15th Saint-Petersburg Conference *in* Spectral Theory and Mathematical Physics, dedicated to M. S. Birman

Euler Institute, St. Petersburg, Russia

June 24–28, 2024

Book of Abstracts

Qualitative spectral properties of the involutive Friedrichs model

Mon, Jun 24
17:50–18:15

Grigorii Agafonkin

Moscow State University

The Friedrichs model deals with a compact perturbation of the operator

$$A_0 f(x) = x f(x)$$

in $L_2[-1,1]$. It is known (see [1], [2]) that in the case of self-adjoint integral perturbation with kernel $K(x,y)$ being Hölder with exponent $\alpha > \frac{1}{2}$ the perturbed operator can only have finite number of discrete eigenvalues, and if $\alpha = \frac{1}{2}$ then its spectral measure can have arbitrary discrete and singular continuous parts.

We study the spectral properties of the operator

$$Hf(x) = ixf(-x) + \int_{-1}^{1} K(x,y) f(y)\, dy$$

("multiplication with involution") in $L_2[-1,1]$. We show that if K is Hölder continuous with exponent $\alpha > \frac{1}{2}$ the perturbed operator again can only have finite number of discrete eigenvalues. In addition, for finite-dimensional perturbations

$$K(x,y) = \sum_{j=1}^{n} \tau_j k_j(x) k_j(y),$$

where $\tau_j \neq 0$, we provide some results in localization of the discrete spectrum.

The support of RSF grant no. 20-11-20261 is acknowledged.

References

[1] L. D. Faddeev, *On the Friedrichs model in the theory of perturbations of a continuous spectrum*, Amer. Math. Soc. Transl. Ser. 2, 62 (1967), 177–203.

[2] B. S. Pavlov and S. V. Petras, *The singular spectrum of the weakly perturbed multiplication operator*, Funct. Anal. Appl., 4 (1970), 136–142.

Wed, Jun 26
17:50–18:30

Mode's transformation and short-wave asymptotics in linearized equations of shallow water theory with abruptly varying coefficients

Anna Allilueva

Ishlinsky Institute for Problems in Mechanics RAS

In this work we consider the system of equations of the shallow water theory, linearized in an external field, that has a sharp jump near a certain curve. Short-wave asymptotic solutions are described for this system; far from the specified curve, they are represented as the sum of three modes – vortex and two wave. Near the curve, wave packets are divided into transmitted and reflected ones, and wave modes can turn into vortex modes and on the contrary. The amplitudes of the transmitted and reflected waves are determined from the transfer equations, the initial conditions for which are found from the auxiliary scattering problem for a system of two ordinary differential equations. The result of the work is an asymptotic series for solving the Cauchy problem, the terms of which are expressed in terms of canonical Maslov operators on falling, transmitted and reflected Lagrangian manifolds.

References

[1] V. P. Maslov and M. V. Fedoriuk, *Semi-Classical Approximation in Quantum Mechanics*, Math. Phys. Appl. Math., vol. 7, Dordrecht: Springer, 1981.

[2] V. P. Maslov and G.,A. Omel'yanov, *Asymptotic soliton-form solutions of equations with small dispersion*, Russian Math. Surveys, 36 (1981), no. 3, 73–149.

[3] F. Calogero and A. Degasperis, *Spectral Transform and Solitons*, North-Holland Publishing Company, 1982.

Fri, Jun 28
16:00–16:25

On a Schrödinger operator with a potential having infinitely many "bumps"

Nikolay Andronov

Saint Petersburg State University

In one of his papers D. B. Pearson proved existence of singular continuous spectrum for a Schrödinger operator with a potential having an infinite sequence of "bumps". The distances between the bumps quickly increase with their sequence number, the description of these distances being rather implicit. In this talk we study an explicit model problem.

Reverse Faber–Krahn and Szegö–Weinberger type inequalities for domains with nontrivial topology

Fri, Jun 28
10:50–11:35

Vladimir Bobkov

Institute of Mathematics, Ufa Federal Research Center RAS

Let $\tau_k(\Omega)$ be the k-th eigenvalue of the mixed Robin-Neumann problem

$$\begin{cases} -\Delta u = \tau u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + hu = 0 & \text{on } \partial B_\alpha, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega_{\text{out}}, \end{cases} \tag{$\mathcal{EP}$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain of the form $\Omega_{\text{out}} \setminus \overline{B_\alpha}$, a ball B_α with radius $\alpha > 0$ is compactly contained in Ω_{out}, and $h \in (-\infty, +\infty]$, the limiting case $h = +\infty$ being correspondent to the Dirichlet boundary condition on ∂B_α.

In the case $h > 0$, it is known that the first eigenvalue $\tau_1(\Omega)$ does not exceed $\tau_1(B_\beta \setminus \overline{B_\alpha})$, where $\beta > 0$ is chosen such that $|\Omega| = |B_\beta \setminus \overline{B_\alpha}|$, which can be regarded as a reverse Faber-Krahn type inequality. We establish this result for any $h \in (-\infty, +\infty]$, and we provide related estimates for higher eigenvalues under additional geometric assumptions on Ω, which can be seen as Szegö-Weinberger type inequalities, cf. [1].

As auxiliary information, we investigate shapes of eigenfunctions associated with several eigenvalues $\tau_k(B_\beta \setminus \overline{B_\alpha})$ and show that they are nonradial at least for all positive and all sufficiently negative values of h when $k \in \{2, \dots, N+2\}$. At the same time, we give numerical evidence that already second eigenfunctions can be radial for some $h < 0$. The latter fact provides a geometrically simple counterexample to the Payne nodal line conjecture in the case of the mixed boundary conditions.

The talk is based on the work [2].

References

[1] T. V. Anoop, V. Bobkov, P. Drábek, *Szegö-Weinberger type inequalities for symmetric domains with holes*, SIAM J. Math. Anal., 54 (2022), no. 1, 389–422.

[2] T. V. Anoop, V. Bobkov, P. Drábek, *Reverse Faber–Krahn and Szegö–Weinberger type inequalities for annular domains under Robin–Neumann boundary conditions*, arXiv:2309.15558.

Fri, Jun 28
12:00–12:45

Inverse spectral problems for a fourth-order differential equation

Natalia Bondarenko

Samara National Research University

In this talk, we consider inverse spectral problems for the equation

$$y^{(4)} - (p(x)y')' + q(x)y = \lambda y, \qquad x \in (0,1), \tag{1}$$

where p and q are real-valued functions of $L_1[0,1]$, λ is the spectral parameter.

Barcilon's Problem [1]. *Find p and q from the three spectra $\mathfrak{S}_{1,2}$, $\mathfrak{S}_{1,3}$, $\mathfrak{S}_{2,3}$ for equation* (1) *with the corresponding boundary conditions:*

$$\mathfrak{S}_{i,j}: \quad y^{(i-1)}(0) = y^{(j-1)}(0) = 0, \quad y(1) = y'(1) = 0.$$

McLaughlin's Problem [2]. *Find p and q from a single spectrum and two sets of norming constants.*

These inverse problems were studied since 1970–1980s but, to the best of the author's knowledge, the uniqueness for their solutions was not proved. Recently, Barcilon's and McLaughlin's problems were interpreted within the framework of the general approach [3] to inverse spectral problems for higher-order differential operators. This interpretation implies new uniqueness results for the considered fourth-order inverse problems (see [4]).

References

[1] V. Barcilon, *On the uniqueness of inverse eigenvalue problems*, Geoph. J. Intern., 38 (1974), 287–298.

[2] J. R. McLaughlin, *Higher-order inverse eigenvalue problems*, in Ordinary and Partial Differential Equations. Lecture Notes in Mathematics, Vol. 964, W. Everitt and B. Sleeman, eds., Springer, Berlin, Heidelberg, 1982.

[3] V. A. Yurko, *Method of Spectral Mappings in the Inverse Problem Theory*, Inverse and Ill-Posed Problems Series, Utrecht, VNU Science, 2002.

[4] N. P. Bondarenko, *McLaughlin's inverse problem for the fourth-order differential operator*, arXiv:2312.15988.

Operator estimates for domains with irregularly oscillating boundary: Dirichlet and Neumann condition

Tue, Jun 25
10:00–10:45

Denis Borisov

Institute of Mathematics, Ufa Federal Research Center RAS

We consider general second-order matrix elliptic operator in a multi-dimensional domain with a fast and irregularly oscillating boundary. The only assumption made is that the amplitude of the oscillations, treated in an appropriate sense, is small. On the oscillating boundary we impose either the Dirichlet or Neumann condition. Our main result states that as the amplitude goes to zero, the considered operator converges in the norm resolvent sense to a homogenized one. The homogenized operator is described by the same differential expression with the Dirichlet or Neumann condition on the limiting boundary, which corresponds to the zero amplitude. Apart from the convergence, we also provide estimates for the convergence rate.

The work is supported by RSF, project no. 23-11-00009[†]

This is a joint work with Radim Suleimanov (Ufa University of Science and Technology).

Speed of mixing for the sine-process

Tue, Jun 25
15:50–16:35

Alexander I. Bufetov

Steklov

Inverse-quadratic estimates are obtained for the decay of correlations of multiplicative functionals under the sine-process. Fredholm determinants associated to the sine-kernel, the operator of orthogonal projection onto the Paley–Wiener space, admit a precise asymptotic formula, the scaling limit of the Borodin–Okounkov–Geronimo–Case Identity, or, in other words, of the Szego Strong Theorem in the form of Ibragimov, and with a precise error term given by an explicit Hankel determinant. The proof of the bound on the speed of mixing involves estimating the decorrelation of the Toeplitz and Hankel determinants entering the Widom proof of the Borodin–Okounkov–Geronimo–Case Identity.

On boundary layers in media with O. A. Ladyzhenskaya rheology

Tue, Jun 25
12:00–12:45

Gregory A. Chechkin

Moscow State University

In [1] one can find some new problems in boundary layer theory.

† `https://rscf.ru/project/23-11-00009`

In this talk, we consider boundary layers in media with complex rheological properties of the Prandtl type (see [2]), Marangoni type and with slip (see [1]). By obtaining one nonlinear equation from the system of boundary layer equations, the existence and uniqueness theorems are proved.

References

[1] V. N. Samokhin and G. A. Chechkin, *Nonclassical problems of mathematical theory of hydrodynamical boundary layer*, Vestnik Moskov. Univ. Ser. I Mat. Mekh., 2024, no. 1, 11–20 (in Russian); Moscow Univ. Math. Bull., 79 (2024), no. 1, 11–21 (in English).

[2] R. R. Bulatova, V. N. Samokhin and G. A. Chechkin, *System of equations for boundary layer of rheologically complex media. The Crocco variables*, Dokl. Acad. Nauk 487 (2019), 119–125 (in Russian); Dokl. Math. 100 (2019), 332–338 (in English).

Wed, Jun 26
15:00–15:45

Spectral asymptotics corresponding to degenerate billiards with semi-rigid walls and nonlinear long coastal waves

Sergey Dobrokhotov

Ishlinsky Institute for Problems in Mechanics RAS

In a recent paper [1], time-periodic solutions of a nonlinear system of shallow water equations in basins with shallow gentle shores localized in the vicinity of the coastline were constructed. In this work, the construction of such solutions is associated with special trajectories of a two-dimensional Hamiltonian system with a Hamiltonian $H = D(x_1, x_2)(p_1^2 + p_2^2)$, where the function D is the depth of the basin. We denote the coastline by $\Gamma = \{D = 0\}$ and assume that $\nabla D|_\Gamma \neq 0$. As D turns to zero as $x \to \Gamma$ thus suitable solutions of the Hamiltonian system organize the so-called billiards with semi-rigid walls, woven from trajectories located between the standard caustics and the "non-standard" ones *Gamma* (see also [2]). These billiards implies the asymptotic eigenfunctions of the linear operator $-\nabla(D(x)\nabla)$ for large eigenvalues and turn mentioned coastal long nonlinear waves. The existence of these billiards with semi-rigid walls is possible in the case of integrable Hamiltonian systems with Hamiltonian H, which practically does not happen in real situations. In this talk, we consider degenerate situations where "standard" caustics are very close to the coastline ("non-standard" caustics). The requirement of integrability then disappears and it is always possible to construct effective asymptotic wave solutions having a small number of oscillations normal to the shore [3] (which are analogs of Stokes and Ursell waves). The corresponding trajectories are strongly localized in the narrow vicinity of the coast, while they always enter the coastline and reflect from it at an angle of 90 degrees. Thus, we have asymptotic solutions similar to the "whispering gallery" type solutions known in acoustics, but at the same time for their existence due to a "degenerate" wall (coastline) the convexity of the two-dimensional region $\{x\colon D(x) > 0\}$ in which the pool is located is not required.

Joint work with Dmitrii Minenkov and Maria Votyakova. The research was carried out with the support of RSF, the project 24-11-00213.

References

[1] S. Dobrokhotov, V. Nazaikinskii and A. Tsvetkova, *Nonlinear effects and run-up of coastal waves generated by billiards with semi-rigid walls in the framework of shallow water theory*, Proc. Steklov Inst. Math., 322 (2023), 105–117.

[2] S. Bolotin and D. Treschev, *Another billiard problem*, Russ. J. Math. Phys., 31 (2024), 50–59.

[3] S. Dobrokhotov, D. Minenkov and M. Votiakova, *Asymptotics of long nonlinear coastal waves in basins with gentle shores*, Russ. J. Math. Phys., 31 (2024), 79–93.

High-energy homogenization of a multidimensional nonstationary Schrödinger equation

Tue, Jun 25
12:50–13:15

Mark Dorodnyi

Saint Petersburg State University

Let Γ be a lattice in $\mathbb{R}^d$, and let Ω be the cell of Γ. For any Γ-periodic function $F(\mathbf{x})$, we denote $F^\varepsilon(\mathbf{x}) := F(\varepsilon^{-1}\mathbf{x})$, where $\varepsilon > 0$ is a (small) parameter. In $L_2(\mathbb{R}^d)$, consider a differential operator given by the expression

$$A_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla + \varepsilon^{-2}V^\varepsilon(\mathbf{x}).$$

Here $g(\mathbf{x})$ is a Γ-periodic positive definite and bounded $(d \times d)$-matrix-valued function with real entries, $V(\mathbf{x})$ is a Γ-periodic real-valued function, $V \in L_\rho(\Omega)$ with a suitable ρ (and it is assumed that $\inf \operatorname{spec} A_1 = 0$).

Let $(\mathbf{k}^\circ, \lambda_0)$ be an arbitrary point of the dispersion relation of the operator A_1; and let $\{e^{i\langle \mathbf{k}^\circ, \mathbf{x}\rangle}\varsigma_j(\mathbf{k}^\circ, \mathbf{x})\}_{j=1}^p$ be corresponding Bloch waves, $(\varsigma_j(\mathbf{k}^\circ, \cdot), \varsigma_k(\mathbf{k}^\circ, \cdot))_{L_2(\Omega)} = \delta_{jk}$. We are interested in the behavior of the solutions $u_{j,\varepsilon}(\mathbf{x}, \tau)$, $\mathbf{x} \in \mathbb{R}^d$, $\tau \in \mathbb{R}$, $j = 1, \dots, p$, $\varepsilon \to 0$, of the following Cauchy problems for the nonstationary Schrödinger equation:

$$\begin{cases} i\dfrac{\partial u_{j,\varepsilon}(\mathbf{x}, \tau)}{\partial \tau} = (A_\varepsilon u_{j,\varepsilon})(\mathbf{x}, \tau), \qquad u_{j,\varepsilon}(\mathbf{x}, 0) = e^{i\varepsilon^{-1}\langle \mathbf{k}^\circ, \mathbf{x}\rangle}\varsigma_j^\varepsilon(\mathbf{x}) f_j(\mathbf{x}), \end{cases}$$

where $f_j(\mathbf{x})$, $j = 1, \dots, p$, are given functions and $\varsigma_j^\varepsilon(\mathbf{x}) := \varsigma_j(\mathbf{k}^\circ, \mathbf{x}/\varepsilon)$. We prove the *following estimates*:

$$\|u_{j,\varepsilon}(\cdot,\tau) - u_{j,\varepsilon}^{\mathrm{eff}}(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \le C(1+|\tau|)\varepsilon\|f_j\|_{H^3(\mathbb{R}^d)}, \qquad j = 1, \dots, p,$$

where

$$u_{j,\varepsilon}^{\mathrm{eff}}(\mathbf{x},\tau) = e^{i\varepsilon^{-1}\langle \mathbf{k}^\circ, \mathbf{x}\rangle} \sum_{l=1}^{p} \varsigma_l^\varepsilon(\mathbf{x}) v_{jl,\varepsilon}^{\mathrm{eff}}(\mathbf{x},\tau)$$

and $\mathbf{v}_{j,\varepsilon}^{\mathrm{eff}}(\mathbf{x},\tau) = (v_{j1,\varepsilon}^{\mathrm{eff}}(\mathbf{x},\tau), \dots, v_{jp,\varepsilon}^{\mathrm{eff}}(\mathbf{x},\tau))^{\mathrm{t}}$ is the solution of the "effective" system

$$\begin{cases} i\dfrac{\partial \mathbf{v}_{j,\varepsilon}^{\mathrm{eff}}(\mathbf{x},\tau)}{\partial \tau} = (A_\varepsilon^{\mathrm{eff}} \mathbf{v}_{j,\varepsilon}^{\mathrm{eff}})(\mathbf{x},\tau), \qquad \mathbf{v}_{j,\varepsilon}^{\mathrm{eff}}(\mathbf{x},0) = f_j(\mathbf{x})\mathbf{e}_j. \end{cases}$$

Here $\mathbf{e}_j$ is the element of the canonical basis in $\mathbb{C}^p$, $A_\varepsilon^{\mathrm{eff}}$ is the *effective operator* with *constant coefficients*

$$A_\varepsilon^{\mathrm{eff}} = \begin{pmatrix} A_\varepsilon^{\mathrm{eff},11} & \dots & A_\varepsilon^{\mathrm{eff},1p} \\ \vdots & \ddots & \vdots \\ A_\varepsilon^{\mathrm{eff},p1} & \dots & A_\varepsilon^{\mathrm{eff},pp} \end{pmatrix},$$
$$A_\varepsilon^{\mathrm{eff},ls} := \varepsilon^{-2}\lambda_0 I - i\varepsilon^{-1}\langle \mathfrak{g}^{1,ls}, \nabla\rangle - \operatorname{div} \mathfrak{g}^{2,ls}\nabla,$$

acting in $L_2(\mathbb{R}^d; \mathbb{C}^p)$. The way to construct $\mathfrak{g}^{1,ls}$ and $\mathfrak{g}^{2,ls}$ is described. The results are published in [1].

The work was supported by Young Russian Mathematics award and the Ministry of Science and Higher Education of the Russian Federation, agreement no. 075-15-2022-287.

References

[1] M. A. Dorodnyi, *High-energy homogenization of a multidimensional nonstationary Schrödinger equation*, Russ. J. Math. Phys, 30 (2023), 480–500.

Fri, Jun 28
10:00–10:45

On a quasi-periodic difference Schrödinger operator with a meromorphic potential

Alexander Fedotov

Saint Petersburg State University

We study a one-dimensional quasiperiodic difference Schrödinger operator with a potential obtained by restricting a meromorphic function to the integer lattice. The meromorphic function has two poles per period and tends to constant limits far from the real line. We show that the spectral analysis of the operator is related to an analysis of a finite-dimensional dynamical system. In the case, when the coupling constant is sufficiently small, we asymptotically describe a series of intervals contained in spectral gaps, their centers and lengths. The lengths of these intervals form an exponentially decreasing sequence, and the rate of decrease is determined by the distance from the poles of the potential to the real line.

The talk is partially based on a joint work with Kirill Sedov.

On the Pólya conjecture for the Neumann problem in planar convex domains

Mon, Jun 24
15:00–15:45

Nikolai Filonov

Saint Petersburg Department of Steklov Institute of Mathematics RAS

Denote by $N(\Omega, \lambda)$ the counting function of the eigenvalues of the Laplace operator in a domain Ω with Neumann boundary conditions. In 1954 G. Pólya conjectured that

$$N(\Omega, \lambda) \geq \frac{|\Omega|\lambda}{4\pi}$$

for all domains Ω in $\mathbb{R}^2$ and for all $\lambda \geq 0$. We show that for convex domains

$$N(\Omega, \lambda) \geq \frac{|\Omega|\lambda}{2\sqrt{3}\, j_0^2},$$

where j_0 is the first zero of the Bessel function J_0.

Estimates and asymptotics for the spectrum of compact pseudodifferential operators of variable order

Tue, Jun 25
17:00–17:45

Andrey Karol

Saint Petersburg State University

We consider compact pseudodifferential operators with symbols whose decaying order with respect to the variable ξ depends on the space variable. Such operators arise in the problem of L_2-small ball deviation asymptotics for Gaussian processes with variable Hurst parameter.

We obtain the estimates for singular values as well as validity conditions of the Weyl's asymptotics. The main difficulty is related to nonsmoothness of the symbol. The results are formulated in terms of the symbol belonging to the classes of multipliers of integral operators.

Wed, Jun 26
12:50–13:15

Asymptotics of solutions for $n \times n$ systems of ordinary differential equations with summable coefficients

Alexey Kosarev

Moscow State University

We deal with a $n \times n$ system of differential equations of the form

$$y' - By - C(\cdot, \lambda)y = \lambda Ay, \quad y = y(x), \quad x \in [0,1], \tag{1}$$

where $A = \operatorname{diag}\{a_1(x), \ldots, a_n(x)\}$, $B = \{b_{jk}(x)\}_{j,k=1}^n$, $C = \{c_{jk}(x,\lambda)\}_{j,k=1}^n$. All elements of matrices A, B and C are summable and complex-valued, and in addition $\|c_{jk}(\cdot,\lambda)\|_{L_1} \to 0$ as $\lambda \to \infty$.

We introduce two key conditions, that turn out to be sufficient to obtain asymptotics of solutions in two shifted sectors $\pm\Lambda_{\alpha,\kappa}$.

Let's say that *Birkhoff's condition* is satisfied on the arc $\lambda = e^{i\theta}$, $\theta \in [\theta_0, \theta_1]$ if there is a permutation σ of indices $\{1, \ldots, n\}$ of the functions a_j such that the inequalities

$$\Re\, e^{i\theta} a_{\sigma(1)}(x) \le \Re\, e^{i\theta} a_{\sigma(2)}(x) \cdots \le \Re\, e^{i\theta} a_{\sigma(n)}(x) \qquad \text{a. e. } x \in [0,1] \tag{2}$$

hold for all $\theta \in [\theta_0, \theta_1]$.

Let's say that the *nondegenerate condition* is satisfied if for each difference $a_j - a_k$ the function $(a_j - a_k)^{-1} = 1/(a_j - a_k)$ is defined almost everywhere, and for some $1 \le p \le \infty$ we have

$$a_j - a_k \in L_p[0,1], \quad (a_j - a_k)^{-1} \in L_{p'}[0,1], \quad 1/p + 1/p' = 1, \quad \forall\, 1 \le j < k \le n. \tag{3}$$

We now state our main result.

Theorem. *Suppose that Birkhoff's condition* (2) *on the arc* $e^{i\theta}, \theta \in [-\alpha, \alpha]$ *and the nondegenerate condition* (3) *are satisfied. Then for any fixed* $\kappa \ge 0$ *there is a fundamental matrix* $Y(x,\lambda)$ *of solutions of the system* (1)*, which in the left-shifted sector*

$$\Lambda_{\alpha,\kappa} = \Big\{ \lambda \in \mathbb{C} \;\Big|\; \lambda = -\kappa + re^{i\theta},\ r \ge 0,\ \theta \in [-\alpha, \alpha] \Big\}$$

has an asymptotic representation

$$Y(x,\lambda) = M(x)\,(I + R(x,\lambda))\,E(x,\lambda), \tag{4}$$

where matrices $M(x), E(x,\lambda)$ *are defined by*

$$E(x,\lambda) = \operatorname{diag}\Big\{ e^{\lambda \int_0^x a_1(t)dt}, \ldots, e^{\lambda \int_0^x a_n(t)dt} \Big\},$$
$$M(x) = \operatorname{diag}\Big\{ e^{\int_0^x b_{11}(t)dt}, \ldots, e^{\int_0^x b_{nn}(t)dt} \Big\},$$

and the elements r_{jk} of the matrix $R(x,\lambda)$ uniformly tend to zero $|r_{jk}(x,\lambda)| \to 0$ as $\lambda \to \infty$ in the sector $\Lambda_{\alpha,\kappa}$.

The statement of the theorem remains valid if the sector $\Lambda_{\alpha,\kappa}$ is replaced by the opposite sector $-\Lambda_{\alpha,\kappa}$.

In the last part of the talk we will propose criterion whether Birkhoff condition is satisfied, and formulate theorems, that replace the nondegenerate condition with restrictions on the structure of the functions $a_j - a_k$.

The report is based on the joint work with A. A. Shkalikov. The work was supported by the RSF grant no. 20-11-20261.

Acoustic eigenmodes localised near a "shoreline" of a vessel covered by a thin elastic membrane

Wed, Jun 26
17:00–17:45

Mikhail A. Lyalinov

Saint Petersburg State University

The report deals with formal short-wavelength asymptotic solutions describing acoustic eigen-oscillations in a vessel, having a hard bottom B, filled in by an acoustic medium W and covered by a thin elastic membrane F, Fig. 1. The solutions are localised in the medium near the line of the rigid contact of the membrane, covering the vessel, with the edge of the vessel, see Fig. 1. The coefficients in the asymptotic expansion of the solutions satisfy a recurrent sequence of solvable problems, whereas the frequencies, for which such non-trivial formal solutions exist, obey an asymptotic "quantizationi"-type condition.

To our knowledge, as "an asymptotic semi-classical phenomenon", the localised near a shoreline solutions (Stokes-type waves localised near shorelines) have been first described in the works by S. Yu. Dobrohotov and his colleagues [1] and [2]. Some experimental results on the water waves localised near a shoreline are discussed in [3] and [4].

The *main* result in this report is in construction of the leading terms (and the first corrections) for the eigen-modes $u = u_{mn}$ of acoustic pressure in the vessel,

$$u_{mn}(s,r,\varphi;\gamma_{mn}) \sim e^{i\{\sqrt{1+\kappa_m^2}\gamma_{mn}s\}} v_m(\gamma_{mn}r,\varphi) \exp\Big(-i\frac{V_m}{2\sqrt{1+\kappa_m^2}}\int_0^s k(\tau)d\tau\Big),$$

where $-\kappa_m^2, v_m$ are the mth eigenvalue and eigenfunction of an auxiliary problem [5] in a wedge W_* which is an orthogonal cross-section W_* of the vessel near the edge-line L. V_m is a constant, s,r,φ are the coordinates near the "shoreline" L, $\vec{r}(s,r,\varphi) = \vec{r}_0(s) + \vec{n}(s)r\cos[\varphi - \Phi] + \vec{\mathcal{N}}r\sin[\varphi - \Phi]$. The "quantization" conditions

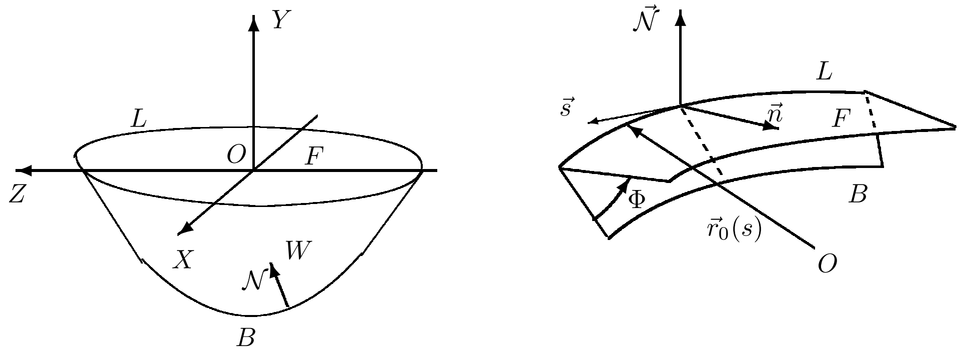

Figure 1. *An acoustically filled in vessel W with the "shoreline" L, the covering membrane F and the rigid bottom B, left. The coordinate system near the edge-line of the vessel, right.*

for the large parameter $\gamma = \gamma_{mn}$ reads,

$$l\sqrt{1+\kappa_m^2}\gamma_{mn} = 2\pi\left(\frac{V_m}{2\sqrt{1+\kappa_m^2}} + n\right), \qquad n \gg 1,\ n \in \mathbb{Z},\ m = 1, 2 \ldots, N_\Phi.$$

The obtained formal asymptotic expressions describe the desired series of the localised asymptotic eigen-modes near the edge-line of an acoustic "drum" with covering by a thin elastic membrane. The properties of the acoustic medium and of the membrane are assumed to be uniform, however, with obvious modifications the analogous results are also valid for the non-uniform case.

References

[1] S. Yu. Dobrohotov, *Asymptotic behaviour of water surface waves trapped by shores and irregularities of the bottom relief*, Soviet Phys. Dokl., 31 (1986), 537–539.

[2] S. Yu. Dobrohotov, P. Zhevandrov and K. Simonov, *Stokes edge waves in closed aqua-basins*, in Theoretical and Experimental Studies of the Long-Wave Processes, Far East Scientific Center of RAS USSR, Vladivostok, 1985, 13–19 (in Russian).

[3] D. Huntley, R. Guza and E. Thornton, *Field observations of surf beat: 1. Progressive edge waves*, J. of Geophys. Research, 86 (1981), 6451–6466.

[4] Y. Yeh, *Experimental study of standing edge waves*, J. Fluid Mech., 169 (1986), 291–304.

[5] M. A. Lyalinov *Localized waves propagating along an angular junction of two thin semi-infinite elastic membranes terminating an acoustic medium*, Russ. J. Math. Phys., 30 (2023), 345–359.

Recurrence relations with gaps of length $2n$ for Bernoulli and Euler polynomials

Wed, Jun 26
10:00–10:45

Karakhan Mirzoev

Moscow State University

Let $B_n(z)$ and $E_n(z)$ be Bernoulli and Euler polynomials and let $n \geq 2$ and $\varepsilon = e^{i\pi/n}$. The cardinality of the set of all possible mappings of the set $\{1, 2, \ldots, n-1\}$ onto the set $\{0, 1\}$ is equal to 2^{n-1}. We number the elements of this set by symbols m_s and define the numbers a_s, putting

$$a_s = \sum_{j=1}^{n-1} (-1)^{m_s(j)} \varepsilon^j, \qquad s = 1, 2, \ldots, 2^{n-1}.$$

In this report, using the methods of the spectral theory of ordinary differential operators, we obtain recurrence relations with gaps of length $2n$ for Bernoulli and Euler polynomials in terms of the numbers a_s.

The main results are the following theorems.

Theorem. *For $n = 1, 2, \ldots$ and $m = 0, 1, \ldots$ the following equalities are true:*

$$\sum_{k=0}^{[m/(2n)]} 2^{m-2nk} C_{m+n}^{2nk+n} B_{m-2nk}(z) \sum_{s=1}^{2^{n-1}} (-1)^{\nu(s)} (1 + a_s)^{2nk+n-1}$$
$$= \left(1 + \frac{m}{n}\right) \sum_{s=1}^{2^{n-1}} (-1)^{\nu(s)} (2z - 1 + a_s)^{m+n-1},$$

where $[a]$ is the integer part of the number a and $\nu(s) = 0$ if the number of terms with a plus sign in the number a_s is even, and $\nu(s) = 1$ otherwise for $n \geq 2$, and, in addition, $\nu(1) = 0$ for $n = 1$.

Theorem. *For $n = 1, 2, \ldots$ and $m = 0, 1, \ldots$ the following equalities are true:*

$$2^{m+n-1} E_m(z) + n \sum_{k=1}^{[m/(2n)]} 2^{m-2nk} C_m^{2nk} E_{m-2nk}(z) \sum_{s=1}^{2^{n-1}} (1 + a_s)^{2nk-1}$$
$$= \sum_{s=1}^{2^{n-1}} (2z - 1 + a_s)^m.$$

Part of the work is devoted to clarifying the nature of the coefficients

$$\sum_{s=1}^{2^{n-1}} (-1)^{\nu(s)} (1 + a_s)^{2nk+n-1} \qquad \sum_{s=1}^{2^{n-1}} (1 + a_s)^{2nk-1}$$

from the identities given in the formulations of Theorems 1 and 2. In particular, it is shown that the sums of k-fold series can be expressed through them and the number π^{2nk}

$$\sum_{j_1,j_2,\dots,j_k=1,\,j_1<j_2<\dots<j_k}^{+\infty} \prod_{l=1}^{k} \frac{1}{(2j_l)^{2n}}, \qquad \sum_{j_1,j_2,\dots,j_k=1,\,j_1<j_2<\dots<j_k}^{+\infty} \prod_{l=1}^{k} \frac{1}{(2j_l-1)^{2n}}.$$

The Bernoulli and Euler numbers are defined as the values of the corresponding polynomials at the points $z = 0$ and $z = 1/2$. In addition, the values of the polynomials $B_n(z)$ and $E_n(z)$ at the points $z = 1$, $z = 1/2$, $z = 1/3$, $z = 1/4$ and $z = 1/6$ are also expressed in terms of these numbers. Thus, Theorems 1 and 2 immediately imply the well-known (see [1], [2] and [3]) and some new recurrence relations with gaps of length $2n$ for them.

This work was supported by RSF under grant 20-11-20261.

References

[1] S. Ramanujan, *Some properties of Bernoulli's numbers*, J. Indian Math. Soc., 3 (1911), 219–234.

[2] D. H. Lehmer, *Lacunary recurrence formulas for the numbers of Bernoulli and Euler*, Ann. of Math., 36 (1935), 637–649.

[3] S. Wagstaff, *Ramanujan's paper on Bernoulli numbers*, J. Indian Math. Soc., 45 (1981), 49–65.

Fri, Jun 28
15:30–15:55

Homogenization of a multidimensional periodic elliptic operator at the edge of a spectral gap: Operator estimates in the energy norm

Arseniy Mishulovich

Saint Petersburg State University

In $L_2(\mathbb{R}^d)$, we consider a second-order elliptic differential operator

$$\mathcal{A}_\varepsilon = \mathbf{D}^* g(\boldsymbol{x}/\varepsilon)\mathbf{D} + \varepsilon^{-2} p(\boldsymbol{x}/\varepsilon), \quad \varepsilon > 0, \quad \boldsymbol{x} \in \mathbb{R}^d, \quad \mathbf{D} = -i\nabla,$$

with periodic coefficients. For small ε, we study the behavior of the resolvent of $\mathcal{A}_\varepsilon$ in a regular point close to the edge of a spectral gap. We will discuss the results on approximation of this resolvent in the "energy" norm (i.e., the norm of operators acting from $L_2(\mathbb{R}^d)$ to the Sobolev space $H^1(\mathbb{R}^d)$) with error $O(\varepsilon)$.

In [1], for $d = 1$ the approximation of the resolvent was obtained. It turned out that the methods employed in this work are not suitable for the multidimensional case. So, to achieve our goal, the contour integration method [2, Sec. 4.2, the third method] was used.

The work is supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075–15–2022–287).

References

[1] A. A. Mishulovich, V. A. Sloushch and T. A. Suslina, *Homogenization of a one-dimensional periodic elliptic operator at the edge of a spectral gap: operator estimates in the energy norm*, Zap. Nauchn. Sem. POMI, 519 (2022), 114–151.

[2] A. L. Piatnitski, V. A. Sloushch, T. A. Suslina and E. A. Zhizhina, *On operator estimates in homogenization of nonlocal operators of convolution type*, J. Differential Equations, 352 (2023), 153–188.

Inverse resonance problems for energy-dependent potentials on the half-line

Fri, Jun 28
12:50–13:15

Dmitrii Mokeev

HSE University – Saint Petersburg

We consider a Schrödinger equation on the half-line

$$-y''(x) + V(x,k)y(x) = k^2 y(x), \qquad (x,k) \in \mathbb{R}_+ \times \mathbb{C}, \tag{1}$$

where $k \mapsto V(\cdot, k)$ is a linear function

$$V(x,k) \coloneqq q(x) + 2kp(x)$$

which coefficients q, p have compact support in $\mathbb{R}_+$. When suitable boundary conditions are assigned at the origin, (1) corresponds to an eigenvalue/resonance problem for an energy-dependent perturbation of the Laplacian. We consider a specific class of energy-dependent Schrödinger equations without eigenvalues, defined with Miura potentials and boundary conditions at the origin. We solve the inverse resonance problem in this case and describe sets of isoresonance potentials and boundary condition parameters.

Our strategy consists in exploiting the solution of the inverse resonance problem for Dirac operators from [1] and a correspondence between energy-dependent Schrödinger and Dirac equations on the half-line, which was obtained in [2].

As a byproduct, we describe sets of isoresonance potentials and boundary condition parameters for Dirac operators and show that the scattering problem for Schrödinger equation or Dirac operator with an arbitrary boundary condition can be reduced to the scattering problem with the Dirichlet boundary condition.

The talk is based on joint work with Evgeny Korotyaev and Andrea Mantile [3].
E. K. and D. M. are supported by RSF Grant 19-71-30002.

References

[1] E. Korotyaev and D. Mokeev, *Inverse resonance scattering for Dirac operators on the half-line*, Anal. Math. Phys., 11 (2021).

[2] R. O. Hryniv and S. S. Manko, *Inverse scattering on the half-line for energy-dependent Schrödinger equations*, Inverse Problems, 36 (2020).

[3] E. Korotyaev, A. Mantile and D. Mokeev, *Inverse resonance problems for energy-dependent potentials on the half-line*, SIAM J. Math. Anal., 56 (2024).

Wed, Jun 26
15:50–16:35

On asymptotic solutions of the nonlinear shallow water equations in a basin with a sloping beach

Vladimir Nazaikinskii

Ishlinsky Institute for Problems in Mechanics RAS

Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary $\partial\Omega$, and let $D(x)$ be a smooth function defined in a neighborhood of the closed domain $\overline{\Omega}$ and satisfying the following conditions: (i) $D(x) > 0$ in Ω; (ii) $D(x) < 0$ outside Ω; (iii) $D(x) = 0$ and $\nabla D(x) \neq 0$ on $\partial\Omega$. Consider the Cauchy problem for the nonlinear shallow water equations in a basin described by the depth function $D(x)$:

$$\begin{cases} \eta_t + \langle \nabla, (D(x)+\eta)u \rangle = 0 & \text{in } \overline{\Omega}_T, \\ u_t + \langle u, \nabla \rangle u + g\nabla\eta = 0 & \text{in } \overline{\Omega}_T, \\ \eta_{t=0} = \eta^{(0)}(x), \quad u\big|_{t=0} = u^{(0)}(x), & \end{cases} \tag{$*$}$$

where $\overline{\Omega}_T = \{(x,t) \mid t \in [0,T],\ D(x) + \eta(x,t) \geqslant 0\}$, the unknown functions $\eta(x,t)$ (the free surface elevation) and $u(x,t) = {}^t(u_1(x,t), u_2(x,t)$ (the horizontal flow velocity) belong to $C^\infty(\overline{\Omega}_T)$, and the initial conditions $\eta^{(0)}(x)$ and $u^{(0)}(x)$ lie in $C_0^\infty(\Omega)$.

Our aim is to construct a formal asymptotic solution of problem $(*)$ treating the nonlinear terms as a small perturbation of the linear equations. A straightforward construction of regular perturbation theory is not possible here, because the domain is solution-dependent. An additional difficulty is that the limit linear operator (which can be reduced to the scalar second-order wave operator $\partial_{tt} - \langle \nabla, gD(x)\nabla)$ is degenerate.

In the talk, I will give a definition of formal asymptotic solution, introduce a solution-dependent change of variables (which is actually a simplification and generalization of the Carrier–Greenspan transform) that effectively "stops" the domain (i.e., the new domain is $\Omega \times [0, T]$), study the limit linear operator, and establish its properties permitting one to use regular perturbation theory in the new variables. A comparison with experimental results will be presented.

The talk presents the results in the joint papers [1]–[3], supported by RSF grants no. 21-11-00341 and no. 21-71-30011.

References

[1] S. Yu. Dobrokhotov, D. S. Minenkov and V. E Nazaikinskii, *Asymptotic solutions of the Cauchy problem for the nonlinear shallow water equations in a basin with a gently sloping beach*, Russ. J. Math. Phys., 29 (2022), 28–36.

[2] V. E. Nazaikinskii, *On an elliptic operator degenerating on the boundary*, Funct. Anal. Appl., 56 (2022), 324–326.

[3] S. Yu. Dobrokhotov, V. A. Kalinichenko, D. S. Minenkov and V. E. Nazaikinskii, *Asymptotics of long standing waves in one-dimensional pools with shallow banks: Theory and experiment*, Fluid Dyn., 58 (2023), 1213–1226.

Minimizers of the fractional Hardy–Sobolev inequality, and their nondegeneracy

Mon, Jun 24
12:00–12:45

Alexander I. Nazarov

Saint Petersburg Department of Steklov Institute of Mathematics RAS & Saint Petersburg State University

We consider ground state solutions to the equation generated by the fractional Hardy–Sobolev inequality. We prove their nondegeneracy and investigate some perturbed equations.

This talk is based on the joint paper with R. Musina [1].

References

[1] R. Musina and A. I. Nazarov, *Complete classification and nondegeneracy of minimizers for the fractional Hardy–Sobolev inequality, and applications*, J. Differential Equations, 280 (2021), 292–314.

Thu, Jun 27
10:50–11:35

Singularities of the Painlevé monodromy manifolds

Victor Novokshenov

Institute of Mathematics, Ufa Federal Research Center RAS

The nonlinear Painlevé differential equation on function $u(z)$ is integrated using the isomonodromic deformation method. It provides an auxiliary linear system of two matrix ODEs of first order in the variable $\lambda \in \mathbb{C}$ such that their deformation with respect to z and $u(z)$ preserves monodromy while circling singular points. Sometimes the only singular point is infinity, then the Stokes are chosen as monodromy data. Thus, the monodromy data play the role of integrals of motion or conservation laws for the corresponding Painlevé equation.

The set of monodromy data forms a *monodromy manifold* of complex dimension 2, since all second-order Painlevé equations and their solutions $u(z)$ depend on two arbitrary constants. There is a one-to-one correspondence between the points of the monodromy manifold and solutions of the Painlevé equation. So the structure of monodromy manifold describes the properties of solutions $u(z)$.

Painlevé equations of the first and second kind

$$u'' = 6u^2 - z, \tag{P_1}$$

$$u'' = zu - u^3, \tag{P_2}$$

have the monodromy manifolds of the form

$$\mathcal{P}_1 = \Big\{(s_1, s_2, s_3) \in \mathbb{C}^3 \;\Big|\; F = s_1 s_2 s_3 + s_1 - s_2 + 1,\ F = 0\Big\}, \tag{1}$$

$$\mathcal{P}_2 = \Big((s_1, s_2, s_3) \in \mathbb{C}^3 \;\Big|\; F = s_1 s_2 s_3 + s_1 - s_2 + s_3,\ F = 0\Big). \tag{2}$$

For other Painlevé equations, the monodromy manifold is also realized as an affine cubic Klein surface in $\mathbb{C}^3$. They also come from other areas of mathematics and physics. In particular, Sklyanin algebras arising from the solution of the Yang-Baxter equations are parameterized by the manifold $\mathcal{P}_2$ [2].

We relate the geometry of monodromy manifolds to the distribution of singularities of solutions to the Painlevé equations. It is well known that all solutions of the equations (P_1) and (P_2) are meromorphic functions in the complex plane z. The distribution of their poles was studied in the early works of P. Boutroux [1], where classes of so-called *truncated solutions* (intégrale tronquée) were found. They correspond to the absence of poles at infinity in critical sectors bounded by rays $\arg z = \pi k/6$, $k = 1, \ldots, 6$. A solution is called k-truncated if it has no poles on k critical rays.

The singularities of manifolds (1) and (2) are determined by the equation $dF = 0$. It turns out that these submanifolds of singularities parametrize truncated solutions of the Painlevé equations [3].

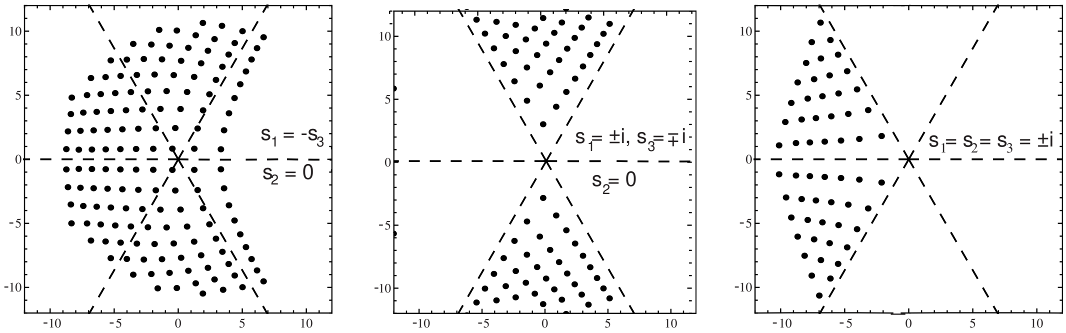

Figure 1. *Poles of truncated solutions to* (P_2)*,* 1*-truncated* (*left*)*,* 2*-truncated* (*center*) *and* 3*-truncated* (*right*) *and their monodromy data. The borders of critical sectors are shown in dashed lines.*

Theorem. *All 1-truncated solutions of the equations* (P_1) *and* (P_2) *correspond to one-dimensional singularity submanifolds* $\mathcal{P}_1$ *and* $\mathcal{P}_2$*.* 2*- and* 3*-truncated solutions correspond to zero-dimensional submanifolds of singularities.*

References

[1] P. Boutroux, *Recherches sur les transcendentes de M. Painlevé et l'étude asymptotique des équations différentielles du seconde ordre*, Ann. École Norm., 30 (1913). 265–375; Ann.École Norm., 31 (1914) 99–159.

[2] M. Mazzocco, L. Chekhov and V. Rubtsov, *Painlevé Monodromy manifolds, decorated character varieties, and cluster algebras*, Int. Math. Res. Not. IMRN, 2017 (2017), no. 24, 7639–7691.

[3] V. Yu. Novokshenov, *Special sotutions of the first and second Painlevé equations and singularities of the monodromy data*, Proc. Steklov Inst. Math., 281 (Suppl. 1) (2013), 105–117.

Homogenization of spectral problem for periodic high contrast convolution-type operators

Tue, Jun 25
10:50–11:35

Andrey Piatnitski

Institute for Information Transmission Problems RAS & UiT The Arctic University of Norway

The talk will focus on homogenization of spectral problems for high contrast convolution-type operators with integrable kernels in the media with a periodic microstructure. Our goal is to show that the spectrum of the original operator converges in the Hausdorff sense, to describe the corresponding limit set and to compare it with the spectrum of the homogenized operator.

Mon, Jun 24
15:50–16:35

On sharp estimates of functions derivatives in the Sobolev spaces with uniform norm

Igor Sheipak

Moscow State University

We consider the Sobolev space $\mathring{W}_p^n[0;1]$, consisting of real-valued functions y having absolutely continuous derivatives up to the order $n-1$ with $y^{(n)} \in L_p[0;1]$ $(1 \le p \le \infty)$ and the boundary conditions $y^{(j)}(0) = y^{(j)}(1) = 0$ $(j = 0, 1, \ldots, n-1)$.

For an arbitrary point $a \in (0;1)$, we investigate the functions $A_{n,k,p}(a)$ that are the best possible values in the inequalities

$$y^{(k)}(a) \le A_{n,k,p}(a)\|y^{(n)}\|_{L_p[0;1]}, \qquad y \in \mathring{W}_p^n[0;1], \qquad k = 0, 1, \ldots, n-1.$$

Our goal is also to obtain the exact constant in the embedding of the space $\mathring{W}_p^n[0;1]$ into the space $\mathring{W}_\infty^k[0;1]$:

$$\Lambda_{n,k,\infty} \coloneqq \max_{a\in[0;1]} A_{n,k,p}(a).$$

Define $\mathcal{P}_m$ to be the space of real polynomials

$$\mathcal{P}_m = \left\{ \sum_{j=0}^{m} c_j x^j, \quad x, c_j \in \mathbb{R}, 0 \le j \le m \right\}$$

of degrees not higher than m. Also consider splines of the form

$$S_{n,k,a}(x) \coloneqq \begin{cases} \dfrac{(x-a)^{n-k-1}}{(n-k-1)!}, & x \in [0;a], \\ 0, & x \in (a;1]. \end{cases}$$

We now state our main result.

Theorem. *For the values $A_{n,k,p}(a)$ the equality*

$$A_{n,k,p}(a) = \min_{u\in P_{n-1}} \|S_{n,k,a} - u\|_{L_{p'}[0;1]}$$

is valid.

Note that the case of $p = 2$ is studied in [1] for all $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n-1$. Next, we are interested in the case where $p = \infty$. Based on the results on the best approximation by polynomials of a characteristic function in $L_1[0;1]$ (see [2]), we prove the following theorem.

Theorem. *If $k = n - 1$ is even, then the point $a = 1/2$ is the point of the global maximum of the function $A_{n,n-1,\infty}$. If $k = n - 1$ is odd, then the point $a = 1/2$ is the point of a local minimum of the function $A_{n,n-1,\infty}$.*

Theorem. *We have that*

$$A_{n,0,\infty}(a) = \frac{V_n(2a-1)}{2^n}, \qquad a \in [0;1],$$

where

$$V_n(t) = \frac{2\sqrt{2}(n+1)}{\pi n!} \left(\frac{1-t^2}{2}\right)^{n+1/2} \int_{-1}^{1} \frac{(1-x^2)^n}{1-(t+ix\sqrt{1-t^2})^{2(n+1)}} dx$$

is the Peano kernel.

References

[1] T. A. Garmanova and I. A. Sheipak, *On Sharp estimates of even-order derivatives in Sobolev spaces*, Funct. Anal. Appl., 55 (2021), 34–44.

[2] M. V. Deikalova, *Integral approximation of the characteristic function of a spherical cap by algebraic polynomials*, Proc. Steklov Inst. Math., 273 (Suppl. 1) (2011), 74–85.

[3] H. Fiedler and W. B. Jukart, *Best L_1-approximation by polinomials*, J. Approx. Theory, 37 (1983), 260–292.

On the unconditional basis property for some general operators and operators generated by systems of O D E

Mon, Jun 24
10:00–10:45

Andrei A. Shkalikov

Moscow State University

We present in the talk some results on preserving the Riesz basis property under non-self-adjoint perturbations of normal operators. Similar problems will be raised up for operators generated by systems of ordinary differential equations.

Wed, Jun 26
12:00–12:45

Spectral multiplicity of Schrödinger operators on a star-graph with non-Kirchhoff interface conditions

Sergey Simonov

Saint Petersburg Department of Steklov Institute of Mathematics RAS

We consider a star-graph with a finite number n of finite or infinite edges and a Schrödinger operator on it with a self-adjoint interface condition at the inner vertex of the form $Au(0) + Bu'(0) = 0$ (where $u(0) = (u_1(0), ..., u_n(0))^t$, $u'(0) = (u'_1(0), ..., u'_n(0))^t$, A and B are $n \times n$ matrices such that $AB^* = BA^*$ and $\mathrm{rank}(A, B) = n$). We study the local multiplicity of the singular spectrum in terms of the spectral data of n decoupled Dirichlet operators on edges.

This setting is a particular case of a more abstract problem which is described in terms of boundary triples and concerns pasting of self-adjoint operators with simple spectra.

The talk is based on a joint work with Harald Woracek.

Mon, Jun 24
10:50–11:35

On a spectrum of the fourth-order differential operator with integral conditions

Alexander L. Skubachevskii

RUDN University

We consider an ordinary fourth-order differential operator

$$Au + \lambda^4 u = a_0(t)u^{(4)}(t) + \sum_{i=1}^{4} a_i(t)u^{(4-i)}(t) + \lambda^4 u = f_0(t) \qquad (t \in (0,1))$$

with integral conditions

$$B_{\rho k}u = \int_0^1 \Big(g_{\rho,k-1}(t)u^{(k-1)}(t) + h_{\rho k}(t)u^{(k)}(t)\Big)\,dt = f_{\rho k} \qquad (\rho = 1,2,\ k = 1,2).$$

Here a_i $(i = 0,\dots,4)$ are real-valued functions, $a_0(t) \geq m > 0$ $(0 \leq t \leq 1)$ and $a_1, a_2, a_3, a_4 \in C[0,1]$, $f_0 \in L_2(0,1)$ is a complex-valued function, $f_{\rho k} \in \mathbb{C}$ $(\rho = 1,2,\ k = 1,2)$ are constants, $\lambda \in \mathbb{C}$ is a spectral parameter; $g_{\rho,k-1}, h_{\rho k}$ $(\rho = 1,2,\ k = 1,2)$ are linearly independent real-valued functions.

We introduce determinants depending on the values of the weight functions at the points 0 and 1:

$$\Delta_h^1 = \begin{vmatrix} h_{11}(0) & h_{11}(1) \\ h_{21}(0) & h_{21}(1) \end{vmatrix}, \qquad \Delta_h^2 = \begin{vmatrix} h_{12}(0) & h_{12}(1) \\ h_{22}(0) & h_{22}(1) \end{vmatrix}.$$

In the Sobolev space $W_2^4(0,1)$ and in the space $\mathcal{W}[0,1] = L_2(0,1)\times\mathbb{C}^4$ we introduce the equivalent norms depending on a spectral parameter:

$$|||u|||_{W_2^4(0,1)} = \left(\|u\|^2_{W_2^4(0,1)} + |\lambda|^8\|u\|^2_{L_2(0,1)}\right)^{1/2},$$

$$|||f|||_{\mathcal{W}[0,1]} = \left(\|f_0\|^2_{L_2(0,1)} + |\lambda|^7(|f_{11}|^2 + |f_{21}|^2) + |\lambda|^5(|f_{12}|^2 + |f_{22}|^2)\right)^{1/2},$$

where $f = (f_0, f_{11}, f_{21}, f_{12}, f_{22})$, $\quad |\lambda| \geq 1$.

We set $\omega_\varepsilon = \{\gamma \in \mathbb{C} : |\arg\gamma| \leq \varepsilon\}$, $\quad \omega_{\varepsilon,q} = \{\gamma \in \omega_\varepsilon : |\gamma| \geq q\}$, where $\varepsilon > 0$.

Let $C^\alpha[a,b]$ be the Hölder space and $W^3_\infty(a,b)$ be the space of absolutely continuous functions $v(t), t \in [a,b]$, such that all derivatives up to the 3rd order belong to $L_\infty(a,b)$, i.e., $v^{(k)} \in L_\infty(a,b)$ $(k = 1,2,3)$.

Set

$$C^\alpha_\beta[0,1] = \{v \in L_2(0,1) : v \in C^\alpha[0,\beta],\ v \in C^\alpha[1-\beta,1]\},$$
$$W^3_{\infty,\beta}(0,1) = \left\{v \in C[0,1] : v \in W^3_\infty(0,\beta),\ v \in W^3_\infty(1-\beta,1)\right\},$$

where $0 < \beta < 1/2$.

We assume that $a_0 \in W^3_{\infty,\beta}(0,1)$ and $h_{\rho k} \in C^\alpha_\beta[0,1]$ $(\alpha \in (1/2,1])$, $g_{\rho,k-1} \in L_2(0,1)$.

In terms of equivalent norms in Sobolev space a priori estimates for solutions of the problem are obtained for sufficiently large values of the parameter λ.

Theorem. *Let $\Delta_h^1 \neq 0$ and $\Delta_h^2 \neq 0$. Then, for any $0 < \varepsilon < \pi/4$, there is a $q_0 > 1$ such that for $\lambda \in \omega_{\varepsilon,q_0}$ every $u \in W_2^4(0,1)$ satisfies the inequality*

$$|||u|||_{W_2^4(0,1)} \leq C|\lambda|^{1/2}|||f|||_{\mathcal{W}[0,1]},$$

where $C > 0$ does not depend on λ, u.

For a proof, see [1]. Using these estimates, sufficient conditions for the discreteness of the spectrum of the corresponding operator are obtained.

Joint work with R. D. Karamyan. This research is supported by the Ministry of Science and Higher Education of the Russian Federation (megagrant agreement no. 075-15-2022-1115).

References

[1] R. D, Karamyan, A. L. Skubachevskii, *Spectral properties of the fourth-order differential operator with integral conditions*, Lobachevskii J. Math., 45 (2024), no. 4, 1561–1577.

Mon, Jun 24
17:00–17:45

Hankel operators with band spectra

Alexander V. Sobolev

University College London

We consider the class of bounded self-adjoint Hankel operators $\mathbf{H}$, realised as integral operators on the positive semi-axis, that commute with dilations by a fixed factor. By analogy with the spectral theory of periodic Schrödinger operators, we develop a Floquet–Bloch decomposition for this class of Hankel operators $\mathbf{H}$, which represents $\mathbf{H}$ as a direct integral of certain compact fiber operators. As a consequence, $\mathbf{H}$ has a band spectrum. We establish main properties of the corresponding band functions, i.e. the eigenvalues of the fiber operators in the Floquet–Bloch decomposition. A striking feature of this model is that one may have flat bands that co-exist with non-flat bands; we consider some simple explicit examples of this nature. Furthermore, we prove that the analytic continuation of the secular determinant for the fiber operator is an elliptic function; this link to elliptic functions is our main tool.

This is a joint work with A. Pushnitski.

Fri, Jun 28
15:00–15:25

On operator estimates for elliptic equations in two-dimensional domains with frequently alternating boundary conditions on rapidly oscillating boundary

Radim Suleymanov

Ufa University of Science and Technology

We consider a second-order semilinear elliptic equation in an arbitrary two-dimensional domain with a rapidly oscillating boundary with a small amplitude of oscillations. The structure of oscillations is arbitrary and we suppose no periodicity or local periodicity in any form. On the oscillating boundary we impose frequent alternation of the Dirichlet and Neumann conditions. We consider the case, when the homogenization leads to the Dirichlet problem for the same equation. The main result is W_2^1- and L_2- operator estimates.

This is a joint work with D. I. Borisov (Institute of Mathematics, Ufa Federal Research Center RAS)

The work is supported by RSF (grant no. 23-11-00009‡).

On *PT*-symmetric Schrödinger operators

Thu, Jun 27
10:00–10:45

Iskander A. Taimanov

Sobolev Institute of Mathematics, Novosibirsk

We discuss PT-symmetric Schrödinger operators, from the point of view of the data of inverse spectral problems.

By definition, the potentials $u(x)$ of PT-symmetric operators satisfy the condition:

$$u(x) \to \overline{u(Px)},$$

where the operator P reverses the orientation and $P^2 = \mathrm{Id}$.

For example, in the one-dimensional case, the PT-symmetry of the potential means that

$$u(x) = \overline{u(-x)}.$$

In our joint article with R.G. Novikov [1] we showed that for rapidly decreasing (short range) potentials of N-dimensional Schrödinger operators with unitary S-matrices for $N > 1$ are real-valued. We remark that for $N = 1$ there are examples of complex-valued exponentially decreasing PT-potentials.

In [2] we gave a classification of algebraic-geometric (finite-zone) one-dimensional PT-symmetric Schrödinger operators in terms of theta-functional formulas for them.

References

[1] R. G. Novikov and I. A. Taimanov, *On unitarity of the scattering operator in non-Hermitian quantum mechanics*, Ann. Henri Poincare (2024) (published online).

[2] I. A. Taimanov, *Finite-zone PT-potentials*, arXiv:2404.11971.

On Molchanov's criterion for the compactness of the resolvent for the non-selfadjoint Sturm–Liouville operator

Wed, Jun 26
10:50–11:35

Sergey Tumanov

Moscow Center of Fundamental and Applied Mathematics at Moscow State University

Consider complex-valued $q \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ and the differential expression

$$l(y) = -y'' + qy \tag{1}$$

‡ https://rscf.ru/project/23-11-00009

and lineals

$$D = \Big\{y \in L_2(\mathbb{R}_+) \,\Big|\, y, y' \in AC_{loc}(\mathbb{R}_+),\ l(y) \in L_2(\mathbb{R}_+)\Big\},$$
$$D_0 = \Big\{y \in D \,\Big|\, y(0) = y'(0) = 0,\ \exists x_0 > 0\ \forall x \geq x_0\ y(x) = 0\Big\},$$
$$D_U = \Big\{y \in D \,\Big|\, U(y) = 0\Big\},$$

where U is some form of boundary conditions at $x = 0$:

$$U(y) = Ay(0) + By'(0), \qquad A, B \in \mathbb{C},\ |A| + |B| > 0.$$

Let us define differential operators $L_0 \subset L_U$ in $L_2(\mathbb{R}_+)$ on the corresponding domains $D_0 \subset D_U$ by the differential expression (1). Refer to L_0 as the minimal operator.

We say that q satisfies the Molchanov condition if for any $a > 0$

$$\lim_{x\to+\infty} \int\limits_x^{x+a} |q(\xi)|\, d\xi = +\infty.$$

Theorem. *In order for the minimal operator L_0 in $L_2(\mathbb{R}_+)$ to have an extension with a compact resolvent, it is necessary that q satisfies the Molchanov condition.*

We say that q satisfies the $\mathbb{R}_-$-condition if for all sufficiently large $x > x_0 \geq 0$ the values of $q(x)$ lie in the sector $\alpha \leq \arg(q(x) - q_0) \leq \beta$ for some $-\pi < \alpha \leq \beta < \pi$ and $q_0 \in \mathbb{C}$.

We call the potential q sectorial if q satisfies the $\mathbb{R}_-$-condition with $\beta - \alpha < \pi$.

Theorem. *Let the potential q be sectorial. Then the operator L_U has a compact resolvent if and only if q satisfies Molchanov's condition.*

The condition $\beta - \alpha < \pi$ cannot be weakened:

Theorem. *There exist potentials q taking imaginary values $q(x) \in i\mathbb{R}$ for all $x \in \mathbb{R}_+$ such that $|q| \to +\infty$ as $x \to +\infty$, but the minimal operator L_0 has no extensions with compact resolvents.*

In this case $\beta - \alpha = \pi$, and obviously such potentials satisfy the Molchanov condition. Moreover L_U with Dirichlet boundary conditions $U(y) = y(0)$ has a bounded non-compact resolvent, at least in the left half-plane.

The following theorem gives a sufficient condition for the compactness of the resolvent when q satisfies the $\mathbb{R}_-$-condition with $\beta - \alpha > \pi$. In this case, the sectoriality property of the operators themselves is lost; in particular, the numerical range of L_U may cover the entire complex plane.

Theorem. *Suppose that for some* $x_0 > 0$ *and all* $x \geq x_0 > 0$ *we have* $|q(x)| \geq 1$*, and additionally:*

- $q \in AC_{\mathrm{loc}}[x_0, +\infty)$,
- *for some* $0 < \varkappa < \pi$
$$-\pi + \varkappa < \arg q(x) < \pi - \varkappa, \qquad x \geq x_0,$$
- *for some* $0 < \delta < 1$
$$\left| \frac{q'(x)}{q^{3/2}(x)} \right| < 4\delta \sin \frac{\varkappa}{2}, \qquad x \geq x_0.$$

Then for the compactness of the resolvent L_U *it is sufficient that for any* $a > 0$

$$\lim_{x \to +\infty} \int\limits_{x}^{x+a} |q(x)|^{1/2} \, dx = +\infty.$$

Homogenization of nonlocal high-contrast elliptic type operators and Markov jump processes in high-contrast periodic media

Tue, Jun 25
15:00–15:45

Elena Zhizhina

Institute for Information Transmission Problems RAS & UiT The Arctic University of Norway

The talk will focus on homogenization problem for nonlocal convolution type elliptic operators in a periodic high-contrast environment. These operators can be considered as generators of Markov jump processes in periodic high-contrast media. For such processes we construct the related semigroup and prove that the limit dynamics remains Markovian, if we consider the jump process on an extended state space. We will also discuss in the talk how memory appears in the evolution of the spatial component of the limit process. We derive the memory kernel for this evolution equation and describe the spectrum of the limit operator. My talk is based on the recent joint work [1] with Andrei Piatnitski.

References

[1] A. Piatnitski and E Zhizhina, *High-contrast periodic random jumps in continuum and limit Markov process*, arXiv:2402.06769, 2024.

Mon, Jun 24
12:50–13:15

Diffraction of large-number whispering gallery mode by jump of curvature

Ekaterina Zlobina

Saint Petersburg State University

We develop a systematic boundary layer approach [1] in a problem of diffraction by non-smooth obstacle, aiming at construction of asymptotic formulas for wavefield. Diffraction of a high-frequency large-number whispering gallery mode is considered, which runs along the concave part of the surface to its straightening point, where the curvature of the surface undergoes a jump. The "ray skeleton" of the wavefield is investigated in detail. Basing on the Leontovich–Fock "parabolic equation method" [1, 2], we obtain asymptotics for all waves arising in the vicinity of the non-smoothness point of the surface, both inside and outside emerging boundary layers [3].

References

[1] V. M. Babich and N. Ya. Kirpichnikova, *The Boundary Layer Method in Diffraction Problems*, Springer, Berlin, 1979.

[2] V. A. Fock, *Electromagnetic Diffraction and Propagation Problems*, Pergamon Press, Oxford, 1965.

[3] E. A. Zlobina, *Diffraction of a large-number whispering gallery mode by a jump of curvature*, Zap. Nauchn. Sem. POMI, 521 (2023), 95–122.