Transcendental Numbers in Quantum Spin Chains

Vladimir Korepin and Kun Hao

C.N. Yang Institute for Theoretical Physics
Stony Brook University

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The XXX spin chain

The most famous spin chain was discovered by Werner Heisenberg in 1928. The Hamiltonian was diagonalised by H. Bethe in 1931. The chain has multiple applications to *condensed matter* [Hubbard model describes interaction of electrons in solids], *statistical physics* [6 vertex model with periodic boundary conditions, E.Lieb 1967], *high energy physics* [SYM; Deep Inelastic Scattering by BFKL]. The Hamiltonian

$$H = \sum_{j=1}^{N} (\sigma_j^x \sigma_{j+1}^x \; + \; \sigma_j^y \sigma_{j+1}^y \; + \; \sigma_j^z \sigma_{j+1}^z \; - 1 \;)$$

is an Hermitian matrix in $\otimes_1^N C_j^2$. Here N is the length of the lattice and $\sigma_j^x, \sigma_j^y, \sigma_j^z$ are Pauli matrices [in a local 2 dimensional complex space j.

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Together with identical matrix, they form the standard basis in linear space of 2 dimensional matrices C_j^2. The Pauli matrices form spin 1/2 representation of $SU(2)$

algebra.

Emptiness formation probability

We consider thermodynamics limit $N \to \infty$, n is fixed, $n << N$. We are considering the anti-ferromagnetic case at zero temperature. Generalized [time independent] correlation functions are defined as

$$\langle \prod_{k=1}^{n} \sigma_k^{a_k} \rangle = \lim_{N\to\infty} \langle GS| \prod_{k=1}^{n} \sigma_k^{a_k} |GS\rangle$$

where $a_k = \{0, x, y, z\}$; and $\langle GS|GS\rangle = 1$. The ground state $|GS\rangle$ was constructed by L. Hulthén. Arkiv. Mat. Astron. Fysik **26 A** (1938) No. 11.

An example of non-local correlation function is the emptiness formation probability. It was introduced in 1989. The Journal reference is

A.R. Its, A.G.Izergin, V.Korepin, N.A.Slavnov, Int. J. Mod. Phys. **B 4**, (1990), 1003.

https://www.worldscientific.com/doi/abs/10.1142/S0217979290000504

$$P(n) = \langle GS| \prod_{j=1}^{n} P_j |GS\rangle$$

where $P_j = (1 + \sigma_j^z)/2$ is a projector on the state with spin up in j-th lattice site. It is a probability of formation of ferromagnetic block in anti-ferromagnetic ground state.

Notations

Positive integers are denoted by n of $a_.$ Rational numbers are

$$Z = \frac{n}{a}$$

Roots of polynomials with rational coefficients

$$Z_n x^n + Z_{n-1} x^{n-1} + \cdots + Z_1 x + Z_0 = 0$$

are called algebraic numbers. It is important that n is finite. Example $\sqrt{2}$.
Transcendental numbers are non-algebraic real numbers. They are limits of infinite sequences of algebraic numbers. Examples are : $ln2$ and π. Majority of real numbers are transcendental.
The numbers x, y are algebraically dependent over the field of rational numbers if

$$Z_{nm} x^n y^m + \sum_{a=0}^{n-1} \sum_{b=0}^{m-1} Z_{ab} x^a y^b = 0.$$

Here a and b are positive integers and each $Z_{a_1 \cdots a_k}$ is a rational number.

Leonhard Euler 1737 product with respect to primes; Bernhard Riemann's 1859 analytical continuation:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{for } \Re(s) > 1.$$

Special values of the function at integer arguments.
For the even positive integers $s = n$,

$$\zeta(n) = (-1)^{\frac{n}{2}+1} \frac{(2\pi)^n B_n}{2n!} .$$

Here B_n are rational numbers: Bernoulli numbers.
https://en.wikipedia.org/wiki/Bernoulli_number
The values of Riemann zeta function at even arguments are algebraically dependent transcendental numbers.

Apéry's conjecture

In 1979 he published a proof of the irrationality of $\zeta(3)$.
See for example:
https://scgp.stonybrook.edu/video/video.php?id=5545
Many mathematicians have since worked on the so-called Apéry sequences to seek alternative proofs that might apply to other odd powers (Frits Beukers, Alfred van der Poorten, Marc Prévost, Keith Ball, Tanguy Rivoal, **Wadim Zudilin**, and Don Zagier).

Apéry's conjecture: the values Riemann zeta function at odd arguments are algebraically independent transcendental numbers $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, $\cdots$, $\zeta(2n+1)$.

$$\sum_{\{a_j\}} Z_{a_1 \cdots a_k} \zeta(\overset{a_1}{3}) \zeta(\overset{a_2}{5}) \cdots \zeta(2\overset{a_n}{n} - 1) \cdots \zeta(2\overset{a_k}{k} - 1) \neq 0$$

Here each of $a_.$ is a positive integer and each $Z_{a_1 \cdots a_k}$ is a rational number.

Examples of the emptiness formation probability

Let us come back to correlations in the spin chain.
The four first values of the emptiness-formation probability look as follows:

$$
\begin{aligned}
P(1) &= \frac{1}{2} = 0.5, \\
P(2) &= \frac{1}{3}(1 - \ln 2) = 0.102284273, \\
P(3) &= \frac{1}{4} - \ln 2 + \frac{3}{8}\zeta(3) = 0.007624158, \\
P(4) &= \frac{1}{5} - 2\ln 2 + \frac{173}{60}\zeta(3) - \frac{11}{6}\zeta(3)\ln 2 - \frac{51}{80}\zeta^2(3) \\
&\quad - \frac{55}{24}\zeta(5) + \frac{85}{24}\zeta(5)\ln 2 = 0.000206270
\end{aligned}
$$

We shall put together $\ln 2$ and $\zeta(s)$ in a moment.

The analytic formula for $P(5)$

$$
\begin{aligned}
P(5) &= \frac{1}{6} - \frac{10}{3}\ln 2 + \frac{281}{24}\zeta(3) - \frac{45}{2}\ln 2\cdot\zeta(3) - \frac{489}{16}\zeta(3)^2 \\
&\quad - \frac{6775}{192}\zeta(5) + \frac{1225}{6}\ln 2\cdot\zeta(5) - \frac{425}{64}\zeta(3)\cdot\zeta(5) - \frac{12125}{256}\zeta(5)^2 \\
&\quad + \frac{6223}{256}\zeta(7) - \frac{11515}{64}\ln 2\cdot\zeta(7) + \frac{42777}{512}\zeta(3)\cdot\zeta(7) \\
&= 2.011725953\times 10^{-6},
\end{aligned}
$$

$P(5)$ is expressed as a polynomial of $\ln 2, \zeta(3), \zeta(5)$ and $\zeta(7)$ with rational coefficients.

H. Boos, V. Korepin, J. Phys. A: Math. Gen. 34:5311, (2001)
https://iopscience.iop.org/article/10.1088/0305-4470/34/26/301
H. Boos, V. Korepin, Y. Nishiyama, M. Shiroishi, J. Phys. A: Math. Gen. 35:4443, (2002)
https://iopscience.iop.org/article/10.1088/0305-4470/35/20/305

The alternating zeta series

The alternating zeta series (the polylogarithm at root of unity)

$$\zeta_a(s) = \sum_{n>0} \frac{(-1)^{n-1}}{n^s} = -\mathrm{Li}_s(-1)$$

Here $\mathrm{Li}_s(x)$ is the polylogarithm.
The alternating zeta series is related to the Riemann zeta function as follows

$$\zeta(s) = \frac{1}{1-2^{1-s}}\zeta_a(s), \quad s \neq 1.$$

The alternating zeta has a limit as $s \to 1$.

$$\zeta_a(1) = \ln 2$$

Spin-spin correlation functions

Spin $S = \frac{1}{2}\sigma$. The spin-spin correlation functions are polynomials in terms of the alternating zeta series

$$\begin{aligned}
\left\langle S_j^z S_{j+1}^z \right\rangle &= \frac{1}{12} - \frac{1}{3}\zeta_a(1) = -0.147715726853315 \\
\left\langle S_j^z S_{j+2}^z \right\rangle &= \frac{1}{12} - \frac{4}{3}\zeta_a(1) + \zeta_a(3) = 0.0606797699564 35 \\
\left\langle S_j^z S_{j+3}^z \right\rangle &= \frac{1}{12} - 3\zeta_a(1) + \frac{74}{9}\zeta_a(3) - \frac{56}{9}\zeta_a(1)\zeta_a(3) - \frac{8}{3}\zeta_a(3)^2 \\
&\quad - \frac{50}{9}\zeta_a(5) + \frac{80}{9}\zeta_a(1)\zeta_a(5) = -0.050248627257235 \\
\left\langle S_j^z S_{j+4}^z \right\rangle &= \frac{1}{12} - \frac{16}{3}\zeta_a(1) + \frac{290}{9}\zeta_a(3) - 72\zeta_a(1)\zeta_a(3) - \frac{1172}{9}\zeta_a(3)^2 - \frac{700}{9}\zeta_a(5) \\
&\quad + \frac{4640}{9}\zeta_a(1)\zeta_a(5) - \frac{220}{9}\zeta_a(3)\zeta_a(5) - \frac{400}{3}\zeta_a(5)^2 \\
&\quad + \frac{455}{9}\zeta_a(7) - \frac{3920}{9}\zeta_a(1)\zeta_a(7) + 280\zeta_a(3)\zeta_a(7) \\
&= 0.034652776982728
\end{aligned}$$

where $\langle S_j^z S_{j+2}^z \rangle = 2\,P(3) - 2\,P(2) + \frac{1}{2}\,P(1)$

In 2001 Boos and Korepin conjectured that:
Each correlation function of the XXX spin chain can be represented as a polynomial in $\ln 2$ **and values of Riemann zeta function at odd arguments with rational coefficients.**
Journal of Phys. A Math. and General, vol 34, pages 5311-5316, 2001
The hypothesis was proved by means of quantum Knizhnik- Zamolodchikov equation.
H. Boos, M. Jimbo, T. Miwa, **F. Smirnov**, Y. Takeyama,
Lett. Math. Phys. **75** 201 (2006) [hep-th/0506171]
https://link.springer.com/article/10.1007/s11005-006-0054-x

Jun Sato did some work on the coefficients. J. Phys. A: Math. Theor. 40, 4253 (2007)
String correlation functions of the spin-1/2 Heisenberg XXZ chain
https://iopscience.iop.org/article/10.1088/1751-8113/40/16/001

Coefficients need finalization.

The proof initiated progress in specific correlations. For non-zero temperature, the asymptotics of the partition function in the thermodynamic limit is

$$Z \; = < \; e^{\frac{-H}{kT}} \; > \sim \; e^{\frac{-Nf}{kT}}$$

the asymptotics of $P(n)$ when n tends to infinity

$$P(n) \; = \; \frac{< \; \prod_{j=1}^{n} \frac{(1+\sigma_j^z)}{2} \;\; e^{\frac{-H}{kT}} \; >}{Z} \; \sim \; \frac{e^{\frac{(N-n)f}{kT}}}{Z} \; = \; e^{-\frac{nf}{kT}}$$

For zero temperature we expect Gaussian decay.

A. Abanov, V. Korepin, Nucl. Phys. B 647 (2002) 565-580
https://www.sciencedirect.com/science/article/pii/S0550321302008994
V. Korepin, S. Lukyanov, Y. Nishiyama, M. Shiroishi, Phys. Lett. A 312: 21-26, (2003)
https://linkinghub.elsevier.com/retrieve/pii/S0375960103006169

At zero temperature the asymptotic form of $P(n)$ is Gaussian,
A. Abanov, V. Korepin, Nucl. Phys. B 647 (2002) 565-580
https://www.sciencedirect.com/science/article/pii/S0550321302008994
V. Korepin, S. Lukyanov, Y. Nishiyama, M. Shiroishi, Phys. Lett. A 312: 21-26, (2003)
Clarify
F. Smirnov, T Miwa, Lett. Math. Phys. 109, 675–698 (2019)
https://link.springer.com/article/10.1007/s11005-018-01143-x

$$P(n) \simeq A n^{-\frac{1}{12}} \left(\frac{\Gamma^2(1/4)}{\pi\sqrt{2\pi}} \right)^{-n^2}, \quad (n \to \infty).$$

Evaluation of the coefficient A is still an open problem.
Note that

$$\frac{-1}{12} = \zeta(-1)$$

the sum of all integers.

The Hamiltonian of the integrable isotropic spin 1 chain on a lattice of N sites with periodic boundary conditions, [Zamolodchikov, Fateev,Takhataja, Babujan].

$$H = \frac{J}{4} \sum_{j=1}^{N} \left[\vec{S}_{j-1} \cdot \vec{S}_j - (\vec{S}_{j-1} \cdot \vec{S}_j)^2 \right].$$

Here $\vec{S}_j$ are 3-dimensional matrices: forming spin 1 representation of $SU(2)$ algebra. Correlations in spin 1 XXX are polynomials in π.

C. Babenko, F. Smirnov, Int. J. Mod. Phys. A 34, 15 (2019) 1950075
https://www.worldscientific.com/doi/abs/10.1142/S0217751X19500751

A. Klümper, D. Nawrath, J. Suzuki, J. Stat. Mech. P08009 (2013)
https://iopscience.iop.org/article/10.1088/1742-5468/2013/08/P08009

G. Ribeiro, A. Klümper, J. Phys. A 49, 254001 (2016)
https://iopscience.iop.org/article/10.1088/1751-8113/49/25/254001
Maybe for higher spins correlations are polynomials of the values of Riemann zeta at integer arguments.

G. Ribeiro, A. Klümper, J. Stat. Mech. 013103 (2019),
Correlation functions of the integrable $SU(N)$ spin chain,
https://iopscience.iop.org/article/10.1088/1742-5468/aaf31e
Hurwitz zeta function is defined for complex variables s and a:

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \text{Re}(s) > 1, \text{ and } a \neq 0, -1, -2, \cdots$$

The $a = 2/N$

Application to Statistical Physics

Also Bernard Julia.

Statistical Theory of Numbers. Number theory and physics (Les Houches, 1989), 276–293, Springer Proc. Phys., 47, Springer, Berlin, 1990.

https://link.springer.com/chapter/10.1007/978-3-642-75405-0_30

He suggested a dictionary between number theory and special cases of quantum statistical mechanics by studying several interesting gases of free particles.

Bernard's work had been triggered several years earlier by an unpublished observation of R Shankar that 1/zeta is related to a partition function of a Fermi gas.
Phase transitions are usually associated with zeroes of the partition functions. In particular the pole of the Riemann zeta function at $s = 1$ can be understood as a "phase transition".

Another important contribution is the thermodynamic limit in number theory: Riemann-Beurling gases. Physica A203 1994 pp425-436. Julia finds generalized Hagedorn-Prigogine singularities. The Beurling zeta function is an analogue of the Riemann zeta function where the ordinary primes are replaced by a set of Beurling generalized primes: any sequence of real numbers greater than 1 that tends to infinity. These were introduced by Beurling (1937) .

We get rid of infinities, we specify momentum dependence (maybe ln p ...) and then there are coefficients. The multiple zeta value is defined by the nested series:

$$\zeta(n_1,\ldots,n_r)=\sum_{0<k_1<k_2<\ldots<k_r}\frac{1}{k_1^{n_1}\ldots k_r^{n_r}},\qquad n_1,\ldots,n_r\in\mathbb{N}\text{ and }n_r\geq 2.$$

These are important objects of number theory [motivic interpretation].
We can linearise the expression for correlations in XXX with spin 1/2.
D.J. Broadhurst, D. Kreimer, Int. J. Mod. Phys. C 06, 04 (1995) 519-524
Knots and Numbers in ϕ^4 Theory to 7 Loops and Beyond
https://doi.org/10.1142/S012918319500037X

Some references

Francis Brown, Depth-graded motivic multiple zeta values
https://arxiv.org/abs/1301.3053

Specific zeta values in physics
https://empslocal.ex.ac.uk/people/staff/mrwatkin/zeta/zetavalues.htm

Dirk Kreimer, *"Knots and Feynman Diagrams"*, Cambridge University Press, 2000.
https://doi.org/10.1017/CBO9780511564024

Number Theory and Physics

So far we covered a small corner of a much larger subject:
Number theory and physics archive
https://empslocal.ex.ac.uk/people/staff/mrwatkin/zeta/physics.htm

[introduction mystery new search home guestbook]
[quantum mechanics] [statistical mechanics] [p-adic and adelic physics] [Selberg trace formula] [string theory and quantum cosmology] [scattering] [dynamical and spectral zeta functions] [trace formulae and explicit formulae] [1/f noise and signal processing] [supersymmetry] [QCD] [renormalisation] [symmetry breaking and phase transitions] [quantum fields] [integer partitions] [time] [biologically-inspired and similarly unconventional methods for finding primes] [dynamical systems] [entropy] [specific zeta values] [logic, languages, information, etc.]

[probability and statistics] [noncommutative geometry] [random matrices] [Fourier theory] [fractal geometry] [Bernoulli numbers] [Farey sequences] [Beurling g-primes] [Golden mean] [directory of zeta functions] [directory of L-functions] [conferences] [miscellaneous]

[Riemann Hypothesis: FAQ and resources Riemann's original paper proposed proofs reformulations]
Communications in Number Theory and Physics (journal founded in 2007)
Critical Strip Explorer applet
prime numbers FAQ and resources (for beginners)

p-Adic strings by B. Drgovich and I. Volovich