Bass' NK groups in positive and mixed characteristics.

jt w/
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(in progress.)

Dream: understand the algebraic K-theory of non-smooth schemes.

<u>Thm</u> (Quillen) X = regular, noetherian scheme then.

$$K(X) \cong K(X \times \mathbb{A}^1)$$

(homotopy invariance)

Levine's coniveau tower, as a way to get the MotSS uses this heavily.

$$S_X^{(p)}(r) \qquad X \in Sm_F$$

$$= \Big\{ \underset{\text{closed int subsch.}}{W \hookrightarrow X \times \Delta^r} : \underset{X\times F}{\operatorname{codim}} (W \cap X \times F) \geq p \Big\}$$

fiber :

$$K^{(p)}(X,r) = \operatorname*{colim}_{S_X^{(p)}(r)} \left(\begin{array}{c} K(X) \\ \downarrow \\ K(X \times \Delta^r \setminus W) \end{array} \right)$$

$$K_W(X \times \Delta^r)$$

$$r \longmapsto K^{(p)}(X,r)$$

is a simp. spectrum

$$|K^{(p)}(X,\cdot)| = K^{(p)}(X).$$

$$\underbrace{\cdots \to K^{(p)}(X) \to \cdots \to}_{\text{filt. spectrum}} K^{(0)}(X)$$

$$K^{(0)}(X) \overset{\wr}{\longrightarrow} K(X) \cong |K(X \times \Delta^{\bullet})|$$

$\wr$

E_1 page s.s.

whose E_2

is not s.s.

"Vorst's conjecture": $\mathbb{A}^1$-inv for K $\Longleftrightarrow$ X is regular.

Humble goal: understand obstruction for K-thy to be $\mathbb{A}^1$-invariant.

<u>Def</u>: $\overset{\text{Bass}}{NK}(X) :=$

$$\mathrm{fib}\left(\begin{tikzcd} K(X\times\mathbb{A}^1) \arrow[r] & K(X) \arrow[l, bend left, "\text{split}"] \end{tikzcd}\right).$$

<u>Obs</u>:

- LES $\to NK_j(X) \to K_j(X\times\mathbb{A}^1) \to K(X)$

- $NK(X) \oplus K(X) \simeq K(X\times\mathbb{A}^1)$

NK is a quite remarkable object.

Thm (Weibel) $A = \mathbb{Z}/p^i\mathbb{Z}$-algebra (commutative)

then $NK_*(A)$ is a p-group.

$(\forall$ elt x, $\exists N$ s.t. $p^N x = 0)$

- $NK(A)[\frac{1}{p}] \simeq 0$

$\Rightarrow K(A)[\frac{1}{p}] \simeq KH(A)[\frac{1}{p}]$.

- This is proved using an action that Weibel defined. on $NK_*(A)$; action is by $W(A)$ = Big Witt vectors of A.
- Another result: A is an $\mathbb{Q}$-alg then $NK_*(A)$ is a $\mathbb{Q}$-module.

$$A \xrightarrow{[-]} W(A)$$

Teichmüller lift, ring map

char 0.

$NK_*(A)$ is sensitive to the characteristic of A.

$$K_0(\mathbb{F}_p) \cong \mathbb{Z}$$

<u>Thm</u> (Farrell) A comm. ring,

$NK_*(A) \neq 0 \Rightarrow$ not finitely generated.

NK is very intrinsic

to K-thy of singular schemes

1) A ring, $A[x]/(x^N)$

$$K_1(A[x]/(x^N)[t]) \supseteq \left(A[x,t]/(x^N)\right)^\times \ni 1+tx$$

$$\Big\downarrow t$$

$$K_1(A[x]/(x^N))$$

2) $A = \mathbb{F}_p[x]/(x^2)$

Vdk: 1971.

$$0 \to x\mathbb{F}_p[x] \to NK_2(A) \to \Omega^1_{\mathbb{F}_p[t]/\mathbb{F}_p} \to 0$$

ab gp.

differential forms

Thm (van der Kallen.)

$$X \longmapsto NK(X)$$

is an étale sheaf.

$A \to B$ finite étale

- $W_n(A) \to W_n(B)$ étale.
- $NK_n(B) \cong NK_n(A) \otimes_{W(A)} W(B)$.

Question: $A = \mathbb{Z}/p^j\mathbb{Z}$ - alg.

Even though know $NK_n(A)$ is a p-group, do we know if it's of bounded torsion?

$\mathbb{Z}/p^j\mathbb{Z}$ (bdd torsion)

$\mathbb{Q}_p/\mathbb{Z}_p$ (unbdd torsion).

This is a natural question, from the pov of other "pathologies of K-thy"

Thm (Geisser-Hesselholt) $I \triangleleft A$ nilpotent ideal, $A = \mathbb{Z}/p^j\mathbb{Z}$-algebra.

then $K_*(A, I)$ is bdd torsion.

$$\left(K(A, I) \to K(A) \to K(A/I) \right)$$

Thm (Hesselholt-Madsen)

A = smooth $\mathbb{F}_p$-algebra,

$B = A[x]/(x^N)$

$NK_*(B)$ is of bounded

p-torsion

Here:
- explicit computation in terms de Rham W.
- as $* \to \infty$, torsion grows

Thm (E.-Speirs)

A = comm. $\mathbb{Z}/p^j\mathbb{Z}$-algebra,

noetherian and fin. dimensional.

(finite val. dimension.)

$NK_*(A)$ is bounded

p-torsion.

Thm (E. - Speirs)

$X/\mathbb{Z}_p$ a noeth, fin. dimensional,

$X[\frac{1}{p}]$ is smooth.

$NK_*(X)$ is bounded

p-torsion.

(Conjectured by M. Morrow.)

Methods of proofs

- $NK / K_*(A, I)$ sensitive to the char.
- $K_*(A, I)$ is bounded p-torsion
- $K_*(A[t], (t)) = NK_*$

t - not nilpotent.

Black magic.

$$I \to A \twoheadrightarrow B \quad \underline{\text{sq zero.}} \quad \text{ext}$$

nilp.

$$\ker\left(GL(A) \to GL(B)\right)$$

$$= \underset{\text{(mult.)}}{GL(I)} = \underset{\text{(add.)}}{M(I)} \Big\} \text{ sensitive to the char.}$$

∞-matrices.

▶ problem is that

$K(A[t], (t))$

(t) not <u>nilpotent</u>.

<u>Thm</u> (Haesemeyer char 0, Cisinski, Kerz-Strunk-Tamme, Land-Tamme, E.-Hoyois-Iwasa-Kelly)

X qcqs, finite val. dim

(noeth, fin. dim.)

then:

$$NK(X)$$

$$\simeq$$

more homological.

$$N\big(\mathrm{Fib}(TC \longrightarrow L_{can}TC)\big)(X).$$

<u>Proof:</u> I claim that cart.

$$\begin{tikzcd}
K^{inv} \arrow[r] \arrow[d] & K \arrow[r, "tr"] \arrow[d] & TC \arrow[d] \\
L_{can}K^{inv} \arrow[r] & L_{can}K \arrow[r] & L_{can}TC.
\end{tikzcd}$$

Land-Tamme:

K^{inv} has cdh descent.

Idea: $A = \mathbb{E}_1$-ring spectrum. (conn.)

$$A \longrightarrow \pi_0 A. \quad \left(\begin{array}{c}\text{a nilpotent} \\ \text{extension.}\end{array}\right)$$

if a localizing inv. is insensitive

to this map, i.e.,

$$E(A) \simeq E(\pi_0 A)$$

$\Rightarrow E$ has cdh descent.

$$
\begin{tikzcd}
\cancel{\text{“}\lim\text{”}}\, E(F_n) & E(\tilde{X}) \arrow[l] \\
\cancel{\text{“}\lim\text{”}}\, E(Z_n) \arrow[u] \arrow[ur, phantom, "\square"] & E(X) \arrow[l] \arrow[u]
\end{tikzcd}
$$

K^{inv} truncating

$\Updownarrow$

Dundas–Goodwillie–McCarthy.

K^{inv} is nilinvariant.

fiber

$$\begin{tikzcd}
NF_K \arrow[r, "\sim"] \arrow[d, "\wr"] & NF_{TC} \arrow[d] \\
NK \arrow[r, "tr"] \arrow[d] & NTC \arrow[d] \\
NL_{cdh}K \arrow[r] \arrow[d, phantom, "\wr"] & NL_{cdh}TC. \\
NKH = 0.
\end{tikzcd}$$

∎

trace methods.

$$\begin{tikzcd} K \arrow[r] \arrow[d] & TC \arrow[d] \\ KH \arrow[r] & L_{cdh} TC \end{tikzcd}$$

motivic homotopy theory

$\wr$

- cdarc sheaf.
- étale sheaf.

can show.

$$NK \simeq N(\mathrm{Fib}(TC \to L_{cdh} TC))$$

Thm. $A = \mathbb{Z}/p^j\mathbb{Z}$ - algebra.

then $NK_*(A)$ is bdd

p - torsion.

Pf) $\mathbb{Z}/p^n\mathbb{Z}[\frac{1}{p}] = 0$.

Bhatt: E is a bdd below

spectrum. Then tfae:

- $\pi_* E$ is bdd p-torsion
- $E^\wedge_p[\frac{1}{p}] \simeq 0$.

$$\boxed{\begin{aligned}\text{WTS: }& NK(A;\mathbb{Q}_p)\\ &= NK(A)^{\wedge}_p[\tfrac{1}{p}] = 0.\end{aligned}}$$

$$\| $$

$$N\left(\mathrm{Fib}\left(TC(A) \to L_{cdh} TC(A)\right)^{\wedge}_p[\tfrac{1}{p}]\right.$$

Claim: follows if we can show that $TC(-;\mathbb{Q}_p)$ is a cdh sheaf.

Suffices to use Land-Tamme:

the map $A \to \pi_0 A$

conn. $\mathbb{E}_1$-ring - $\mathbb{Z}/p;\mathbb{Q}$-alg induces iso on

$$\begin{tikzcd} TC(A; \mathbb{Q}_p) \arrow[r] & TC(\pi_0 A; \mathbb{Q}_p) \\ TC(A, \pi_0 A; \mathbb{Q}_p) \arrow[u] & \end{tikzcd}$$

Reduce to: $A = \mathbb{E}_1$-ring, $M^{[1]}$ is A-module; form $A \oplus M_{[1]}$ = square zero extension.

$$TC(A, A\oplus M)^{\wedge}_p[\tfrac{1}{p}]$$

can remove p-compl; already complete

$$\simeq TC(A, A\oplus M)[\tfrac{1}{p}]$$

$THH(R)$

$$\simeq \quad R\otimes R\otimes R \;\substack{\to\\ \to\\ \to}\; R\otimes R \;\substack{\to\\ \to}\; R$$

$$a\otimes b\otimes c \longmapsto \begin{matrix} ab\otimes c \\ a\otimes bc \\ ca\otimes b. \end{matrix}$$

If R is p-nilpotent, the resulting geom. realization is p-complete.

Fact: $|-|$ of bdd p-complete obj's is p-complete. (Check on π_*; $\mathrm{Ext}^{0,1}(p\text{-local}, \pi_* E) = 0$.) (SAG)

$$TC(A, A\oplus M)[\tfrac{1}{p}]$$

$$\simeq TC(A[\tfrac{1}{p}], A\oplus M[\tfrac{1}{p}])$$

Raskin's writeup of DGM-thm.

$$\simeq 0$$

$\blacksquare$

$THH(A)[\tfrac{1}{p}]$

Mixed characteristic situation.

Tool: Beilinson fiber square.

Beilinson, Antieau-Mathew-Morrow-Nikolaus.

Thm R is a ring

$$
\begin{tikzcd}
 & TC(R;\mathbb{Q}_p) \arrow[r] \arrow[dl] \arrow[d] \arrow[dr, phantom, "\square"] & TC(\underline{R/p};\mathbb{Q}_p) \arrow[d, "\text{crys. chern character.}"] \\
TC^- \arrow[r, "\text{nat. map}"] & HC^-(R;\mathbb{Q}_p) \arrow[r] & HP(\underline{R};\mathbb{Q}_p)
\end{tikzcd}
$$

R : $\mathbb{Z}_p$-algebra.

$$N\left[\mathrm{fib}\left(TC \longrightarrow L_{can}TC\right)\right)^{\wedge}_{p}\left[\tfrac{1}{p}\right]$$

w/ something more homological.

__Lemma__ In this case.

$$\left(\mathrm{fib}\left(TC \longrightarrow L_{can}TC\right)\right)^{\wedge}_{p}\left[\tfrac{1}{p}\right]$$

$$\simeq \left(\mathrm{fib}\left(\underline{\Sigma HC} \longrightarrow \underline{L_{can}}\Sigma HC\right)\right)^{\wedge}_{p}\left[\tfrac{1}{p}\right]$$

Cyclic homology.

Pf sketch.

has coh descent.

$$\begin{tikzcd}
\mathcal{F}_{\Sigma HC}\ TC(R) \arrow[r] \arrow[d] & \mathcal{F}\ \cancelto{0}{TC(R/p)} \arrow[d] \\
\mathcal{F}\ \underline{HC^-(R)} \arrow[r, "can"'] & \mathcal{F}\ \underline{HP(R)}
\end{tikzcd}$$

$$\Sigma HC \to HC^- \to HP.$$

<u>Lemma</u> (SBI-sequence). SES

$$0 \to NHC_{n-1}(R;\mathbb{Q}_p) \xrightarrow{B} NHH_n(R;\mathbb{Q}_p) \xrightarrow{I} NHC_n(R;\mathbb{Q}_p) \to 0.$$

- traded

$$NK \iff NJ_{TC}$$

rationally

$$J_{TC} \underset{\text{Bedlinson}}{\iff} J_{HC}.$$

$$NHC \iff N\underline{HH}$$

HH has Künneth!

$\Rightarrow$ Can compute.

∎

$$HH_n(R[t])$$

$$\cong HH_n(R) \otimes_{\mathbb{Z}} \mathbb{Z}[t]$$

$$\oplus$$

$$HH_{n-1}(R) \otimes_{\mathbb{Z}} \Omega^1_{\mathbb{Z}[t]/\mathbb{Z}}.$$

- K-book page 355

6.7.4 Farrell.

- Char 0 : Cortiñas - Haesemeyer - Weibel - Walker.