 - 8\alpha_4 u\right).$$

Thus the Novikov's 1-equation coincides with the Newton equation

$$u'' = 3u^2 + 8\alpha_4$$

with cubic potential.

The remaining first integrals of this equation are polynomials in $H_{3,3}$.

Klein problem.

The problem of constructing a multidimensional analogue of the Weierstrass σ-function is classical.

In 1886, F.Klein proposed the following problem:
Modify multidimensional θ-function $\theta(\mathbf{z}; \Gamma_V)$ in order to obtain an entire function which is:

(1) independent of a choice of basis in the lattice Γ_V;
(2) covariant with respect to the Möbius transformations of the curve V.

On this problem, Klein published 3 works (1886–1890).

In 1923, a 3-volume collection of Klein's scientific works was published.
In the preface to the works on the problem under discussion, he emphasized that the theory of hyperelliptic functions is still far from complete.

The covariance requirement (2) immediately led to the need to confine ourselves to the class of hyperelliptic curves. But even this case caused artificial difficulties.

H.F.Baker (1903) disregarded requirement (2) and showed that in the case of curves of genus 2, it is possible to construct analogues of elliptic σ-functions without using θ-functions.

Since 1990, in a cycle of works, V.M. Buchstaber, V.Z. Enolskii and D.V. Leykin have been developed a theory of multidimensional σ-functions associated with given models of plane algebraic curves.

Hyperelliptic sigma function

V.M. Buchstaber, V.Z. Enol'skii, D.V. Leikin constructed the theory of sigma functions $\sigma(\mathbf{z}, \lambda)$ of non-singular hyperelliptic curves

$$y^2 = x^{2g+1} + \lambda_4 x^{2g-1} + \ldots + \lambda_{4g} x + \lambda_{4g+2}, \ |x| = 2, \ |y| = 2g+1, \ |\lambda_{2s}| = 2s,$$

where $\mathbf{z} = (z_1, \ldots, z_{2g-1}), \ \lambda = (\lambda_4, \ldots, \lambda_{4g+2})$.

Let's put $|z_{2k-1}| = -2k+1, \ |\lambda_{2k}| = 2k, \ \omega = (i_1, \ldots, i_g)$.
The function $\sigma(\mathbf{z}, \lambda)$ is an entire function given by the homogeneous series

$$\sigma(\mathbf{z}, \lambda) = \sum p_\omega(\lambda) \mathbf{z}^\omega, \ |\sigma(\mathbf{z}, \lambda)| = -\frac{1}{2} g(g+1),$$

where $\mathbf{z}^\omega = z_1^{i_1} \cdots z_{2g-1}^{i_g}$, and $p_\omega(\lambda)$ is a homogeneous polynomial of degree

$$|p_\omega(\lambda)| = i_1 + \cdots + (2g-1) i_g - \frac{1}{2} g(g+1).$$

Initial condition: $\sigma(\mathbf{z}_*, 0) = z_1^{\frac{1}{2}g(g+1)}$, where $\mathbf{z}_* = (z_1, 0, \ldots, 0)$.

Hyperelliptic $\wp$-function

Denote by $\mathcal{F}_g$ the field of meromorphic functions on the Jacobian of the general nonsingular hyperelliptic curve,

and by $\mathcal{P}_g \subset \mathcal{F}_g$ the subring generated over $\mathbb{C}$ by all logarithmic derivatives of the function $\sigma(\mathbf{z}, \lambda)$ of order 2 and higher.

The field $\mathcal{F}_g$ is a fractions field of the ring $\mathcal{P}_g$.

Let's put

$$\wp_{2k} = \wp_{2k}(\mathbf{z}, \lambda) = -\frac{\partial^2}{\partial z_1 \partial z_{2k-1}} \ln \sigma(\mathbf{z}, \lambda), \ |\wp_{2k}| = 2k, \ k = 1, \ldots, g.$$

Denote by $\mathcal{B} \subset \mathcal{P}_g$ the subring generated by $3g$ functions

$$\wp_{2k}, \quad \wp_{2k}', \quad \wp_{2k}'', \quad k = 1, \ldots, g,$$

where $f' = \frac{\partial f}{\partial z_1}$.

Let $\wp_{2i-1,2k-1} = \wp_{2i-1,2k-1}(\mathbf{z}) = -\partial_{2i-1}\partial_{2k-1} \ln \sigma(\mathbf{z})$ where $i \neq 1$ or $k \neq 1$.

Theorem 2.

All algebraic relations between hyperelliptic functions $\wp_\omega$ of genus g follow from the relations

$$\wp''_{2i} = 6(\wp_{2i+2} + \wp_2\wp_{2i}) - 2(\wp_{3,2i-1} - \lambda_{2i+2}\delta_{i,1}). \qquad (4)$$

$$\begin{aligned} \wp'_{2i}\wp'_{2k} = 4(\wp_{2i}\wp_{2k+2} + \wp_{2i+2}\wp_{2k} + \wp_2\wp_{2i}\wp_{2k} + \wp_{2i+1,2k+1}) - \\ - 2(\wp_{2i}\wp_{3,2k-1} + \wp_{2k}\wp_{3,2i-1} + \wp_{2i-1,2k+3} + \wp_{2i+3,2k-1}) + \\ + 2(\lambda_{2i+2}\wp_{2k}\delta_{i,1} + \lambda_{2k+2}\wp_{2i}\delta_{k,1}) + 2\lambda_{2(i+j+1)}(2\delta_{i,k} + \delta_{i,k-1} + \delta_{i-1,k}). \end{aligned} \qquad (5)$$

Here $\delta_{i,k}$ is the Kronecker symbol, $\deg \delta_{i,k} = 0$.

Corollary.

For all $g \geqslant 1$, we have the following relations:

1. Setting $i = 1$ in (4), we obtain
$$\wp_2'' = 6\wp_2^2 + 4\wp_4 + 2\lambda_4. \tag{6}$$
2. Setting $i = 2$ in (4), we obtain
$$\wp_4'' = 6(\wp_2\wp_4 + \wp_6) - 2\wp_{3,3}.$$
3. Setting $i = k = 1$ in (5), we obtain
$$(\wp_2')^2 = 4\left[\wp_2^3 + (\wp_4 + \lambda_4)\wp_2 + \wp_{3,3} - \wp_6 + \lambda_6\right].$$

We have $\wp_{2i}' = \partial_{2i-1}(\wp_2)$. Then from (6) we obtain:

Corollary.

For any $g > 1$, the function $u = 2\wp(\mathbf{z})$ is a solution of KdV equation
$$u''' = 6uu' + 4\dot{u}, \quad \text{where} \quad \dot{u} = 2\partial_3(\wp_2).$$

Theorem 3.

• The functions $\wp_{2k}^{(s)}$, $k = 1, \ldots, g$, $s = 0, 1, 2$, are algebraically independent, and so $\mathcal{B} \simeq \mathbb{C}[\wp_{2k}^{(s)}, \ k = 1, \ldots, g, \ s = 0, 1, 2]$.

• The embedding $\mathcal{B} \subset \mathcal{P}_g$ is an isomorphism.

• The embedding $\mathbb{C}[\lambda] \subset \mathcal{B}$ is defined.

Examples:

For any $g \geqslant 2$, $\lambda_4 = \frac{1}{2}\wp_2'' - 3\wp_2^2 - 2\wp_4$,
for $g = 1$ we must put $\wp_4 = 0$.

For any $g \geqslant 3$, $\lambda_6 = -2\wp_6 + \frac{1}{2}\wp_4'' - 4\wp_4\wp_2 + \frac{1}{4}[(\wp_2')^2 - 4\wp_2^3 - 4\lambda_4\wp_2]$,
for $g = 2$ we must put $\wp_6 = 0$; and for $g = 1$ we must put $\wp_6 = 0$ and $\wp_4 = 0$.

Denote by $\mathfrak{A}_\Lambda \subset \mathcal{B}$ the subring $\mathbb{C}[\lambda][\wp_2, \wp_2', \wp_2'', \dots]$.

Theorem 4.

The embedding $\mathfrak{A}_\Lambda \subset \mathcal{B}$ is an isomorphism.

For example: for any $g \geqslant 2$, $2\wp_4 = \frac{1}{2}\wp_2'' - 3\wp_2^2 - \lambda_4$.

Theorem 5.

The homomorphism

$$\eta\colon \mathfrak{A} \to \mathfrak{A}_\Lambda \;:\; \eta(u_k) = 2\wp_2^{(k)},\ k = 1, \dots, 2g-1,$$

induces an isomorphism $\mathfrak{A}/\mathfrak{J}_g \to \mathfrak{A}_\Lambda$, where the image of the parameters α_{2k}, $k = 2, \dots, 2g+2$, is given by recursion

$$2\alpha_{2k+2} = \lambda_{2k+2} - \sum_{i=1}^{k-2} \alpha_{2i+2}\alpha_{2k-2i},\ k \geqslant 3,\ 2\alpha_4 = \lambda_4,\ 2\alpha_6 = \lambda_6.$$

Corollary.

The constructed dynamical systems have the solutions of the form

$$x_{1,2k-1} = \wp_{2k}, \quad x_{2,2k-1} = \wp_{2k}', \quad x_{3,2k-1} = \wp_{2k}'',\ k = 1, \dots, g.$$

Polynomial integrable system in $\mathbb{C}^6$

The system in $\mathbb{C}^6$:

$$x'_{1,1} = x_{2,1}, \quad x'_{2,1} = x_{3,1}, \quad x'_{3,1} = 4(3x_{1,1}x_{2,1} + x_{2,3}),$$
$$x'_{1,3} = x_{2,3}, \quad x'_{2,3} = x_{3,3}, \quad x'_{3,3} = 4(2x_{1,1}x_{2,3} + x_{2,1}x_{1,3}).$$

The solution to this system:

$$x_{1,1} = \wp_2, \quad x_{1,3} = \wp_4$$

where $\wp_{2k},\ k = 1, 2$, are the hyperelliptic functions defined by the sigma function of the hyperelliptic curve

$$y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}.$$

The intersection of four hypersurfaces in $\mathbb{C}^6$, which are the level surfaces of the values of homogeneous polynomials

$$\lambda_{2k}(\mathbf{x}) = \lambda_{2k},\ \mathbf{x} = (x_{s,2i-1}),\ s = 1, 2, 3,\ i = 1, 2,\ k = 2, \dots, 5,$$

defines the realization of the Jacobian variety of a curve of genus 2 in $\mathbb{C}^6$.

The hyperelliptic σ-function of genus g.

Theorem 6.

For any $g \geqslant 1$, there exists the function $\sigma(\mathbf{z}, \lambda)$ such that:

(a) $\sigma(\mathbf{z}, \lambda)$ is an entire quasiperiodic function of $\mathbf{z} \in \mathbb{C}^g$, $\lambda \in \mathcal{M} = \mathbb{C}^{2g} \setminus \Sigma_{\mathcal{D}}$.

(b) $\sigma(\mathbf{z}; 0)$ coincides with Adler-Moser polynomial up to a constant factor.

(c) $\sigma(\mathbf{z}, \lambda)$ is a solution to the system $Q_{2j}(\sigma(\mathbf{z}, \lambda)) = 0$, $j = 0, \dots, 2g-1$, where $Q_{2j} = \ell_{2j} - \frac{1}{2} H_{2j} - \delta_{2j}(\lambda)$, with $\ell_{2j} \in \mathcal{L}_g$ and

$$H_{2j} = \alpha_j^{kl}(\lambda)\partial_{2k-1}\partial_{2l-1} + 2\beta_{jk}^{l}(\lambda) z_{2k-1}\partial_{2l-1} + \gamma_{jkl}(\lambda) z_{2k-1} z_{2l-1},$$

$$\delta_{2j}(\lambda) = \frac{1}{8}\ell_{2j} \log\det T(\lambda) + \frac{1}{2}\beta_{jk}^{k}(\lambda),$$

where (k, l)-summation from 1 to g extends over the repeated indices. Here $\alpha_j^{kl}(\lambda) = \alpha_j^{lk}(\lambda)$, $\beta_{jk}^{l}(\lambda)$ and $\gamma_{jkl}(\lambda) = \gamma_{jlk}(\lambda)$ are polynomials of λ.

(d) The initial condition $\sigma(\mathbf{z}_*; 0) = z_1^{\frac{g(g+1)}{2}}$, $\mathbf{z}_* = (z_1, 0, \dots, 0)$, uniquely determines the entire function $\sigma(\mathbf{z}, \lambda)$ as the solution to the system $Q_{2j}(\sigma(\mathbf{z}, \lambda)) = 0$, $j = 0, 1, 2$.

The annihilators Q_{2j} of the σ-function and a quantum oscillator.

Write the system of equations $Q_{2j}(\sigma(\mathbf{z},\lambda)) = 0,\ j = 1,\dots,2g-1$, in the form

of Schrödinger equations $\ell_{2j}(\sigma) = \left\{\frac{1}{2}H_{2j} + \delta_{2j}(\lambda)\right\}(\sigma)$,

of a multidimensional quantum harmonic oscillator with multiple 'times'.

The formalism of quantum oscillator:

H_{2j} is a set of 'quadratic Hamiltonians',

ℓ_{2j} are derivatives over 'times',

δ_{2j} is 'the energy of an oscillator mode'.

The realization of the sigma-function in the form of an average of the 'ground state wave-function' (a multidimensional Gaussian function) over a lattice suggests a natural interpretation of sigma-function as the 'wave-function of the coherent state' of the oscillator.

Polynomial Lie algebras.

Set $\Lambda = \mathbb{C}[\lambda_0, \ldots, \lambda_n]$.

A polynomial Lie algebra over Λ is called an infinite-dimensional Lie algebra L with the structure of a free left Λ-module with a basis $\ell_0, \ldots, \ell_n$ and a Lie bracket such that

$$[\ell_i, \ell_j] = \sum_{k=0}^{n} c_{i,j}^{k}(\lambda)\ell_k, \quad [\ell_i, \lambda_q] = v_{i,q}(\lambda), \quad [\lambda_q, \lambda_r] = 0,$$

where $v_{i,q}(\lambda)$, $c_{i,j}^{k}(\lambda) \in \Lambda$.

In the Lie algebra L, the Jacobi identity is equivalent to the system

$$\sum_{q=0}^{n} v_{i,q}\frac{\partial c_{j,k}^{m}}{\partial \lambda_q} + v_{k,q}\frac{\partial c_{i,j}^{m}}{\partial \lambda_q} + v_{j,q}\frac{\partial c_{k,i}^{m}}{\partial \lambda_q} + c_{j,k}^{q}c_{i,q}^{m} + c_{i,j}^{q}c_{k,q}^{m} + c_{k,i}^{q}c_{j,q}^{m} = 0,$$

$$\sum_{q=0}^{n} c_{i,j}^{q}v_{q,k} + v_{j,q}\frac{\partial v_{i,k}}{\partial \lambda_q} - v_{i,q}\frac{\partial v_{j,k}}{\partial \lambda_q} = 0.$$

Hyperelliptic curves.

Consider the curve

$$V_\lambda = \left\{ (x,y) \in \mathbb{C}^2 \, : y^2 = f(x;\lambda) = x^{2g+1} + \sum_{k=2}^{2g+1} \lambda_{2k} x^{2g-k+1} \right\},$$

where $g \geqslant 1$ and $\lambda = (\lambda_4, \ldots, \lambda_{4g+2}) \in \mathbb{C}^{2g}$ are the parameters.

Set $\Sigma_\mathcal{D} = \{\lambda \in \mathbb{C}^{2g} \, : f(x;\lambda)$ has multiple roots$\}$ and $\mathcal{M} = \mathcal{M}_g = \mathbb{C}^{2g} \setminus \Sigma_\mathcal{D}$.

For any $\lambda \in \mathcal{M}$ we have the Jacobian variety $Jac(V_\lambda) = \mathbb{C}^g / \Gamma_\lambda$ and the field of meromorphic functions F_λ on $Jac(V_\lambda)$.

Let us denote by $\mathcal{L}_g$ the infinite-dimensional Lie algebra of polynomial vector fields in $\mathbb{C}^{2g}$ that are tangent to the discriminant $\Sigma_\mathcal{D} \subset \mathbb{C}^{2g}$ of the universal hyperelliptic curve V_λ.

Hyperelliptic polynomial Lie algebras.

The Lie algebra $\mathcal{L}_g$ is a polynomial Lie algebra over $\Lambda_g = \mathbb{C}[\lambda_4, \dots, \lambda_{4g+2}]$ with the Λ-basis

$$\ell_{2k} = \sum_{s=2}^{2g+1} v_{2k+2,2s-2}(\lambda) \frac{\partial}{\partial \lambda_{2k}}, \ k = 0, 2, \dots, 4g-2, \ v_{2k+2,2s-2}(\lambda) \in \mathbb{C}[\lambda],$$

$$[\ell_{2i}, \lambda_{2q}] = v_{2i+2,2q-2}(\lambda), \quad [\lambda_{2q}, \lambda_{2r}] = 0.$$

Set $\lambda_s = 0$ for $s \notin \{0, 4, 6, \dots, 4g, 4g+2\}$ and $\lambda_0 = 1$.
For $k, m \in \{1, 2, \dots, 2g\}$, $k \leqslant m$

$$v_{2k,2m}(\lambda) = \sum_{s=0}^{k-1} 2(k+m-2s)\lambda_{2s}\lambda_{2(k+m-s)} - \frac{2k(2g-m+1)}{2g+1}\lambda_{2k}\lambda_{2m},$$

and $v_{2k,2m}(\lambda) = v_{2m,2k}(\lambda)$ for $k > m$.

We have

$$[\ell_2, \ell_{2k}] = 2(k-1)\ell_{2k+2} + \frac{4(2g-k)}{(2g+1)}(\lambda_{2k+2}\ell_0 - \lambda_4 \ell_{2k-2}).$$

Lie algebras of Schrödinger operators.

Let $Q_{2j} = \ell_{2j} - \frac{1}{2} H_{2j} - \delta_{2j}(\lambda)$ be our Schrödinger operators.

Theorem 7.

There exists a polynomial Lie algebra $Sch\Lambda_g$ over Λ_g with the basis $Q_{2j},\ j = 0, \dots, 2g-1$.

The correspondence $Q_{2j} \to \ell_{2j}$ defines an isomorphism of polynomial Lie algebras $Sch\Lambda_g \to \mathcal{L}_g$ over Λ_g.

Corollary.

For $k = 3, \dots, 2g-1$.

$$Q_{2k} = \frac{1}{2(k-2)}[Q_2, Q_{2k-2}] - 2\frac{2g-k+1}{(k-2)(2g+1)}(\lambda_{2k} Q_0 - \lambda_4 Q_{2k-4}).$$

For $g = 1$

$$H_0 = z_1\partial_1 - 1; \quad H_2 = \frac{1}{2}\partial_1^2 - \frac{1}{6}\lambda_4 z_1^2.$$

For $g = 2$

$$H_0 = z_1\partial_1 + 3z_3\partial_3 - 3;$$

$$H_2 = \frac{1}{2}\partial_1^2 - \frac{4}{5}\lambda_4 z_3\partial_1 + z_1\partial_3 - \frac{3}{10}\lambda_4 z_1^2 + \left(\frac{3}{2}\lambda_8 - \frac{2}{5}\lambda_4^2\right) z_3^2;$$

$$H_4 = \partial_1\partial_3 - \frac{6}{5}\lambda_6 z_3\partial_1 + \lambda_4 z_3\partial_3 - \frac{1}{5}\lambda_6 z_1^2 + \lambda_8 z_1 z_3 + \left(3\lambda_{10} - \frac{3}{5}\lambda_4\lambda_6\right) z_3^2 - \lambda_4;$$

$$H_6 = \frac{1}{2}\partial_3^2 - \frac{3}{5}\lambda_8 z_3\partial_1 - \frac{1}{10}\lambda_8 z_1^2 + 2\lambda_{10} z_1 z_3 - \frac{3}{10}\lambda_4\lambda_8 z_3^2 - \frac{1}{2}\lambda_6.$$

Explicit form for Schrödinger operators H_0 and H_2

$$H_0 = \sum_{s=1}^{g} (2s-1) z_{2s-1} \partial_{2s-1} - \frac{g(g+1)}{2};$$

$$H_2 = \frac{1}{2}\partial_1^2 + \sum_{s=1}^{g-1} (2s-1) z_{2s-1} \partial_{2s+1} - \frac{4}{2g+1}\lambda_4 \sum_{s=1}^{g-1} (g-s) z_{2s+1} \partial_{2s-1} +$$
$$+ \sum_{s=1}^{g} \left(\frac{2s-1}{2}\lambda_{4s} - \frac{2(g-s+1)}{2g+1}\lambda_4 \lambda_{4s-4} \right) z_{2s-1}^2;$$

Explicit form for Schrödinger operator H_4

$$H_4 = \partial_1\partial_3 + \sum_{s=1}^{g-2}(2s-1)z_{2s-1}\partial_{2s+3} + \lambda_4\sum_{s=1}^{g-1}(2s-1)z_{2s+1}\partial_{2s+1} -$$

$$- \frac{6}{2g+1}\lambda_6\sum_{s=1}^{g-1}(g-s)z_{2s+1}\partial_{2s-1} +$$

$$+ \sum_{s=1}^{g}\left((2s-1)\lambda_{4s+2} - \frac{3(g-s+1)}{2g+1}\lambda_6\lambda_{4s-4}\right)z_{2s-1}^2 +$$

$$+ \sum_{s=1}^{g-1}(2s-1)\lambda_{4s+4}z_{2s-1}z_{2s+1} - \frac{g(g-1)}{2}\lambda_4.$$

The talk based on the papers:

[1.] V.M. Buchstaber,
"Polynomial dynamical systems and the Korteweg–de Vries equation",
Proc. Steklov Inst. Math., 294, 2016, 176–200.

[2.] V.M. Buchstaber, V.Z. Enolski, D.V. Leykin,
"σ-functions: old and new results",
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[3.] V.M. Buchstaber, E.Yu. Bunkova,
"Sigma Functions and Lie Algebras of Schrödinger Operators",
Funct. Anal. Appl., 54:4 (2020), 229–340.

[4.] V.M. Buchstaber, A.V. Mikhailov,
"KdV hierarchies and quantum Novikov's equations",
Open Commun. in Nonlin. Mathem. Physics, no. 1 (2024), 1–36.

THANK YOU FOR YOUR ATTENTION!