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ABSTRACTS

Contents

Tits-type alternative for $\mathbb{G}_a$-generated groups of automorphisms

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In 1972, Tits has proved that any subgroup of the general linear group over a field of characteristic zero either contains a nonabelian free subgroup or is a finite extension of a solvable group. In the first part of the talk we survey known facts on the Tits alternative for groups of regular and birational automorphisms of algebraic varieties.

The second part of the talk is based on joint works with Mikhail Zaidenberg. Let X be an affine algebraic variety defined over an algebraically closed field of characteristic zero and $\mathbb{G}_a$ be the additive group of the ground filed. Consider a subgroup H of the automorphism group $\mathrm{Aut}(X)$ generated by a finite collection of $\mathbb{G}_a$-subgroups. We conjecture that either H contains a nonabelian free subgroup or H is a unipotent affine algebraic group. We prove this conjecture when X is an affine toric variety and the $\mathbb{G}_a$-subgroups are normalized by the acting torus. Also we show that if X is a complex affine surface then either H contains a nonabelian free subgroup or H is a metabelian unipotent algebraic group.

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Root subgroups on horospherical varieties

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In the study of automorphism groups of toric varieties, a key role is played by one-parameter additive groups normalized by the acting torus. Such subgroups are called root subgroups and each of them is uniquely determined by its weight, called a Demazure root of the corresponding toric variety. Moreover, the set of all Demazure roots admits an explicit combinatorial description in terms of the fan defining the toric variety, and this description is especially simple in the case where the variety is affine.

In the setting of arbitrary connected reductive groups acting on algebraic varieties, a natural generalization of toric varieties is given by spherical varieties. A spherical variety is an algebraic variety X equipped with an action of a connected reductive group G in such a way that a Borel subgroup B of G has a dense open orbit in X. A proper generalization of root subgroups for spherical varieties is given by one-parameter additive groups normalized by B, which are called B-root subgroups.

The most accessible for study class of spherical varieties is given by horospherical ones. In the talk we shall discuss B-root subgroups on horospherical varieties.

The talk is based on joint works of the speaker with I. Arzhantsev and with V. Zhgoon. The work is supported by the RSF-DST grant 22-41-02019.

Translates of a line

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It is readily apparent that any coordinate of $k^{[2]}$ is a line of $k^{[2]}$ and any translate of a coordinate of $k^{[2]}$ is again a coordinate of $k^{[2]}$. In a landmark result, Abhyankar-Moh and Suzuki, established that over a field k of characteristic zero any line of $k^{[2]}$ is a coordinate of $k^{[2]}$, known a the Epimorphism Theorem. Consequently, it follows that in a polynomial algebra in two indeterminates over a field of characteristic zero, lines and coordinates are synonymous; and translate of a line is a line too.

However, it is a well-established fact that over fields of positive characteristic, the equivalence between lines and coordinates no longer holds true; a line need not be a coordinate. This leads to an intriguing, yet unresolved problem posed by Sathaye which asks whether any translate of line remains a line.

In this talk, which forms part of a collaborative work with Animesh Lahiri, shall discuss on the captivating question posed by Sathaye.

On commuting locally nilpotent derivations and isotropy subgroups in $k[X, Y, Z]$

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For a locally nilpotent derivation on an affine algebraic variety, its isotropy subgroup is defined to be the group of algebraic automorphisms which commute with the locally nilpotent derivation. Let k be a field of characteristic zero. In this talk, I would like to discuss the properties of commuting locally nilpotent derivations on the polynomial ring $k[X, Y, Z]$ and their isotropy subgroups.

This talk is based on recent works with S. Gayfullin and A. Lahiri.

Trinomial varieties

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Trinomial variety is an affine algebraic variety given by a system of equations of some special form, where each equation has the form

$$a_0T_{01}^{l_{01}}\dots T_{0n_0}^{l_{0n_0}}+a_1T_{11}^{l_{11}}\dots T_{1n_1}^{l_{1n_1}}+a_2T_{21}^{l_{21}}\dots T_{2n_2}^{l_{2n_2}}.$$

This class of varieties is interesting since each rational normal affine variety with a torus action of complexity one can be obtained as a good quotient of a trinomial variety by an action of quasitorus. Moreover this is the Cox realization of this variety.

In the talk we prove the criterion of rigidity of a trinomial variety and describe the automorphism group of a rigid trinomial variety and obtain some partial results about Makar-Limanov invariant of such varieties. Also we apply Bhatwadekar's technique to compute stable Makar-Limanov invariant of trinomial varieties.

This talk is based on a joint works with Polina Evdokimova, Anton Shafarevich, Viktoriia Borovik, and Mikhail Petrov.

The work is supported by the RSF-DST grant 22-41-02019.

On the generalisation of linear planes of Sathaye and Russell

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In this talk we shall present several examples where the Abhyankar-Sathaye Conjecture for linear hyperplanes has been confirmed.

On affine forms

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Let k be a field with algebraic closure $\bar{k}$ and A be a k-algebra. We say that A is an $\mathbb{A}^n$-form over k if $A \otimes_k \bar{k} = \bar{k}^{[n]}$. It is well-known that separable $\mathbb{A}^1$-forms are trivial (i.e., $k^{[1]}$) and that there exist non-trivial purely inseparable $\mathbb{A}^1$-forms over fields of positive characteristic. An extensive study of such algebras was made by T. Asanuma. T. Kambayashi established that separable $\mathbb{A}^2$-forms over a field k are trivial. Over any field of positive characteristic, the non-trivial purely inseparable $\mathbb{A}^1$-forms can be used to give examples of non-trivial $\mathbb{A}^n$-forms for any integer $n > 1$. However, the problem of existence of non-trivial separable $\mathbb{A}^3$-forms over a field is still open in general.

Now let R be a ring containing a field k. An R-algebra A is said to be an $\mathbb{A}^n$-form over R with respect to k if $A \otimes_k \bar{k} = (R \otimes_k \bar{k})^{[n]}$, where $\bar{k}$ denotes the algebraic closure of k. A.K. Dutta has investigated separable $\mathbb{A}^1$-forms over any ring R containing a field k and as a corollary obtained that the separable $\mathbb{A}^2$-forms over a PID containing $\mathbb{Q}$ are the trivial ones.

In this talk, we will discuss a partial result on separable $\mathbb{A}^3$-forms over a field k and show an extension of the results on $\mathbb{A}^2$-forms to any one-dimensional Noetherian $\mathbb{Q}$-algebra and to any $\mathbb{Q}$-algebra having a fixed point free locally nilpotent derivation.

A survey on finite generation of kernel of locally nilpotent derivations of polynomial rings

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Hilbert's fourteenth problem, in context of locally nilpotent derivations, asks whether kernel of a locally nilpotent derivation (LND) of a finitely generated k-algebra is finitely generated. In this talk, we give a survey of all the known results and discuss results on LNDs of $R[X, Y, Z]$, where R is a UFD and finitely generated k-algebra.

On rigidity of Pham-Brieskorn surfaces

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An irreducible affine variety $\mathbb{V}$ over a field k is called rigid if it admits no non-trivial $\mathbb{G}$-action. Any $\mathbb{G}$-action on $\mathbb{V}$ has an equivalent formulation as an exponential map on the corresponding coordinate ring $k[\mathbb{V}]$. Thus a k-algebra A is said to be rigid if it has no non-trivial exponential map. We define a k-algebra A to be stably rigid if for any $n \geq 0$ and for any exponential map ϕ on the polynomial ring $A[X_1, \ldots, X_n]$, we have $A \subseteq A[X_1, \ldots, X_n]^{\phi}$.

For any three integers $a, b, c \geq 1$, the rigidity of Pham-Brieskorn surfaces,

$$B_{(a,b,c)} := \frac{k[X, Y, Z]}{(X^a + Y^b + Z^c)},$$

has been studied by several experts including G. Freudenburg, A. J. Crachiola, S. Maubach, when k is an algebraically closed field of characteristic zero.

In this talk we are going to discuss the rigidity of Pham-Brieskorn rings over an arbitrary field k of characteristic $p \geq 0$. We give some sufficient conditions on (a, b, c) for which any Pham-Brieskorn domain $B_{(a,b,c)}$ is rigid. This gives an alternative approach to show that there does not exist any non-trivial exponential map on $k[X, Y, Z, T]/(X^m Y + T^{p^r q} + Z^{p^e}) = k[x, y, z, t]$, for $m, q > 1$, $p \nmid mq$ and $e > r \geq 1$, which fixes y, a crucial result used in [1] to show that the Zariski Cancellation Problem does not hold for the affine 3-space.

We also provide a sufficient condition for $B_{(a,b,c)}$ to be stably rigid.

This is a joint work with Neena Gupta.

References

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On the Grothendieck–Serre conjecture in the mixed characteristic case

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Let R be a local regular unramified ring of mixed characteristic $(p, 0)$. Let D be an Azumaya R-algebra. Then the mentioned conjecture is true for the special linear group $\mathrm{SL}_1(D)$. That is for the group $Nrd = 1$.

Structure of connected nested automorphism groups

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A nested group is an increasing union of algebraic groups. It is well known that any algebraic subgroup of the automorphism group $\mathrm{Aut}(X)$ of an affine variety X is closed with respect to the ind-topology. The closedness of connected nested subgroups in $\mathrm{Aut}(X)$ is an open question.

In this talk, we answer this question positively. Moreover, we establish the structure of nested subgroups of $\mathrm{Aut}(X)$. In the course of the proof, we describe maximal nested unipotent subgroups of $\mathrm{Aut}(X)$ and their relation to $\mathbb{G}_a^k$-actions on X.

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Tits construction and the Rost invariant

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Simple Lie algebras over an algebraically closed field of characteristic 0 are described by Dynkin diagrams. Over a non-closed field, the same Dynkin diagram can correspond to many simple algebras, so it is interesting to study constructions of simple Lie algebras and invariants that make it possible to recognize their isomorphism or reflect some of their properties. One such construction of exceptional (i.e., types E_6, E_7, E_8, F_4 or G_2) Lie algebras was proposed by Jacques Tits; the Jordan algebra and an alternative algebra are given as input, and the output is a Lie algebra, and all real forms of Lie algebras can be constructed in this way. One of the most useful invariants (with meaning in the third Galois cohomology group) was constructed by Markus Rost. We show that a Lie algebra of (outer) type E_6 is obtained by the Tits construction if and only if the Rost invariant is a pure symbol. As an application of this result we prove a Springer-type theorem for an E_6-homogeneous manifold.

On modified Derksen and Makar-Limanov invariants

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The talk is based on joint work with Sergey Gaifullin.

The modified Derksen invariant $\mathrm{HD}^*(X)$ of an affine algebraic variety X is the subalgebra in $\mathbb{K}[X]$ generated by kernels of all locally nilpotent derivations of $\mathbb{K}[X]$ with slices. If there is a locally nilpotent derivation of $\mathbb{K}[X]$ with a slice then $X \simeq Y \times \mathbb{A}^1$ where Y is an affine variety. In my talk we will show that there are three possibilities: A) $\mathrm{HD}^*(X) = \mathbb{K}[X]$; B) $\mathrm{HD}^*(X)$ is a proper infinitely generated subalgebra; C) $\mathrm{HD}^*(X) = \mathbb{K}[Y]$. We give examples for each case, and also provide sufficient conditions for the variety Y so that the variety $X = Y \times \mathbb{A}^1$ belongs to one of the types.

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Local triviality of principal G-bundles over the relative projective line

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Let U be a semilocal noetherian integral scheme. Let G be a reductive group scheme over U. We show that a principal G-bundle over the relative projective line $\mathbb{P}^1_U$ is Zariski locally trivial, once it is trivial at the infinity section. The talk is based on the joint work with Ivan Panin.

Rigid hypersurfaces with separable variables

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Let us consider a rigid hypersurface with separable variables defined by the polynomial F. Such varieties are considered, for example, in [1] and [2]. Then it is known that in the group of automorphisms there exists a torus T containing all other tori. We will show that the group of automorphisms of such a hypersurface is isomorphic to the semidirect product of the group of permutations of variables preserving F on the torus T.

The work is supported by the RSF-DST grant 22-41-02019.

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Algebraic monoid structures on the affine 3-space

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The talk is based on joint work with Ivan Arzhantsev and Roman Avdeev. We have completed the classification of algebraic monoids on the affine 3-space. The most challenging case was that of non-commutative monoids with a group of invertible elements of rank one, which we reduced to the corresponding commutative case.

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