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Book of Abstracts

On some properties of eigenvalues of the involutive Friedrichs model

Tue, Jun 24
17:00–17:25

Grigorii Agafonkin

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Consider the spectral problem for the operator

$$Af(x) = ixf(-x) + \int_{-1}^{1} K(x,y)f(y)\,dy$$

in $L_2[-1,1]$. It was shown in [1] that if K is Hölder continuous with exponent $\alpha > 1/2$ and vanishes on $\partial[-1,1]^2$ then the operator A can only have finite number of discrete eigenvalues.

In this talk, we suppose that the perturbation is finite-dimensional; that is,

$$K(x,y) = \tau \sum_{j=1}^{n} k_j(x)k_j(y).$$

We show that for small $\tau > 0$ the discrete spectrum of the operator is empty. Additionally, the problem of the eigenvalues lying on the essential spectrum $[-1,1]$ is studied.

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Wed, Jun 25
16:10–16:40

Short-wave solutions of the wave equation with localized velocity perturbations

Anna Allilueva

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To describe the propagation of short waves in media of various natures, asymptotic solutions of hyperbolic equations are often used that are expressed in terms of WKB-exponentials or, in the presence of focal points, in terms of the Maslov canonical operator on the corresponding Lagrangian manifolds (see, for example, [1, 2]). The small parameter in these asymptotics is the ratio of the wavelength to the characteristic scale of change in the characteristics of the medium. If the medium contains localized inhomogeneities (for example, narrow underwater ridges or pycnoclines in the ocean, layers with sharply changing optical or acoustic density, etc.), the coefficients of the equations turn out to depend on (generally speaking, another) small parameter, and the dependence is not regular: the weak limits of the coefficients have singularities at points corresponding to the specified inhomogeneities. Generally speaking, the standard theory of short-wave asymptotics does not work in such a situation; in particular, the geometric objects determining the solution are reconstructed. For the case in which the characteristic scale of the inhomogeneity is comparable to the wavelength (i. e., the corresponding small parameters are of the same order), asymptotic solutions of a number of hyperbolic problems are described in the papers [3]–[7]; in these papers, in particular, it is shown that the Lagrangian manifolds determining the asymptotics are separated at the points specifying the singularities of the coefficients into several connected components describing the transmitted and reflected waves. The amplitudes of these waves are determined by the one-dimensional scattering problem for the "model" equation, which does not contain a small parameter; thus, this auxiliary problem is not related to the theory of geometric asymptotics.

If the scale of the localized inhomogeneity is much larger than the wavelength, then the construction of the asymptotics changes: geometric objects must arise in the "model" problem as well. In this paper, we analyze the Cauchy problem for a one-dimensional wave equation, in which the velocity experiences a smoothed jump, and the characteristic wavelength of the initial perturbation is described by the small parameter ε, while the characteristic width of the jump is described by the parameter $\varepsilon^{1/n}$, where n is integer.

Thus, we consider the problem

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{x-x_0}{\varepsilon^{1/n}}, x\Big)\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \quad u|_{t=0} = \varphi^0(x)e^{iS_0(x)/\varepsilon}, \quad u_t|_{t=0} = 0, \quad \varepsilon \to 0. \tag{1}$$

Here the function $c(y,x)\colon \mathbb{R}^2 \to \mathbb{R}$ ($y = (x-x_0)/\varepsilon^{1/n}$) is smooth and strictly positive, and $c(y,x) \to c^{\pm}(x)$ as $y \to \pm\infty$ faster than any power of y together with all derivatives. The last condition reflects the localized nature of the inhomogeneity; we assume that the functions $c^{\pm}(x)$ are also smooth and positive. We assume that S_0 and φ^0 are smooth functions, and φ^0 is compactly supported, $\partial S_0/\partial x|_{\operatorname{supp}\varphi^0} \neq 0$, and the initial wave packet is outside the localized inhomogeneity, i. e., $x|_{\operatorname{supp}\varphi^0} < 0$. Our objective is to describe the scattering of a packet as it passes through the point x_0, which is the support of the inhomogeneity. The absence of focal points enables us to write out simple analytical formulas for the asymptotics, as well as to calculate the coefficient of wave transmission through the inhomogeneity. The reflected wave packet is absent in our case.

Equations with rapidly changing coefficients are the subject of many works (see, for example, [7, 8]); the introduction of the fast variable $y = (x - x_0)/\varepsilon^{1/n}$ transforms the right-hand side of the wave operator into an operator of the form

$$\frac{1}{\varepsilon^2} c^2(y,x)\Big(\varepsilon\frac{\partial}{\partial x} + \varepsilon^{(n-1)/n}\frac{\partial}{\partial y}\Big)^2,$$

whose leading ε-symbol (after the multiplication by ε^2) represents the operator $-c^2(x,y)(p - \frac{i}{\varepsilon^{1/m}}\partial/\partial y)^2$ The scheme of semiclassical asymptotics for equations with an operator-valued symbol has been applied many times to equations with rapidly varying coefficients (see, for example, [7, 8]); the key point of this scheme is the use of eigenvalues of the symbol as classical Hamiltonians. In our case, the symbol is an operator with a small parameter, and its spectrum contains a continuous component. There is no general semiclassical theory in such a situation; to describe the asymptotics, we use the considerations given in [10], as well as the technique developed for constructing soliton-like asymptotics of nonlinear equations (see, for example, [9]). The result of the work is an asymptotic series for the solution to the Cauchy problem (1).

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Mon, Jun 23
15:00–15:45

On geometric approximations of point interactions for general two-dimensional and three-dimensional operators

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Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be an arbitrary non-empty domain, bounded or unbounded, with a C^2-boundary. The case $\Omega = \mathbb{R}^n$ is possible. By $\omega \subset \mathbb{R}^n$ we denote a bounded simply-connected domain in $\mathbb{R}^n$ containing the origin and having C^3-boundary. Choosing a fixed point $x_0 \in \Omega$, denoting by ε a small positive parameter, we define a small cavity in Ω around x_0, namely, $\omega_\varepsilon := \{x \in \mathbb{R}^n : (x - x_0)\varepsilon^{-1} \in \omega\}$, and consider a perforated domain $\Omega_\varepsilon := \Omega \setminus \omega_\varepsilon$.

Let $\mathrm{A} = \mathrm{A}(x)$ be an $n \times n$ matrix function with complex-valued entries, $\mathrm{B} = \mathrm{B}(x)$ be an n-dimensional vector function with complex-valued components, and $V = V(x)$ be a complex-valued function. The matrix A, vector B, and function V are defined on $\bar{\Omega}$ and are smooth enough, namely, $\mathrm{A} \in W^1_\infty(\Omega)$, $\mathrm{B} \in L_\infty(\Omega)$, $A_0 \in L_\infty(\Omega)$. We suppose that the matrix A is Hermitian and uniformly positive definite, that is,

$$\mathrm{A}^*(x) = \mathrm{A}(x), \qquad (\mathrm{A}(x)z, z)_{\mathbb{C}^n} \geqslant c_0|z|^2, \qquad z \in \mathbb{C}^n, x \in \bar{\Omega},$$

where c_0 is a fixed positive constant independent of x and z. In addition we suppose that there exists a subdomain $\Omega_0 \subseteq \Omega$ such that $x_0 \in \Omega_0$ and $\mathrm{A} \in C^4(\bar{\Omega}_0)$, $\mathrm{B} \in C^3(\bar{\Omega}_0)$, $V \in C^2(\bar{\Omega}_0)$.

The main object of our study is a second order differential operators $\mathcal{H}_\varepsilon$ with the differential expression

$$\hat{\mathcal{H}} := -\operatorname{div} \mathrm{A}(x)\nabla + \mathrm{B}(x)\cdot\nabla + V(x)$$

in Ω_ε subject to boundary conditions

$$\mathcal{L}u = 0 \quad \text{on} \quad \partial\Omega, \qquad \frac{\partial u}{\partial \mathrm{n}} + \ell_\varepsilon u = 0 \quad \text{on} \quad \partial\omega_\varepsilon, \qquad \frac{\partial}{\partial \mathrm{n}} := \nu\cdot\mathrm{A}(x)\nabla.$$

Here $\mathcal{L}$ is an arbitrary boundary operator, ν is the unit normal on $\partial\omega_\varepsilon$ directed inside ω_ε. The main feature of the operator $\mathcal{H}_\varepsilon$ is the boundary operator ℓ_ε. In the case $n=2$ it is defined as

$$\begin{aligned}
&\ell_\varepsilon u := \varepsilon^{-1}\alpha\Big(\varepsilon^{-1}(\cdot - x_0), \ln^{-1}\varepsilon\Big)u + \varepsilon^{-1}\ln^{-1}\varepsilon\,\alpha_2^\varepsilon\Big(u - \langle\alpha_3^\varepsilon u\rangle_{\partial\omega_\varepsilon}\ln\varepsilon\Big),\\
&\alpha(x,\mu) := \alpha_0(x) + \mu\alpha_1(x), \qquad \langle u\rangle_{\partial\omega_\varepsilon} := \frac{1}{|\partial\omega_\varepsilon|}\int\limits_{\partial\omega_\varepsilon} u\,ds,\\
&\int\limits_{\partial\omega}\alpha_3(x)\,ds = 0, \qquad \int\limits_{\partial\omega}\alpha_3(x)\ln|\mathrm{A}_0^{-\frac{1}{2}}x|\,ds = 1, \qquad \alpha_2 = \alpha_2^0 + \bar{\alpha}_3, \qquad \alpha_2^0 \in \mathbb{C}.
\end{aligned}$$

In the case $n=3$ we let

$$\ell_\varepsilon u := \alpha_0(\cdot - x_0)u + \alpha_1^\varepsilon u + \varepsilon^{-2}\alpha_2^\varepsilon(\alpha_3^\varepsilon, u)_{L_2(\partial\omega_\varepsilon)}, \qquad \alpha_j^\varepsilon(x) := \alpha_j(\varepsilon^{-1}(x - x_0)),$$

where α_j are some complex-valued continuous functions on $\partial\omega$. In both cases

$$\begin{aligned}
\alpha_0(x) &:= -\frac{1}{G_{-1}(x)}\frac{\partial G_{-1}}{\partial \mathrm{n}}(x) = \frac{\nu\cdot x}{|\mathrm{A}_0^{-1/2}x|^2},\\
G_{-1}(x) &:= \begin{cases} -\ln|\mathrm{A}_0^{-1/2}x|, & n=2,\\ \dfrac{1}{|\mathrm{A}_0^{-1/2}x|}, & n=3,\end{cases}
\end{aligned}$$

where $\mathrm{A}_0 := \mathrm{A}(0)$.

Our main aim is to study the asymptotic behavior of the operator $\mathcal{H}_\varepsilon$ and its spectral characteristics for small ε.

The boundary value problem

$$(\hat{\mathcal{H}} + c_1)G = 0 \quad \text{in} \quad \Omega\setminus\{x_0\}, \qquad \mathcal{L}G = 0 \quad \text{on} \quad \partial\Omega,$$

with some constant c_1 has a unique solution with the differentiable asymptotics

$$
\begin{aligned}
&G(x) = G_{-1}(x-x_0) + G_0(x-x_0) + a_0 + O(|x-x_0|), \qquad x \to x_0,\\
&G_0 = 0, \quad n = 2,\\
&G_0(x) \coloneqq \sum_{i,j=1}^{3} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} |\mathrm{A}_0^{-1/2} x| + \sum_{i,j,k=1}^{3} a_{ijk} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} |\mathrm{A}_0^{-1/2} x|^3\\
&\qquad\quad + \sum_{j=1}^{3} a_j \frac{\partial}{\partial x_j} |\mathrm{A}_0^{-1/2} x|, \quad n = 3,
\end{aligned}
$$

where a_{ij} are homogeneous polynomials of order 1 and a_{ijk}, a_j, a_0 are some complex constants.

Let $\mathcal{H}_{0,\beta}$ be the operator in $L_2(\Omega)$ with the differential expression $\hat{\mathcal{H}}$, the boundary conditions $\mathcal{L}u = 0$ and a point interaction at x_0. The domain of this operator and its action are defined as

$$
\begin{aligned}
\mathfrak{D}(\mathcal{H}_{0,\beta}) &\coloneqq \left\{ u = u(x) \colon u(x) = v(x) + (\beta - a_0)^{-1} v(x_0) G(x),\ v \in \mathfrak{D}(\mathcal{H}_\Omega) \right\},\\
\mathcal{H}_{0,\beta} u &= \mathcal{H}_\Omega v - c_3 (\beta - a_0)^{-1} v(x_0) G,
\end{aligned}
$$

with the constant c_1, where $\mathcal{H}_\Omega$ is the operator in Ω with the differential expression $\hat{\mathcal{H}}$ and the boundary conditions $\mathcal{L}u = 0$.

Our main result states that the operator $\mathcal{H}_\varepsilon$ converge in the norm resolvent sense to the operator $\mathcal{H}_{0,\beta}$, where β is determined by the functions α_j. This means that the operator $\mathcal{H}_{0,\beta}$ with the point interaction can be approximated by the operator $\mathcal{H}_\varepsilon$ in the perforated domain with a non-local Robin boundary condition on $\partial\omega_\varepsilon$.

If $\alpha_2 = 0$, then the operator $\mathcal{H}_\varepsilon$ has a classical Robin condition on $\partial\omega_\varepsilon$ and in this case we have to impose additional restrictions to α_1 in order to ensure the resolvent convergence. This gives certain restrictions to the value of β and restrict the set of possible β. In other words, the classical Robin condition allows us to approximate the operator with the point interactions with coupling constant from some subset in the complex plane.

The presence of the non-local term in the boundary condition in the case $\alpha_2 \neq 0$ allows us to get an arbitrary complex value β, that is, the non-local term allows us to approximate the operator with an arbitrary point interaction.

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Convergence in distribution of random fields

Wed, Jun 25
12:00–12:45

Alexander I. Bufetov

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For a random field on a compact metric space, convergence in distribution follows from the convergence of the corresponding operators. To establish the latter, the formalism is used of Birman–Solomyak and of Russo.

The Nonlinear Carleson Conjecture for OPUC: new development

Mon, Jun 23
17:00–17:45

Sergey A. Denisov

University of Wisconsin–Madison

The Nonlinear Carleson conjecture for OPUC (advocated by T. Tao, C. Thiele and Y. Tsai in [1]) asks to prove asymptotics for polynomials $\Phi_n(z, \sigma)$ orthogonal on the unit circle $\mathbb{T}$ when $n \to \infty$ for a. e. $z \in \mathbb{T}$ in the case when the measure of orthogonality σ belongs to the Szegő class, i. e., when

$$\int_{\mathbb{T}} \log w \, dm > -\infty$$

where $d\sigma = w\, dm + d\sigma_s$, m is the normalized Lebesgue measure on $\mathbb{T}$: $dm = (2\pi)^{-1}\, d\theta$, and σ_s is a singular part. This problem is related to the Schrödinger Conjecture in quantum mechanics and to the Steklov Conjecture in approximation theory.

In this talk, we will discuss some recent developments and relate the problem to a fine behavior of roots on Φ_n. This work is based on a joint project with Roman Bessonov [2] and more recent work of the speaker [3].

References

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Wed, Jun 25
10:50–11:35

Non-compact Lagrangian manifolds with fold-type singularity and uniform asymptotics of the eigenfunctions of the Laplace operator in an ellipse

Sergey Dobrokhotov

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The asymptotics of the eigenfunctions of the Laplace operator in an elliptic domain with various boundary conditions has been studied in the works of many scientists. The results of studies of such (semi-classical) asymptotics based on the construction of compact Lagrangian manifolds with an edge associated with Birkhoff billiards and the subsequent construction of the canonical Maslov operator on them are described in the papers and books of V. F. Lazutkin. In this talk we present a different approach summarizing some of the considerations proposed in the work of S. Y. Dobrokhotov, D. S. Minenkov, and V. E. Nazaikinskii on the asymptotics of the Bessel function and recent results by A. Yu. Anikin, V. E. Nazaikinskii, A. A. Tolchennikov, and A. V. Tsvetkova, concerning an effective description of the asymptotic solution in a wide neighborhood of fold-type caustics. First, using some caustics (smooth and closed), we construct families of non-compact invariant Lagrangian manifolds and asymptotic generalized eigenfunctions of the Laplace operator, defined as canonical Maslov operators on them, which are globally implemented as Airy functions of a complex argument. Then we select curves on the plane on which the obtained functions or normal derivatives vanish, and thus are solutions to the Dirichlet or Neumann problem in the regions bounded by such a curve. The choice of these curves is related, on the one hand, to the points formed by a stretched thread spanning the caustic curve, and, on the other, to the Bohr-Sommerfeld quantization rule. In the case when the selected caustics are ellipses, we naturally obtain asymptotic eigenfunctions of the Dirichlet or Neumann problem for the Laplace operator in an ellipse co-focal with a given caustic. From the point of view of spectral theory, the new result here is a global description of the asymptotic eigenfunctions of the Laplace operator in the form of Airy functions of a complex argument.

The work was done together with V. E. Nazaikinskii, A. V. Tyurin, A. V. Tsvetkova in the frame of the State assignment (state registration no. 124012500442-3) and with the support of a grant from the Foundation for the Development of Theoretical Physics and Mathematics “BASIS”.

The spectral problem for the Kronig–Penney model of the Dirac operators in terms of Schur's algorithm

Fri, Jun 27
12:50–13:15

Pavel Gubkin

St. Petersburg Department of Steklov Institute of Mathematics RAS

The relativistic Kronig–Penney model describes the one-dimensional Dirac operator $\mathcal{D}_Q$ on the half-line $\mathbb{R}_+$ of the form $\mathcal{D}_Q\colon X \mapsto JX' + QX$, where constant matrix J is a square root of the minus identity matrix and $Q = \sum_{k\geq 0} Q_k \delta_{hk}$ is a measure-valued potential supported on the half-lattice $h\mathbb{Z}_+$ for some $h > 0$. As well as in more classical cases, e. g., $Q \in L^2(\mathbb{R}_+)$, for such a Dirac operator one can define the Weyl function m_Q and the corresponding Schur function $f_Q = \frac{m_Q - i}{m_Q + i}$. The Schur function is completely determined by its recursion coefficients, and there is a bijective correspondence between the recursion coefficients and the potential Q using a simple explicit formula. This correspondence reduces the spectral theory of the Kronig–Penney model to the theory of orthogonal polynomials on the unit circle. In the talk we will show how this reduction can be used to obtain the explicit two-sided uniform stability estimate for the mapping $Q \mapsto m_Q$. The talk is based on the joint work [1] with Roman Bessonov.

References

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Spectral problem for self-balanced stresses in a non-Euclidean continuous medium model

Thu, Jun 26
10:50–11:35

Mikhail Guzev

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When modeling various physical phenomena, researchers often have to solve the problem of calculating the spectral characteristics of the operator corresponding to the problem under consideration. The latter depend on the structure of this operator and the conditions determined by the physical requirements of the problem under study. In this paper, we will study the spectral characteristics of the equilibrium operator in continuum mechanics, leading to self-balanced stress fields. The need to solve such problems for bodies of various shapes is due to the use of the results obtained in various research areas. In engineering, self-balanced fields are called residual stresses, and the need to study them is clearly emphasized in the process of analyzing engineering problems when measuring the level of internal stresses in welded structures [7, 8]. Obtaining analytical solutions is important and relevant for developing methods of refined strength analysis, for testing numerical algorithms for solving problems and other applications.

In this paper, a class of self-balanced stress fields of a continuous medium, parameterized by a stress function, is constructed. It is proposed to consider this function as an element of the spectrum of a biharmonic operator. Spectral characteristics of the operator are constructed for a cylindrical sample and various types of boundary conditions.

The research was carried out within the state assignment for IAM FEB RAS no. 075-00459-25-00.

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On a spectral problem of nonlinear perturbation theory

Mon, Jun 23
12:50–13:20

Vasiliy Kachalov

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The problem of perturbing the spectrum of a linear operator by a linear operator of the form $(A + \varepsilon B)u = \lambda u$, where ε is a small parameter, was usually solved by the small parameter method (the Poincaré method) and the resulting series in degrees of ε converged asymptotically [1, Ch. XII, § 3]. However, for quantum mechanics, the eigenvalues of — are the energy levels of atoms, and it was desirable that the above-mentioned Rayleigh–Schrodinger series converge in the usual sense. It was proved by [1, Ch. XII, § 2] that, provided the operator B is subordinate to the operator A, the proper pairs $\{\lambda(\varepsilon), u(\varepsilon)\}$ are analytic at the point $\varepsilon = 0$.

The presented report presents the conditions of analyticity for a small parameter of eigenvalues and eigenvectors when the operator B is bilinear. Thus, the concepts of analytic families of operators of type (A) and Kato are generalized to the nonlinear case [2, Ch. VII, § 2, p. 2].

Let $\mathcal{A}$ be a commutative Banach algebra with unity equipped with a scalar product. Consider the spectral problem

$$Au + \varepsilon u \cdot Hu = \lambda u, \tag{1}$$

where A and H are unbounded operators, and the operator A has a discrete spectrum. It is a generalized formulation of the stationary problem of hydrodynamic stability [3].

It is proved that, under a certain set of conditions, the eigenvalues and eigenvectors of problem (1) are analytic at the point $\varepsilon = 0$.

It should be noted that the problem of constructing series by degrees of a small parameter that converge in the usual sense began to be systematically studied in the theory of singular perturbations, within the framework of the regularization method of S. A. Lomov [4, 5].

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Thu, Jun 26
11:40–12:10

Unidirectional waves

Aleksei Kiselev

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The talk concerns a class of localized solutions to the wave equation

$$u_{xx} + u_{yy} + u_{zz} - c^{-2}u_{tt} = 0, \qquad c = \text{const} > 0\,, \tag{1}$$

the interest in which is motivated by recent progress in generation of few-cycle optical pulses. In contrast to the high-frequency approximate and exact solutions, see, e. g., [1], that had previously attracted so much attention, the solutions under consideration are not required to be localized more strongly than to have finite energy.

Undirectional solutions to (1) are those whose spacial Fourier spectrum lies in a half-space. Initially, they were introduced for the axisymmetric case in [2] as

$$u = u(\rho, z, t) = \int_0^\infty d\omega e^{i\omega t} \int_0^{\omega/c} dk_z A(k_z, \omega) e^{-ik_z z} J_0(\rho\sqrt{(\omega/c)^2 - k_z^2})\,, \tag{2}$$

where $\rho = \sqrt{x^2 + y^2}$, with an arbitrary weight A.

The simplest unidirectional solution was found through a quite elementary approach in [3] and has the form

$$u = 1/S(S - z_*), \qquad S = S(t, \rho) = \sqrt{(ct_*)^2 - \rho^2}, \tag{3}$$

where $z_* = z + i\zeta, t_* = t + i\tau$, ζ and τ are free real parameters, $\zeta < c\tau$, the root branch is such that $S|_{x=y=0} = ct_*$. The function (3) has finite energy and is useful in modeling a variety of few-cycle optical pulses [4].

The importance of the class of relatively undistorted waves of the form

$$u = f(S - z_*)/S, \tag{4}$$

with f an arbitrary analytical function, is noted in this context in [5] (for the solution, (3), $f(S - z_*) = 1/(S - z_*)$). Relationship between (4) and the classical spherical waves is observed, and the equivalence of several integral representations including (2), (3) is established in [6]. An important tool was representing solutions in terms of their their large-time asymptotics in the far-zone. Unexpected examples of finite-energy solutions having no spherical-wave asymptotics have been found [7].

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Asymptotic expansions of solutions for systems of ODE and basis property of eigenfunctions

Tue, Jun 24
12:50–13:15

Alexey Kosarev

Moscow State University

We deal with a $n \times n$ system of ordinary differential equations of the form

$$y' - By = \lambda Ay, \qquad y = y(x), \qquad x \in [0,1],$$

where λ is a large complex parameter, $A = \operatorname{diag}\{a_1(x), \dots, a_n(x)\}$ and $B = \{b_{jk}(x)\}_{j,k=1}^{n}$. The entries of matrices A and B are complex-valued functions that, for some integer $m \geq 1$, satisfy the smoothness conditions

$$a_j \in W_1^m[0,1], \quad b_{jj} \in W_1^{m-1}[0,1], \quad b_{jk} \in W_1^m[0,1] \text{ for } j \neq k, \quad j,k = 1, \dots, n.$$

We establish sufficient conditions to obtain asymptotic expansion

$$Y(x,\lambda) = M(x)\left(I + \lambda^{-1}R_1(x) + \cdots + \lambda^{-m}R^m(x) + \lambda^{-m}o(1)\right)E(x,\lambda)$$

of fundamental matrix $Y(x,\lambda)$ of solutions in some sectors of the complex λ-plane, where matrices M, E and $R_1, \ldots, \mathbb{R}_m$ are explicitly expressed in terms of the entries A and B. We then outline how these expansions can be continued into larger regions bounded by logarithmic curves.

In the second part of the talk, we apply these asymptotic formulas to investigate the basis property of eigenfunctions of a non-regular spectral problem in the space $(L_p[0,1])^n$ for $1 < p < \infty$.

This report is based on joint work with A. A. Shkalikov. The work was supported by the Russian Science Foundation grant no. 25-11-00304.

Tue, Jun 24
15:00–15:45

On complex-valued β-ensemble integrals

Karol K. Kozlowski

Laboratoire de Physique, ENS de Lyon, UMR 5672 du CNRS, Lyon, France

β-ensembles refer to the following class of N-dimensional integral

$$\mathcal{Z}_{N,\Gamma}[V] = \int_{\Gamma^N} \prod_{a<b}^{N} [(z_b - z_a)(z_a - z_b)]^{\beta/2} \prod_{k=1}^{N} \mathrm{e}^{-N\beta V(z_k)}\, d\mathbf{z}$$

depending on a sufficiently regular, unbounded, curve $\Gamma \subset \mathbb{C}$ and a function $V\colon \Gamma \to \mathbb{C}$ with $\operatorname{Re} V$ growing fast enough, so that the integral is convergent.

These N-fold integrals arise, for instance, in the study of spectra of a large class of random symmetric, Hermitian or self-dual Hermitian matrices. In fact, then one deals with real valued β-ensembles, *viz.* $\Gamma = \mathbb{R}$ and $V\colon \mathbb{R} \to \mathbb{R}$ at $\beta = 1, 2$ or 4. Also, for general β, the real valued $\mathcal{Z}_{N,\mathbb{R}}[V]$ corresponds to the partition function of a gaz of charged particles in two-dimensions confined to the real line. Such integrals arise as well in enumeration problems of various kinds of graphs.

The large-N behaviour of β-ensemble integrals is of prime interest to these occurrences. It also serves as playground for developing techniques allowing one to tackle the large-N behaviour of more complex integrals. After long years of development, nowadays, the large-N asymptotics of β-ensemble integrals in the real valued case is very well understood. However, the large-N behaviour in the genuinely complex valued setting was open for a long time. The interest in such problems is motivated by various applications of such techniques to problems in quantum field theory, and more generally, the theory of correlation functions in integrable systems.

I will review the probabilistic techniques allowing one to study β-ensembles in the real valued setting. This will then lead me present the recently developed techniques allowing one to understand the complex valued case. The talk is based on the joint work with A. Guionnet and A. Little.

On a homogeneous symmetric operator

Fri, Jun 27
12:00–12:45

Konstantin A. Makarov

University of Missouri, USA

In this talk, we discuss homogeneous symmetric operators with deficiency indices $(1, 1)$ that do not admit a homogeneous self-adjoint extension. Using the Livšic function approach, we characterize the set of all such operators up to unitary equivalence and describe the complete set of corresponding unitary invariants.

The talk is based on joint work with Eduard Tsekanovskii.

Scattering matrices for singular perturbations and applications to BV problems

Tue, Jun 24
15:50–16:35

Mark Malamud

St. Petersburg State University

We will discuss the scattering matrix of two selfadjoint operators A_0 and A_1 in a Hilbert space $\mathfrak{H}$ assuming them to be resolvent comparable, i. e. that their resolvent difference is of trace class. We introduce a symmetric operator $S = A_0 \cap A_1$ and will treat A_0 and A_1 as the extensions of S.

To treat the scattering matrix $S(A_0, A_1; \lambda)$ in the framework of extension theory we introduce a special B-generalized boundary triple for the operator S^* and consider the Weyl function $M(\lambda)$ corresponding to this triple. Our main abstract result shows that the scattering matrix $S(A_0, A_1; \lambda)$ is expressed by means of the non-tangential limit values of the Weyl function $M(\lambda + i0)$.

We demonstrate applications of the main abstract result to ordinary differential operators as well as to systems of such operators. Besides, we apply it to different selfadjoint realizations of Schrodinger differential expressions in exterior domains in $\mathbb{R}^2$ and $\mathbb{R}^3$. In particular, if A_D and A_N are the Dirichlet and Neumann realizations, then the scattering matrix of the scattering system $\{A_D, A_N\}$ is expressed by means of the limit values of the Dirichlet-to-Neumann map dependent on the spectral parameter. The latter is the classical object naturally appeared in the theory of boundary value problems of the second order elliptic operators.

A relation with trace formulas and spectral shift function will be discussed too.

The talk is based on results of the joint publication [1] and some new results.

References

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Mon, Jun 23
17:50–18:20

Classical and quantum dynamics of a particle in a narrow angle

Dmitrii Minenkov

Ishlinsky Institute for Problems in Mechanics RAS

The 2D Schrodinger equation with variable potential in the narrow domain diffeomorphic to the wedge with the Dirichlet boundary condition is considered. The corresponding classical problem is the billiard in this domain. In general, the corresponding dynamical system is not integrable. The small angle is a small parameter which allows one to make the averaging and reduce the classical dynamical system to an integrable one modulo exponential small correction (see [1]). The quantum adiabatic approximation (in the form of operator separation of variables [2]) is used to construct the asymptotic eigenfunctions (quasimodes) of the Schrodinger operator. The relation between classical averaging and constructed quasimodes is discussed and the behavior of quasimodes in the neighborhood of the cusp is studied. Also the relation between Bessel and Airy functions that follows from different representations of asymptotics near the cusp is obtained.

The talk is done based on results obtained together with S. Yu. Dobrokhotov, A. I. Neishtadt and S. B. Shlosman [3] and is supported by RSF grant 24-11-00213.

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On an analogue of Orlov's theorem for linear differential equations with distributional coefficients

Tue, Jun 24
10:50–11:35

Karakhan Mirzoev

Moscow State University

In the first part of the report, a construction will be presented that allows us to determine in what sense a differential expression of the form

$$\tau_{2n}(y) \coloneqq \sum_{k=0}^{n}(-1)^{n-k}(p_k^{(k)}y^{(n-k)})^{(n-k)} + i\sum_{k=0}^{n-1}(-1)^{n-k-1}\Big\{(q_k^{(k)}y^{(n-k-1)})^{(n-k)} + (q_k^{(k)}y^{(n-k)})^{(n-k-1)}\Big\}$$

should be interpreted. Here all derivatives are understood in the sense of distribution theory. In the second part of the report, by imposing additional conditions on the functions p_k and q_k, the leading term of the asymptotics at infinity of a certain fundamental system of solutions of the equation $\tau_{2n}(y) = \lambda y$ will be found, where λ is a fixed complex parameter. These formulas allow us to determine the defect numbers of the operator generated by the expression $\tau_{2n}(y)$ in the Hilbert space $L^2[1, +\infty)$.

The theorem on the asymptotics of solutions at infinity for differential equations of the form $l_{2n}(y) = \lambda y$ with real analytic coefficients in some neighborhood of infinity was first obtained by S. A. Orlov in the work [1]. In the work [2] an analogue of this theorem was obtained for differential equations with real locally summable coefficients of the form

$$l_m(y) \coloneqq \sum_{j=0}^{[m/2]}(p_j y^{(j)})^{(j)} + i\sum_{j=0}^{[(m-1)/2]}[(q_j y^{(j+1)})^{(j)} + (q_j y^{(j)})^{(j+1)}],$$

where $[a]$ is the largest integer not exceeding a. And in the work [3] an analogue of the same theorem is given for two-term differential equations of even order with admissible distributional coefficients.

The work was supported by the Russian Science Foundation, project 25-11-00304.

References

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Fri, Jun 27
16:35–17:00

Threshold approximations for the resolvent of a factorized non-selfadjoint operator family. Applications to homogenization of differential operators

Arsenii Mishulovich

St. Petersburg State University

In a Hilbert space, we consider a family of operators admitting a factorization $A(t) = X(t)^*\mathcal{B}X(t)$, where $X(t) = X_0+tX_1$ ($t \in \mathbb{R}$), $\mathcal{B} = I+iB$ and B is selfadjoint. The operator $A(t)$ is defined via a closed sectorial form, and we assume that the subspace $\mathfrak{N} = \operatorname{Ker} A(0)$ is finite-dimensional. For the resolvent $(A(t) + \varepsilon^2 I)^{-1}$ with small ε, we derive an approximation in the operator norm on a fixed interval $|t| \leqslant t^0$. We present the results on an approximation in main order with error $O(\varepsilon^{-1})$ and a more involved estimation with so-called "corrector" and error $O(1)$.

The results are applied to homogenization in the small period limit of a matrix second order differential operator.

Thu, Jun 26
10:00–10:45

On the Maslov index and its computation

Vladimir Nazaikinskii

Ishlinsky Institute for Problems in Mechanics RAS

Local expressions for the Maslov canonical operator contain square roots of some Jacobians and, to correctly define the canonical operator, one must choose a branch of the root, i. e., in fact, a branch of the Jacobian argument, according to a certain rule. In most existing expositions of the theory of the canonical operator, this rule is one of the most mysterious places, at least if one tries to apply it in practice. In this talk, which is based on the paper [1], we remove the veil of mystery and present computationally efficient formulas suitable for computer implementation.

The work was financially supported by the Russian Science Foundation, project no. 24-11-00213 (`https://rscf.ru/en/project/24-11-00213/`).

References

[1] S. Yu. Dobrokhotov and V. E. Nazaikinskii, *On the arguments of Jacobians in local expressions of the Maslov canonical operator*, Math. Notes, 116 (2024), 1264–1276.

Band-gap structure of the spectrum of the Dirichlet problem for the Lamè system in a double-periodic lattice of thin plates

Mon, Jun 23
12:00–12:45

Sergei A. Nazarov

Institute of Mechanical Engineering Problems RAS

Let $\boxplus^0 \subset \mathbb{R}^2 \ni (x_1, x_2)$ be a 1D double-periodic graph composed from the straight lines $\{x\colon x_j = n, x_{3-j} \in \mathbb{R}\}$ with $j = 1, 2$ and $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ while $\boxplus^h$ stands for its $h/2$-neighborhood. In the spatial elastic square lattice $\boxplus^h_H = \boxplus^h \times (0, H) \subset \mathbb{R}^3$ we consider the Dirichlet problem for the Lamè system

$$L(\nabla)u^h(x) \coloneqq -\mu(\nabla \cdot \nabla)u^h(x) - (\lambda + \mu)\nabla\nabla \cdot u^h(x) = \Lambda^h u^h(x),\ x \in \boxplus^h_H,$$
$$u^h(x) = 0,\ x \in \partial \boxplus^h_H .$$

Here, h and H are positive small and fixed parameters, $\lambda \geq 0$ and $\mu > 0$ are the Lamè constants, $u^h = (u^h_1, u^h_2, u^h_3)$ and Λ^h are the displacement vector and the spectral parameter while $\nabla =$ grad and $\nabla \cdot =$ div. The main result concerns asymptotic formulas for the bands and gaps in the low-frequency range of the spectrum of this problem. It is obtained by examining asymptotics as $h \to +0$ of the eigenvalues $\{\Lambda_k(\theta)\}_{k=1}^{\infty}$ of the model problem in the periodicity cell $\{x \in \boxplus^h_H\colon |x_j| < 1/2, j = 1, 2\}$ with the quasi-periodicity conditions at the cell truncation sides involving the Floquet parameters $\theta = (\theta_1, \theta_2) \in [-\pi, \pi]^2$.

The main role in the asymptotic analysis is played by the boundary layers which are described by the scalar and vector Dirichlet problems for the Laplace $-\mu\nabla \cdot \nabla$ and Lamè $L(\nabla)$ operators in the cruciform waveguide $\mathbb{X}$ composed from two unit perpendicular strips. The discrete spectra of the corresponding problems in $\mathbb{X}$ consist of the single eigenvalue M_Δ and M_L in the interval $(0, M_\dagger)$ below the cutoff point $M_\dagger = \mu\pi^2$ of their common continuous spectrum. The important observation $M_\Delta < M_L$ and application of the factorization method in the vector problem of 2D elasticity (unexpected remedy, at least for the author) provide implementation and justification of the dimension reduction procedure to derive the limiting problem, namely the Dirichlet one for an ordinary second-order differential equation in the interval $(0, H)$ whose positive and simple eigenvalues $\{B_k\}_{k=1}^{\infty}$ form the correction terms in the asymptotic expansions of the eigenvalues

$$\Lambda_k(\theta) = M_\Delta h^{-2} + B_k + O(\sqrt{h}), \qquad k = 1, 2, 3, \dots$$

This formula allows to conclude that in the low-frequency range of the spectrum of the model problem consists of bands of exponentially small length while between them spectral bands of width $O(1)$ stand open. The exact formulas for the gaps will be given in the talk.

Mon, Jun 23
10:00–10:45

Homogenization of convolution type operators with heavy tails

Andrey Piatnitski

UiT The Arctic University of Norway & Higher School of Modern Mathematics MIPT

The talk will focus on homogenization problems for nonlocal convolution-type operators of the form

$$L^\varepsilon u(x) = \frac{1}{\varepsilon^{d+\alpha}} \int_{\mathbb{R}^d} p\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)\Big(u(y)-u(x)\Big)\,dy, \qquad 0<\alpha<2,$$

with a small parameter $\varepsilon > 0$. We assume that

1 The kernel $p(\cdot) \in L^1(\mathbb{R}^d)$ is a non-negative even function, $p(z) = p(-z)$.

2 The function $\Lambda(\cdot)$ is symmetric, $\Lambda(\xi,\eta) = \Lambda(\eta,\xi)$, and satisfies the inequality

$$0 < \Lambda^- \leqslant \Lambda(\xi,\eta) \leqslant \Lambda^+ \qquad \text{for all } \xi,\eta \in \mathbb{R}^d.$$

Under these two conditions L^ε is a bounded non-negative self-adjoint operator in $L^2(\mathbb{R}^d)$.

We study the asymptotic behaviour of the resolvent $(-L^\varepsilon + m)^{-1}$, as $\varepsilon \to 0$, in the following two cases:

- $\Lambda(x,y)$ is $\mathbb{Z}^d$ periodic in each of the variables x and y;
- $\Lambda(x,y)$ is a statistically homogeneous ergodic random field.

The normalization factor $\varepsilon^{-(d+\alpha)}$ in the definition of L^ε suggests that, for large $|z|$, $p(z)$ satisfies the estimate

$$\beta_1|z|^{-(d+\alpha)} \le p(z) \le \beta_2|z|^{-(d+\alpha)}, \qquad 0<\beta_1 \leqslant \beta_2.$$

We show that the family $\{L^\varepsilon\}$ admits homogenization, if $p(\cdot)$ satisfies the following two conditions:

(i) in a weak sense $p(z) \sim |z|^{-d-\alpha}k(z/|z|)$, as $|z| \to \infty$, where $k(s)$ is a continuous function on the sphere S^{d-1};

(ii) the local oscillation of $p(z)$ divided by $p(z)$ tends to zero, as $|z| \to \infty$.

The homogenized operator takes the form

$$L^0 u(x) = \int_{\mathbb{R}^d} \bar\Lambda k\Big(\frac{x-y}{|x-y|}\Big)\,\frac{(u(y)-u(x))}{|y-x|^{d+\alpha}}\,dy,$$

where $\bar\Lambda$ is the mean value of Λ.

This is a joint work with Elena Zhizhina (UiT, MIPT).

Radiation and scattering of acoustic waves in discrete waveguides with several cylindrical outlets to infinity

Fri, Jun 27
10:50–11:35

Aleksandr Poretskii

St. Petersburg State University

A discrete waveguide is a graph G consisting of several discrete semi-cylinders (cylindrical outlets) connected by a finite number of edges and vertices. A discrete cylinder is a graph that is periodic when shifted by a given vector and has a finite periodicity cell. On the graph G, we consider an equation of the form

$$-\operatorname{div} a\nabla u + qu = \mu u + f,$$

where the given function f and the unknown function u are functions on the set of graph vertices, div and ∇ are difference analogues of the corresponding differential operators, and μ is a real spectral parameter. The coefficients a and q are given functions on E and V, respectively; here E stands for the set of edges, while V is the set of vertices. We assume that, in each cylindrical outlet, the coefficients stabilize at infinity to periodic functions; the stabilization rate is exponential.

For this problem, we construct incoming and outgoing waves, eigenfunctions of the continuous spectrum (bounded solutions of the corresponding homogeneous problem that do not belong to the space $\ell_2(V)$), introduce a scattering matrix, and describe the radiation principle (a well-posed statement of the problem with intrinsic radiation conditions at infinity).

The talk is based on a joint research with D. S. Smorchkov. The research is supported by the Russian Science Foundation no. 22-11-00070-П (`https://rscf.ru/en/project/22-11-00070/`).

On adiabatic evolution generated by a Schrödinger operator with discrete and continuous spectra

Fri, Jun 27
15:35–16:00

Vasily A. Sergeev

Chebyshev Laboratory, St. Petersburg State University

This talk is based on joint work with A. A. Fedotov. As $\varepsilon \to 0$, consider the Schrödinger equation

$$i\varepsilon\frac{\partial\Psi}{\partial\tau} = -\frac{\partial^2\Psi}{\partial x^2} + v(x,\tau)\Psi, \qquad x>0, \qquad \tau\in\mathbb{R}, \qquad \Psi|_{x=0}=0. \tag{1}$$

Here the potential v is a finite square well that shrinks linearly with time:

$$v(x,\tau) = \begin{cases} -1 & \text{if} \quad 0\le x\le 1-\tau, \\ 0, & \text{otherwise.} \end{cases}$$

One can say that equation (1) describes adiabatic evolution generated by the stationary operator $H(\tau) = -\partial^2/\partial x^2 + v(x,\tau)$ with the Dirichlet boundary condition at zero. The spectrum of $H(\tau)$ consists of the (absolutely) continuous spectrum $[0, +\infty)$ and a finite number of negative eigenvalues. With time, the eigenvalues one by one approach the edge of the continuous spectrum and, having reached it, disappear.

A set of solutions to (1), each close at some moment to an eigenfunction of the stationary operator $H(\tau)$, were constructed in [1] and asymptotically studied in [1, 2, 3]. One can say that these solutions correspond to the eigenvalues of $H(\tau)$. More recently, we constructed and asymptotically studied an analogous solution corresponding to the continuous spectrum of the stationary operator. In this talk we describe these results.

References

[1] A. A. Fedotov, *Adiabatic evolution generated by a one-dimensional Schrödinger operator with decreasing number of eigenvalues*, Math. Notes, 116 (2024), 804–830.

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Wed, Jun 25
10:00–10:45

Semi-classical spectrum for one-dimensional Schroedinger operator with multi-scaled abruptly varying potential

Andrei Shafarevich

Moscow State University

Semi-classical eigenvalues for the wide class of operators with smooth coefficients can be described in terms of quantization conditions and Maslov canonical operator on the corresponding Lagrangian manifold. We study simplest example of operators with “fast-slow” variables – one-dimensional Schroedinger operator with abruptly varying potential. We prove that in this situation semi-classical spectrum can be described in terms of quantization conditions on one-dimensional bundle over the circle. The base of this bundle is associated with slow variable, as well as fibers – with the fast one. The proof relies on generalization of canonical operator for such a bundle.

Invariant subspace problem on Krein space and related problems

Tue, Jun 24
10:00–10:45

Andrey Shkalikov

Moscow State University

We will discuss the problem of the existence of maximal sign-definite invariant subspaces in a Hilbert space with an indefinite metric (Krein space). In general, this is an open, difficult problem, the prospects for solving which are absolutely unclear. The report will deliver the results on the classes of operators for which it has a positive solution. The connection of this problem with the factorization theory of operator pencils, with the so-called "half-range problem", with the theory of diagonalization of operator matrices will be traced. Some applications of the results to concrete problems will be demonstrated.

The research is supported by the Russian Science Foundation, grant no. 25-11-00304.

Asymptotics of spectra for Dirichlet and Dirichlet–Neumann problems for the Sturm–Liouville equation with integral perturbation

Tue, Jun 24
17:30–17:55

Vladimir Sivkin

HSE University

We study the asymptotics for Dirichlet and Dirichlet–Neumann problems of the Sturm–Liouville equation perturbed by an integral operator. We present sharp asymptotic formulas for the eigenvalues for the case of a convolution kernel. These formulas contain information about the Fourier coefficients of the potential and the kernel. The estimates of the asymptotics for the remainder take into account the rate of decrease both for large eigenvalues and the small norms of the potential and kernel. The formulas are new even in the case of the Sturm–Liouville operator, when the convolution kernel is zero.

One theorem on the limit behavior of a branching Wiener process with singular branching rate

Fri, Jun 27
15:00–15:30

Natalya Smorodina

St. Petersburg Department of Steklov Institute of Mathematics RAS & St. Petersburg State University

We construct branching one-dimensional Wiener process $X_x(t)$, starting from single particle at $x \in \mathbb{R}$, the branching rate of which is a distribution $q_\alpha(x) =$

$-|x|^{-1-\alpha}$, where $\alpha \in (0, 1/2)$. The distribution q_α acts on a test function f as

$$\int (f(0) - f(x))|x|^{-1-\alpha}\, dx.$$

For this process we consider operator semigroup P^t, acting on the function $\varphi \in L_2(\mathbb{R})$ as

$$[P^t\varphi](x) = \mathbb{E} \sum_{y \in X_x(t)} \varphi(y),$$

where the summation is carried out over all particles of the branching process $X_x(t)$, existing in the system at time t. It is shown that the generator of the semigroup P^t is the operator $\mathcal{A} = \frac{1}{2}\, d^2/dx^2 + q_\alpha(x)$, understanding in the sense of the paper [1]. The spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A}$ consists of a negative semi-axis $(-\infty, 0]$ and a unique positive eigenvalue λ_0, which corresponds to a positive eigenfunction φ_0, $\|\varphi_0\|_2 = 1$.

We now present our main result.

Theorem. *Suppose that $\varphi \in L_2(\mathbb{R})$, by φ_1 denote its projection onto the orthogonal complement to φ_0. Then for any $t > 0$ the inequality holds*

$$\|P^t\varphi - e^{\lambda_0 t}(\varphi, \varphi_0)\varphi_0\|_2 \leqslant \|\varphi_1\|_2.$$

References

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Fri, Jun 27
10:00–10:45

Discrete models for differential operators

Dmitriy Stolyarov

St. Petersburg State University

I will survey an interesting analogy between martingales and functions. Though such similarities are widely known in harmonic analysis, the particular discrete martingale setting found first by S. Janson in late 70s and then rediscovered by Ayoush, Wojciechowski, and me around 2018, seems to shed light on problems for vectorial functions in L_1 norms and vectorial measures. Though the idea has lead to certain progress on so-called Bourgain–Brezis inequalities, there is no rigorous justification why martingales and functions on Euclidean space behave in a similar manner. In recent years, together with Ayoush, Spector, and Wojceichowski, we found new instances where the analogy works (for example, for rank-one theorems in the spirit of G. Alberti). This naturally settles many interesting questions.

On two-dimensional model elliptic operator in a strip with fast oscillating boundary and frequent alternation of boundary conditions

Fri, Jun 27
16:05–16:30

Radim Suleymanov

Ufa University of Science and Technology

We consider a Laplace equation in a two-dimensional infinite strip with a rapidly oscillating boundary with a small amplitude of oscillations. The oscillations in this model case are assumed to be periodic and stepwise. On the oscillating boundary we impose frequent alternation of the Dirichlet and Neumann conditions. The lengths of the pieces with the Dirichlet conditions are power in the small parameter, which is the characteristic length of the pieces with the Neumann condition. We show that the homogenization leads to the Neumann problem for the same equation. The main result is W_2^1-operator estimates. We stress that for the classical frequent alternation on non-oscillating boundary the homogenized Neumann condition appears only in the case, when the lengths of the Dirichlet pieces are exponentially small in comparison with the Neumann pieces. At the same time, as the present result shows, once frequent alternation is imposed on fast oscillating boundary, this significantly changes the borderline between the homogenized Dirichlet and Neumann condition.

Homogenization of the Lévy-type operators with periodic coefficients

Mon, Jun 23
10:50–11:35

Tatiana Suslina

St. Petersburg State University

In $L_2(\mathbb{R}^d)$, we consider an operator $\mathbb{A}_\varepsilon$, $\varepsilon > 0$, formally given by

$$(\mathbb{A}_\varepsilon u)(\mathbf{x}) := \int_{\mathbb{R}^d} \mu(\mathbf{x}/\varepsilon, \mathbf{y}/\varepsilon) \frac{(u(\mathbf{x}) - u(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^{d+\alpha}}\, d\mathbf{y}, \qquad 0 < \alpha < 2.$$

It is assumed that $\mu \in L_\infty(\mathbb{R}^{2d})$, $0 < \mu_- \leqslant \mu(\mathbf{x},\mathbf{y}) \leqslant \mu_+ < \infty$, $\mu(\mathbf{x},\mathbf{y}) = \mu(\mathbf{y},\mathbf{x})$, and $\mu(\mathbf{x}+\mathbf{m}, \mathbf{y}+\mathbf{n}) = \mu(\mathbf{x},\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$, $\mathbf{m},\mathbf{n} \in \mathbb{Z}^d$. Precisely, $\mathbb{A}_\varepsilon$ is the selfadjoint operator generated by the closed quadratic form

$$a_\varepsilon[u,u] := \frac{1}{2}\int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}^d} d\mathbf{x}\, \mu(\mathbf{x}/\varepsilon, \mathbf{y}/\varepsilon) \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+\alpha}}, \qquad u \in H^{\alpha/2}(\mathbb{R}^d).$$

In [1], it was proved that the resolvent $(\mathbb{A}_\varepsilon + I)^{-1}$ strongly converges to $(\mathbb{A}^0+I)^{-1}$, as $\varepsilon \to 0$. Here $\mathbb{A}^0$ is the effective operator of the same form as $\mathbb{A}_\varepsilon$ with the constant coefficient $\mu^0 = \int_\Omega \int_\Omega \mu(\mathbf{x},\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}$, $\Omega = (0,1)^d$. Note that $\mathbb{A}^0$ can be represented as $\mathbb{A}^0 = c_0(d,\alpha)\mu^0(-\Delta)^{\alpha/2}$.

We prove that the resolvent of $\mathbb{A}_\varepsilon$ converges in the operator norm on $L_2(\mathbb{R}^d)$.

Theorem. *We have*

$$\left\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C \begin{cases} \varepsilon^\alpha, & 0 < \alpha < 1, \\ \varepsilon(1 + |\log \varepsilon|)^2, & \alpha = 1, \\ \varepsilon^{2-\alpha}, & 1 < \alpha < 2. \end{cases}$$

The error $O(\varepsilon^\alpha)$ for $0 < \alpha < 1$ is optimal. For $1 < \alpha < 2$ we improve accuracy by taking correctors into account.

Theorem. *For $k \in \mathbb{N}$ and $2 - 1/k < \alpha < 2$ we have*

$$\begin{aligned} \Big\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1} - \sum_{m=1}^{k} \varepsilon^{m(2-\alpha)} \mathbb{K}_m\Big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ \leqslant C \begin{cases} \varepsilon, & 2 - \frac{1}{k} < \alpha \leqslant 2 - \frac{1}{k+1}, \\ \varepsilon^{(k+1)(2-\alpha)}, & 2 - \frac{1}{k+1} < \alpha < 2. \end{cases} \end{aligned}$$

Here $\mathbb{K}_m = (\operatorname{div} g\nabla)^m(\mathbb{A}^0 + I)^{-m-1}$, and g is a constant symmetric matrix defined in terms of the solutions of some auxiliary problems.

To prove the results, we modify the operator-theoretic approach for the case of nonlocal operators.

The report is based on the joint works [2], [3] with A. Piatnitski, V. Sloushch and E. Zhizhina. The work was supported by the RSF grant no. 22-11-00092-P.

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A semi-classical approximation of solution to the two-dimensional Dirac equation with a linear potential and localized initial condition

Wed, Jun 25
15:00–15:30

Anton Tolchennikov[†], Ilya Bogaevskii[‡]

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We consider the Cauchy problem for the non-stationary two-dimensional Dirac equation with a linear potential $U = x_1$ and localized initial condition (h is a small parameter):

$$\begin{cases} ih\,\dfrac{\partial u}{\partial t} = x_1 u + \Big(-ih\,\dfrac{\partial}{\partial x_1} - h\,\dfrac{\partial}{\partial x_2}\Big)v, \\ ih\,\dfrac{\partial v}{\partial t} = \Big(-ih\,\dfrac{\partial}{\partial x_1} + h\,\dfrac{\partial}{\partial x_2}\Big)u + x_1 v, \end{cases}$$

$$u|_{t=0} = h^{-2}u^0\Big(\frac{x}{h}\Big), \quad v|_{t=0} = h^{-2}v^0\Big(\frac{x}{h}\Big), \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

The solution is localized in the disk $\{|x| \le t\}$ at a fixed time $t \ge 0$. A WKB-approximation outside of the singular diameter $\{x_2 = 0, |x_1| \le t\}$ has been constructed in [1]. We study the approximation in a neighborhood of this diameter. The leading term of the approximation contains logarithm of the small parameter. It is connected with the multiplicity change phenomenon of the Hamiltonians (i. e., of the eigenvalues of the matrix symbol of the differential operator from the right hand side of the equation).

The work is supported by the RSF (the grant 24-11-00213), INI, and LMS.

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Localized Gaussian type beams and asymptotics of solutions of the Helmholtz equation

Wed, Jun 25
12:50–13:20

Anna Tsvetkova

Ishlinsky Institute for Problems in Mechanics RAS

We discuss a geometric approach based on the Maslov canonical operator theory to constructing the asymptotics of Gaussian type beams. In particular, we consider the Laguerre-Gauss beams, which are the solution of the three-dimensional Helmholtz equation in the paraxial approximation (which can be considered as the Shrödinger equation) [1]. The considered beams are the product of the Gaussian exponent and the Laguerre polynomials.

The discussed approach based on the semiclassical approximation and the study of the dynamics of Lagrangian manifolds, which makes it possible to obtain an effective asymptotics of the considered beams in terms of Airy and Bessel functions of compound argument. Obtained formula gives a good approximation even for small indices of the corresponding polynomials [2].

One of the advantages of the discussed approach is that it is quiet universal. In particular, it allows to abandon the paraxial approximation and consider the original Helmholtz equation

$$h^2\tilde{\triangle}u + k^2u = 0, \tag{1}$$

where $\tilde{\triangle} = \partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\partial^2/\partial z^2$ is the three-dimensional Laplace operator, $h > 0$ is a small parameter. In the talk the global asymptotics in terms of special functions of the solution of the Helmholtz equation with "initial" conditions generated by Laguerre–Gauss beams will be given [3]. We will also discuss the type of asymptotics that arises in the case where k is not constant.

The work is supported by the RSF (project 24-11-00213).

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Tue, Jun 24
12:00–12:45

Sectoriality and compactness of the resolvent for a non-self-adjoint Sturm–Liouville operator with singular distributional potential

Sergey Tumanov

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In the classical theory of the Sturm–Liouville operator defined by the differential expression

$$l(y) = -y'' + qy,$$

where $q \in L_{1,\text{loc}}(\mathbb{R}_+)$, general results that do not require the smoothness of the potential q are often formulated in terms of its antiderivative s, where $q = s'$. In particular, the Molchanov criterion for the compactness of the resolvent and its generalizations characterize the growth of the antiderivative on segments of constant length.

The nature of this phenomenon can be explained by the possibility of regularization of the differential equation $l(y) = f$: there is an equivalent system of equations, the coefficients of which are functions of s:

$$\begin{pmatrix} y(x) \\ y^{[1]}(x) \end{pmatrix}' = \begin{pmatrix} s(x) & 1 \\ -s^2(x) & -s(x) \end{pmatrix} \begin{pmatrix} y(x) \\ y^{[1]}(x) \end{pmatrix} - \begin{pmatrix} 0 \\ f(x) \end{pmatrix}.$$

The equivalent notation for the expression $l(y)$ is following:

$$l(y) = -(y^{[1]})' - sy^{[1]} - s^2 y, \qquad \text{where } y^{[1]} = y' - sy. \tag{1}$$

This observation suggests the possibility of generalizing the well-known classical results (the theorems of Molchanov, Brink, Ismagilov, etc.) to the case of the complex-valued $q \in W^{-1}_{2,\text{loc}}(\mathbb{R}_+)$. We managed to do this work, which will be the subject of our talk.

Asymptotics for long nonlinear coastal waves propagating along a sloping beach

Wed, Jun 25
17:05–17:30

Maria Votiakova

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Coastal waves are understood as gravitational waves on water in a basin with depth $D(x, y)$, localized in the vicinity of the coastline. This report presents asymptotic solutions of a nonlinear shallow water equations system over a uniformly sloping bottom $D(x, y) = \gamma x$, describing waves traveling along the shoreline. The asymptotic solutions of the nonlinear shallow water equations system are written in the form of parametrically defined functions determined through exact solutions of the linearized system (see [1, 2]). In the linear problem, the variables are separable, and normal to the coastline the solution is expanded in terms of eigenfunctions of the operator $\hat{L} = -\partial^2/\partial x^2 - x\partial/\partial x + (k^2 x - \omega^2/\gamma)$ with boundedness conditions at the shoreline $|\xi|_{x=0} < \infty$ and decay at infinity. The connection of the constructed solution with the classical (integrable) "billiard with a semi-rigid wall" (see [3]) is also discussed. This work was supported by the Russian Science Foundation under grant no. 24-11-00213.

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Mon, Jun 23
15:50–16:35

Homogenization of parabolic problems for non-local convolution type operators under non-diffusive scaling of coefficients

Elena Zhizhina

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We study homogenization problem for non-autonomous parabolic equations of the form $\partial_t u = L(t)u$ with a non-local convolution type operator $L(t)$ that has a non-symmetric jump kernel which is periodic in spatial variables and in time.

It is assumed that the space-time scaling of the environment is not diffusive, and our goal is to study the asymptotic behaviour of solutions to the Cauchy problem for equation

$$\frac{\partial u^\varepsilon}{\partial t} \;=\; \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\, \mu\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}; \frac{t}{\varepsilon^\alpha}\Big)\, \left(u^\varepsilon(y,t) - u^\varepsilon(x,t)\right) dy,$$

with the initial condition $u^\varepsilon(x,0) = u_0$, $u_0 \in L^2(\mathbb{R}^d)$, as $\varepsilon \to 0$; here $0 < \alpha < 2$.

We show that asymptotically the spatial and temporal evolutions of the solutions are getting decoupled, and the homogenization result holds in a moving frame

$$(x,t) \;\to\; (x - b^\varepsilon(t),\, t) =: (x^\varepsilon, t).$$

If coefficients $\mu(\xi, \eta, s)$ are sufficiently regular in s, then

$$b^\varepsilon(t) = \sum_{j=0}^{k(\alpha)} \varepsilon^{-1+j(2-\alpha)} b_j t + \varepsilon^{\alpha-1} B_0\Big(\frac{t}{\varepsilon^\alpha}\Big),$$

where $k(\alpha) = [\frac{1}{2-\alpha}]$, b_0, $b_1, \ldots$ are vectors in $\mathbb{R}^d$, $B_0(s)$ is a 1-periodic continuously differentiable vector-function.

My talk is based on our joint work [1] with Andrei Piatnitski.

References

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Almost tangential diffraction on a delta potential

Wed, Jun 25
15:35–16:05

Ekaterina A. Zlobina†, Aleksei P. Kiselev‡

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Cauchy problems for the Shrödinger equation with delta potential and various initial data attract attention both of physicists and mathematicians interested in diffraction phenomena that occur in the presence of sharp singularities of the refractive index (see [1, 2, 3]). Developing our research started in [4], we address the following Cauchy problem:

$$2ik\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} + 2k\eta\delta(y)u = 0, \qquad x > 0,\ -\infty < y < \infty, \tag{1}$$

$$u|_{x=0} = e^{-iky\alpha}. \tag{2}$$

Here, δ is the Dirac delta function, k and α are positive constants, $k > 0$, $\alpha > 0$, and the parameter η is an arbitrary but nonzero complex number, $\eta \in \mathbb{C} \setminus \{0\}$. The function $u = u(x, y)$ is assumed to be in $C^1((0, +\infty)) \cap C([0, +\infty))$ with respect to x and in $C^2((-\infty, 0) \cup (0, +\infty)) \cap C((-\infty, +\infty))$ with respect to y.

From the viewpoint of the heuristic parabolic-equation method (see, e. g., [5]), the problem (1), (2) can be considered as an approximation of diffraction of a plane wave almost tangentially incident on a screen. We construct an explicit solution of the problem (1), (2) with the help of integral transformations and carry out a comprehensive asymptotic analysis of the solution for a wide range of the parameters η and α. The results are compared with those found in diffraction problems. Under certain assumptions, the constructed solution of (1), (2) is shown to give an asymptotic solution for the Helmholtz equation with the same delta potential.

The work was supported by the Russian Science Foundation (project 22-11-00070).

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