International Conference
"Friends in Partial Differential Equations"
dedicated to the 90th birthday
of outstanding mathematician
Nina N. Uraltseva

St. Petersburg (online),
May 24-26, 2024

ABSTRACTS

Organizing Committee:
D.E. Apushkinskaya (*RUDN University, Moscow*)
A.I. Nazarov (*St. Petersburg Dept of the Steklov Institute and St. Petersburg University*)
N.A. Petrova (*L. Euler International Mathematical Institute, St. Petersburg*)

Schedule of the Meeting

Time*	24.05.2024, Friday
09.45–09.55	*Opening*
10.00–10.40	A. Arkhipova
10.45–11.25	H. Shagholian
11.30–12.10	E. Panov
12.10–14.30	*Lunch*
14.30–15.10	A. Skubachevskii
15.15–15.55	B. Sirakov
16.00–16.40	G. Mingione
16.40–17.00	*Coffee-break*
17.00–	*Virtual Birthday Party*

Time*	25.05.2024, Saturday
10.00–10.40	N. Filonov
10.45–11.25	R. Musina
11.30–12.10	V. Bobkov
12.15–14.25	*Lunch*
14.30–14.50	A. Poretskii
14.55–15.15	E. Zlobina
15.20–15.40	N. Rastegaev
15.40–16.00	*Coffee-break*
16.00–16.40	G. Weiss
16.45–17.25	N. Krylov

Time*	26.05.2024, Sunday
10.00–10.40	A. Piatnitski
10.45–11.25	K. Pileckas
11.30–12.10	D. Palagachev
12.10–14.30	*Lunch*
14.30–15.10	A. A. Laptev
15.15–15.55	G. Seregin
16.00–16.40	S. Tikhomirov
16.40–16.50	*Closing*

* This is St. Petersburg time (GMT+3).

Regularity results for strongly nonlinear elliptic and parabolic systems

ARINA A. ARKHIPOVA

St. Petersburg State University

We consider elliptic and parabolic systems of equations with strong (quadratic) nonlinearity in the gradient. We discuss the known partial regularity results and the author's results of the last years.

Assuming a one-side restriction on the strongly nonlinear term, we describe conditions that guarantee local smoothness of weak possibly unbounded solutions of quasilinear and nonlinear systems.

Payne nodal set conjecture for the fractional p-Laplacian in Steiner symmetric domains

VLADIMIR E. BOBKOV

Institute of Mathematics of Ufa Federal Research Centre of Russian Academy of Science

Let u be either a second eigenfunction of the fractional p-Laplacian or a least energy nodal solution of the equation $(-\Delta)_p^s u = f(u)$ with superhomogeneous and subcritical nonlinearity f, in a bounded open set Ω and under the nonlocal zero Dirichlet conditions. Assuming that Ω is Steiner symmetric, we show that the supports of positive and negative parts of u intersect $\partial\Omega$, and, consequently, the nodal set of u has the same property.

The proof involves the analysis of certain polarization inequalities related to positive and negative parts of u, and alternative characterizations of second eigenfunctions and least energy nodal solutions.

The talk is based on a joint work with S. Kolonitskii.

On the rate of decay at infinity of solutions to the Schrödinger equation in a cylinder domains

NIKOLAI D. FILONOV

St.Petersburg Department of the Steklov Mathematical Institute

We consider the equation

$$-\Delta u + Vu = 0$$

in the cylinder $\mathbb{R} \times (0, 2\pi)^d$ with periodic boundary conditions on the side surface. A potential V is assumed to be *real-valued* and bounded. We are interested in the possible rate of decay of a non-trivial solution u at infinity. Clearly, a solution can decrease exponentially. For $d = 1$ or $d = 2$, a solution can not decrease faster: if

$$u(x, y) = O\left(e^{-N|x|}\right) \quad \forall\ N,$$

then $u \equiv 0$. Here x is the axial variable. For $d \geq 3$, we construct an example of a non-trivial solution decreasing as $e^{-c|x|^{4/3}}$, and it is known that it is optimal,

$$u(x, y) = O\left(e^{-N|x|^{4/3}}\right) \quad \forall\ N \qquad \Longrightarrow \qquad u \equiv 0.$$

This is a joint work with S. Krymskii.

Once again on evolution equations with monotone operators in Hilbert spaces and applications

NIKOLAI KRYLOV

University of Minnesota

We prove the existence of W^1_2-solutions of uniformly nondegenerate parabolic equations

$$\partial_t u = D_i(a^{ij}_t D_j u_t + \beta^i_t u_t) + b^i_t D_i u_t + c_t u_t + f_t$$

in case that $f_\cdot \in (L_2 + L_1)([0,T], L_2(\mathbb{R}^d))$, $b = b^M + b^B$ and for some r satisfying $2 < r \leq d$ and sufficiently small constant $\hat{b}$

$$\Big(\fint_{B_\rho} |b^M_t|^r \, dx \Big)^{1/r} \leq \hat{b}\rho^{-1}, \quad \rho \leq \rho_0, \quad \int_0^T \sup_x |b^B_t|^2 \, dt < \infty.$$

Similar conditions are imposed on β and c, so that $|b_t| = |\beta_t| = \varepsilon/|x|$, $|c_t| = \varepsilon/|x|^2$ are allowed. Even the case of $b^M = 0$, $\beta = 0, c = 0$, $f \in L_1([0,T], L_2(\mathbb{R}^d))$ seems to be new. Functions $b \in L_q(L_p(\mathbb{R}^d))$ with $p > d, d/p + 2/q = 1$ are in the above described class.

Joint work with I. Gyöngy.

A sharp Lieb-Thirring inequality for functional difference operators

ARI LAPTEV

Imperial College London

We prove sharp Lieb-Thirring type inequalities for the eigenvalues of a class of one-dimensional functional difference operators associated to mirror curves.

We furthermore prove that the bottom of the essential spectrum of these operators is a resonance state.

Nonuniformly elliptic Schauder theory

GIUSEPPE MINGIONE

Università degli Studi di Parma

Schauder estimates are a basic tool in elliptic and parabolic PDE. The idea is to show that solutions are as regular as coefficients allow. They serve as a basic tool in a wide variety of situations: higher regularity of solutions to problems showing any kind of ellipticity, including free boundaries, bootstrap processes, existence theorems and so on.

Their validity in the setting of linear uniformly elliptic problems is classical. First results were obtained by Hopf, Giraud, Caccioppoli and Schauder in the 20/30s of the past century. Extensions were obtained by Agmon, Douglis and Nirenberg. New proofs were achieved over the years by Campanato, Trudinger, Simon (via suitable function spaces, convolution, blow-up, respectively). Nonlinear versions were achieved by Giaquinta and Giusti, DiBenedetto, Manfredi. More recently, nonlocal versions were obtained as well.

As the equations in question are non-differentiable, all these approaches unavoidably rely on perturbation methods, i.e., freezing coefficients and comparing original solutions to solutions with constant coefficients problems. Such approaches, relying on the availability of homogenous estimates for frozen problems, ceases to work in nonuniformly elliptic problems, for which such homogeneity is lost, and for which the validity of Schauder theory has remained an open problem for decades.

We shall present a full solution to the problem of Schauder estimates in the nonlinear, nonuniformly elliptic setting. In particular, we shall present the first direct, non-perturbative approach to pointwise gradient estimates for non-differentiable equations ever.

From recent, joint work with Cristiana De Filippis (University of Parma).

Hardy and Hardy-Sobolev type inequalities involving mixed weights

ROBERTA MUSINA

University of Udine

We report about class of dilation-invariant inequalities for functions having compact support in cones $\mathcal{C} \subseteq \mathbb{R}^{d-k} \times \mathbb{R}^k$. The leading term has the form

$$\int_{\mathcal{C}} \frac{|y|^a}{(|x|^2 + |y|^2)^b} \, |\nabla u|^p \, dxdy \, .$$

We include both classical spherical weights, initially examined by Il'in [Mat. Sb., 1961] and further discussed by Caffarelli-Kohn-Nirenberg [Compositio Math., 1984], as well as cylindrical weights, firstly investigated by Maz'ya in his monograph on Sobolev spaces.

References

1. G. Cora, R. Musina, A.I. Nazarov, *Hardy type inequalities with mixed weights in cones*, Ann. Scuola Normale Superiore, to appear.

2. R. Musina, A.I. Nazarov, *Hardy-Sobolev inequalities with general mixed weights*, in progress.

Quasilinear Venttsel BVPs with discontinuous data

DIAN K. PALAGACHEV

Politecnico di Bari

We will present some very recent results regarding the regularity and solvability theory of second-order quasilinear elliptic equations with discontinuous coefficients coupled with boundary conditions of Venttsel type, given in terms of second-order quasilinear operators with discontinuous data.

The results are obtained in collaboration with Darya Apushkinskaya (Moscow), Alexander Nazarov (St. Petersburg) and Lubomira Softova (Salerno).

On self-similar solutions of a multiphase Stefan problem on the half-line

EVGENY YU. PANOV

Yaroslav-the-Wise Novgorod State University

We study self-similar solutions of a multi-phase Stefan problem for a heat equation on the half-line $x > 0$ with a constant initial data and with Dirichlet or Neumann boundary conditions.

In the case of Dirichlet boundary condition we prove that a nonlinear algebraic system for determination of the free boundaries is gradient one and the corresponding potential is an explicitly written strictly convex and coercive function. Therefore, there exists a unique minimum point of the potential, coordinates of this point determine free boundaries and provide the desired solution.

In the case of Neumann boundary condition we demonstrate that the problem can have solutions with different numbers (called types) of phase transitions. For each fixed type n the system for determination of the free boundaries is again gradient and the corresponding potential is proved to be strictly convex and coercive, but in some wider non-physical domain. On the base of these properties we prove existence and uniqueness of a solution and provide precise conditions to specify the type of this solution.

On the spectrum of nonlocal convolution operators with potential

ANDREY PIATNITSKI

Institute for Information Transmission Problems of the Russian Academy of Sciences (Kharkevich Institute) and The Arctic University of Norway

The talk will focus on the spectrum of a self-adjoint convolution operator with an integrable kernel that is perturbed by an essentially bounded real valued potential, The potential tends to zero at infinity. We show that the essential spectrum of such an operator is the union of the spectrum of the convolution operator and of the essential range of the potential. Then we provide several sufficient conditions for the existence of a countable sequence of discrete eigenvalues. For operators having non-connected essential spectrum we give sufficient conditions for the existence of discrete eigenvalues in the corresponding spectral gaps.

This is a joint work with D. Borisov and E. Zhizhina.

Non-stationary Navier-Stokes equations in 2D power-cusp domain

KONSTANTIN PILECKAS

Vilniaus Universitetas

The initial boundary value problem for the non-stationary Navier-Stokes equations is studied in 2D bounded domain with a power cusp singular point O on the boundary. The case of the boundary value with a nonzero flow rate is considered. In this case there is a source/sink in O and the solution necessary has infinite energy integral.

To find a solution, we first construct the formal asymptotic expansion (U^J, P^J) of it near the singular point, and then we find a solution in the form $u = \zeta U^J + v$, where ζ is cutoff function and v has finite dissipation of energy.

Mathematical scattering theory in electromagnetic waveguides

ALEXANDER PORETSKII

St. Petersburg State University

Waveguide occupies a 3D domain G having several cylindrical outlets to infinity and is described by the non-stationary Maxwell system with conductive boundary conditions. Dielectric permittivity and magnetic permeability are assumed to be positive definite matrices $\varepsilon(x)$ and $\mu(x)$ depending on a point x in G. At infinity, in each cylindrical outlet, the matrix-valued functions converge with an exponential rate to matrix-valued functions that do not depend on the axial coordinate of the cylinder.

For the corresponding stationary problem with spectral parameter we define continuous spectrum eigenfunctions and the scattering matrix. The non-stationary Maxwell system is extended up to an equation of the form $i\partial_t \mathcal{U}(x,t) = \mathcal{A}(x, D_x)\mathcal{U}(x,t)$ with an elliptic operator $\mathcal{A}(x, D_x)$. We associate with the equation a boundary value problem and, for an appropriate couple of such problems, construct the scattering theory. We calculate the wave operators, define the scattering operator and describe its relation to the scattering matrix. From the obtained results we extract information about the original Maxwell system.

Kružkov-type uniqueness theorem for the chemical flood conservation law system

NIKITA V. RASTEGAEV

St.Petersburg Department of the Steklov Mathematical Institute

We consider the system

$$\begin{cases} s_t + f(s,c)_x = 0, \\ (cs + a(c))_t + (cf(s,c))_x = 0, \end{cases}$$

most commonly used to describe the flood of the oil reservoir with a chemical solution. The flow function f is commonly assumed to be S-shaped in s. The adsorption function a is often concave and usually represented by the Langmuir adsorption isotherm. In our work, we limit ourselves to functions f monotone in c.

This system is neither strictly hyperbolic nor genuinely non-linear, therefore known results for strictly hyperbolic genuinely non-linear systems of conservation laws are not directly applicable.

The solutions for some boundary-initial problems for this system were explored, for example, in [1] (Riemann problem), [2] and [3] (slug injection). The last two papers use the Lagrange coordinate transformation to split the equations and the characteristics method to construct solutions. However, the question of the uniqueness of the constructed solutions is not covered.

We use the proposed coordinate change to prove a Kružkov-type uniqueness theorem for the Cauchy problem with several limitations on the initial data and the class of weak solutions under consideration. The vanishing viscosity method is utilized to determine admissible shocks.

This talk is based on the joint work with S. G. Matveenko.

References

1. T. Johansen, R. Winther, *The solution of the Riemann problem for a hyperbolic system of conservation laws modeling polymer flooding*, SIAM journal on mathematical analysis, **19**:3 (1988), 541–566.

2. A.P. Pires, P.G. Bedrikovetsky, and A.A. Shapiro, *A splitting technique for analytical modelling of two-phase multicomponent flow in porous media*, Journal of Petroleum Science and Engineering, **51**:(1-2) (2006), 54–67.

3. P.M. Ribeiro, A.P. Pires, The displacement of oil by polymer slugs considering adsorption effects // SPE Annual Technical Conference and Exhibition, 2008, September, pp. SPE-115272.

Remarks on potential singularities of solutions to certain elliptic systems of PDE's

GREGORY A. SEREGIN

St.Petersburg Department of the Steklov Mathematical Institute and Oxford University

We are going to consider points in space in which solutions to certain elliptic systems of PDE's do not satisfy the usual ε-regularity conditions.

Constraint Maps
(Overview & Recent Developments)

HENRIK SHAHGHOLIAN

Royal Institute of Technology

I will discuss, at a heuristic level, constraint maps that minimize Dirichlet energy, potentially with a forcing term. Specifically, I will focus on their optimal smoothness and the resulting free boundary. Although the study of these maps began in the 1970s and saw further development in the 1980s, it was later neglected. Our recent research has revived this theory, offering deeper insights and uncovering intriguing connections between solution singularities and free boundaries.

This is based on joint works with A. Figalli (ETH), A. Guerra (ETH), and S. Kim (Uppsala).

Uniform (or not) a priori estimates for the Lane-Emden system in the plane

BOYAN SIRAKOV

Pontifícia Universidade Católica do Rio de Janeiro

We prove that positive solutions of the superlinear Lane-Emden system in a two-dimensional smooth bounded domain are bounded independently of the exponents in the system, provided the exponents are comparable. As a consequence, the energy of the solutions is uniformly bounded, a crucial information in their asymptotic study.

On the other hand, the boundedness may fail if the exponents are not comparable, a surprising incidence of a situation in which the sub-critical Lane-Emden system behaves differently from the scalar Lane-Emden equation.

Joint work with Nikola Kamburov (PUC-Chile).

On classical solutions of Vlasov–Poisson system with external magnetic field

ALEXANDER L. SKUBACHEVSKII

RUDN University

We consider the second mixed problem for the Vlasov–Poisson system with external magnetic field in a half–space. This problem models the kinetics of high temperature plasma in a fusion reactor.

If plasma reaches a boundary of the domain, it can lead to the destruction of the reactor. Therefore we study solutions of the above mentioned mixed problem such that the supports of density distribution functions of charged particles are located strictly inside the domain. For this purpose we use an external magnetic field.

It will be obtained sufficient conditions for external magnetic field, which provide existence of global classical solutions having supports of density distribution functions inside the domain [1].

The work is sponsored by the Ministry of Science and Higher Education of the Russian Federation (Megagrant, No. 075-15-2022-1115).

References

1. A.L. Skubachevskii, *On the existence of global solutions for the Vlasov–Poisson system in a half-space and plasma confinement* , Lobachevskii Journal of Mathematics, **45**:2 (2024), 280–292.

Cascade of traveling waves in miscible displacement in porous media

SERGEY TIKHOMIROV

Pontifícia Universidade Católica do Rio de Janeiro

We study the motion of miscible liquids in porous media with the speed determined by Darcy's law. It has a long list of applications in the petroleum industry. The two basic examples are the displacement of viscous liquids and the motion induced by gravity. Such motion often is unstable and creates patterns called viscous fingers.

We concentrate on the important for applications property of viscous fingers - speed of their propagation. The work is inspired by the results of F. Otto and G. Menon for a simplified model, called transverse flow equilibrium (TFE). In this work a rigorous upper bound was proved using the comparison principle. At the same time numerical experiments suggest that the actual speeds are better than Otto-Menon estimates.

We consider a two-tubes model – the simplest model which contains fingering instability. For the case of gravitational fingers we were able to find families of traveling waves and their speeds – it is significantly different from the Otto-Menon estimates. The proof strongly relies on techniques from dynamical systems: heteroclinic orbits, singularly-perturbed systems, normal hyperbolicity, transversality.

This is a joint work with Yulia Petrova and Yalchin Efendiev.

Classification of global solutions to the obstacle problem in the plane

GEORG S. WEISS

Universität Duisburg-Essen

Global solutions to the obstacle problem were first completely classified in two dimensions by Sakai using complex analysis techniques. Although the complex analysis approach produced a very succinct proof in two dimensions, it left the higher dimensional cases, and even closely related problems in two dimensions, unresolved. A complete classification in dimensions $n \geq 3$ was recently given by Eberle, Figalli and Weiss, forty years after Sakai published his proof. In this paper we give a proof of Sakai's classification result for unbounded coincidence sets in the spirit of the recent proof by Eberle, Figalli and Weiss. Our approach, in particular, avoids the need for complex analysis techniques and offers new perspectives on two-dimensional problems that complex analysis cannot address.

This is a joint work with Anthony Salib.

High-frequency diffraction by a jump of curvature. Tangential incidence

EKATERINA A. ZLOBINA

St. Petersburg State University

We are concerned with the construction of asymptotic formulas within the framework of systematic boundary layer method [1]. We consider two-dimensional diffraction by the contour C composed of half-line C_- and part of smooth curve C_+ (see Fig. 1) with a jump in curvature at the conjugation point O. The plane wave e^{ikx} with large wavenumber $k \to \infty$ grazes along C_- to O, see Fig. 1. The outgoing wave u^{out} is governed by the Helmholtz equation and the Neumann boundary condition:

$$u^{\text{out}}_{xx} + u^{\text{out}}_{yy} + k^2 u^{\text{out}} = 0, \quad \partial_n \left(u^{\text{out}} + e^{ikx}\right)\big|_C = 0.$$

Here, ∂_n denotes the derivative along the normal to the contour C.

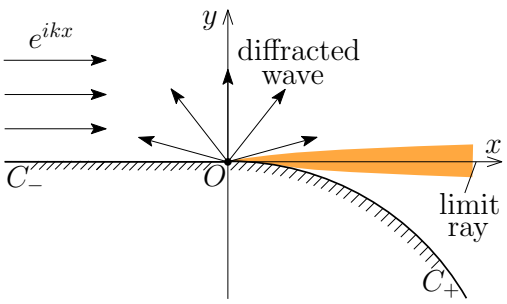

Fig. 1. The geometry of the problem

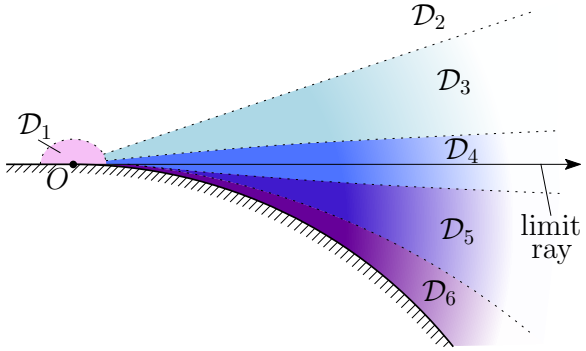

Fig. 2. The sketch of boundary layers

The problem has been previously addressed by A.V. Popov [2] who, using a heuristic approach, derived an expression for the diffracted cylindrical wave diverging from the jump point O (see Fig. 1).

Our aim is to develop a formal boundary layer approach and construct high-frequency approximations for the wavefield in the neighborhood of the limit ray (orange zone in Fig. 1). We base on the Leontovich—Fock parabolic equation method [1,3]. This allows an asymptotic description of the wavefield in boundary layers surrounding the limit ray (see Fig. 2), which involves novel special functions.

The talk is based on the joined work with A.P. Kiselev [4].

References

1. V.M. Babich, N.Ya. Kirpichnikova, *The Boundary Layer Method in Diffraction Problems*, Springer, Berlin, 1979.

2. A.V. Popov, *Backscattering from a line of jump of curvature*, in: Trudy V Vses. Sympos. Diffr. Raspr. Voln, Nauka, Leningrad, 1970, 171–175 (1971).

3. V.A. Fock, *Electromagnetic Diffraction and Propagation Problems*, Pergamon Press, Oxford, 1965.

4. E.A. Zlobina, A.P. Kiselev, *The Malyuzhinets—Popov diffraction problem revisited*, Wave Motion, **121** (2023), Article ID 103172.