Goncharov depth conjecture and volumes of orthoschemes

Plan:

1) Polylogarithms and Mixed Tate Motives.

2) Orthoschemes

3) Formula for QLi and volume.

Periods of mixed Tate motives

are numbers $\int_\gamma \omega \in \mathbb{C}$, where

ω is an algebraic differential form

$[\omega] \in H^\bullet_{dR}(\underline{\underline{X}} \setminus \underline{\underline{D_1}})$

γ is a cycle $[\gamma] \in H_{\bullet, B}(X, D_2)$

so that $H^\bullet(X \setminus D_1, D_2 \setminus (D_1 \cap D_2))$ carries

a mixed Hodge structure of mixed

Tate type.

Examples: 1) $\underline{2\pi i = \int_{\circlearrowleft} \frac{dt}{t}}$

2) $\underline{\log(a) = \int_1^a \frac{dt}{t}}$

$\checkmark \left[\frac{dt}{t}\right] \in H^1(\mathbb{P}^1 \setminus \{0, \infty\})$

$\checkmark \left[{}_1\!\sim\!{}^a\right] \in H_1(\mathbb{P}^1, \{1, a\})$

and $\underbrace{H^1(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, a\})}$ is mixed Tate.

3) $Li_2(a) = -\int_0^a \int_0^{t_2} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2}$ "="

"=" $\sum_{m>0} \frac{a^m}{m^2}$ for $|a|<1$,

4) Classical polylogarithms

$$Li_n(a) = \sum_{m>0} \frac{a^m}{m^n}, \quad |a|<1.$$

5) Iterated integrals on $\mathbb{P}^1$ (also called Hyperlogarithms)

$I(a_0; a_1, \dots, a_n; a_{n+1}) =$

$$\int_{a_0}^{a_{m+1}} \frac{dt}{t-a_1} \circ \frac{dt}{t-a_2} \circ \dots \circ \frac{dt}{t-a_m} =$$

$$\int\limits_{a_0 \le t_1 \le t_2 < t_3 < \dots < t_m \le a_{m+1}} \frac{dt_1}{t_1 - a_1} \wedge \frac{dt_2}{t_2 - a_2} \wedge \dots \wedge \frac{dt_m}{t_m - a_m} .$$

6) Multiple polylogarithms - (introduced by Goncharov)

For $n_1, \dots, n_k \in \mathbb{N}$:

$$Li_{n_1, \dots, n_k}(a_1, \dots, a_k) = \sum_{m_1 > m_2 > \dots > m_k > 0} \frac{a_1^{m_1} a_2^{m_2} \dots a_k^{m_k}}{m_1^{n_1} m_2^{n_2} \dots m_k^{n_k}} .$$

$n_1 + \dots + n_k$ - weight

k - depth

Remark: Iterated integrals and multiple polylogarithms are related by a simple change of variable:

$$Li_{1,1}(a, b) = I(0; 1, a; ab).$$

Conjecture 1 (Universality)

Every period of mixed Tate motives is a linear combination of multiple polylogarithms.

Conjecture 2 (A part of "Depth Conjecture".)

Every multiple polylogarithm of weight n is a linear combination of multiple polylogarithm of depth $\left\lfloor \frac{n}{2} \right\rfloor$.

Example:

$Li_{1,1,1} \quad = \; Li_3(\;)$

$Li_4 \quad Li_{3,1} \;\; Li_4$

$$\underline{Li_{1,1}(a_1, a_2)} = Li_2\left(\frac{1-a_1}{1-a_2^{-1}}\right) - Li_2\left(\frac{a_2}{a_2-a_1}\right) - Li_2(a_1 a_2).$$

Remark: Full Depth conjecture gives a nessessery and sufficent condition for a multiple polylogarithm to be a linear combination of multiple polylogarithms of depth $\leq d$.

It would imply:

1) Zagier conjecture: special values of the zeta function of a number field is a sum of products of classical polylogarithms.

2) Volumes of hyperbolic manifolds are linear combinations of classical polylogarithms.

Goal: explain the proof of Conjecture 2.

Idea of the proof:

Let $\underline{x_0, x_1, \ldots, x_{2n+1}} \in \mathbb{P}^1$ be a collection of distinct points. We will construct a function

$$\underline{QLi_{n,k}(x_0, \ldots, x_{2n+1})} \leftarrow \text{period of mixed Tate motives.}$$

It has weight $\underline{n+k}$ and depth $\underline{n}$.

I) On one hand it is universal: every iterated integral (and thus MP) can be expressed via

$QLi_{n,n}$ and $QLi_{n,n+1}$.

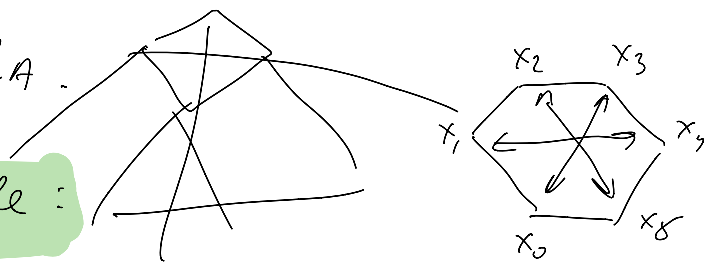
$QLi_{n,n}$: вес $2n$, глубина n

$QLi_{n,n+1}$: вес $2n+1$, глубина n.

II) On the other hand it can be expressed via multiple polylogarithms of depth $= \left\lfloor \frac{\text{weight}}{2} \right\rfloor$.

I) + II) imply conjecture 2.

Part II) is based on explicit formula.



$\mathcal{M}_{0,6}$

Example:

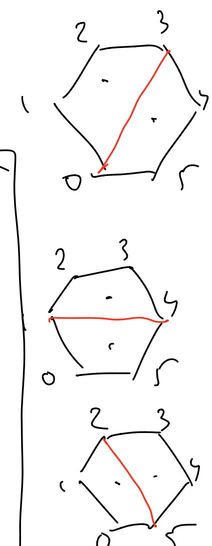
$\mathcal{P}'(\{x_0,\dots,x_5\})$.

$$QLi_{2,0}(x_0,\dots,x_5) =$$

$$= Li_{1,1}([x_0,x_3,x_4,x_5],[x_0,x_1,x_2,x_3])$$

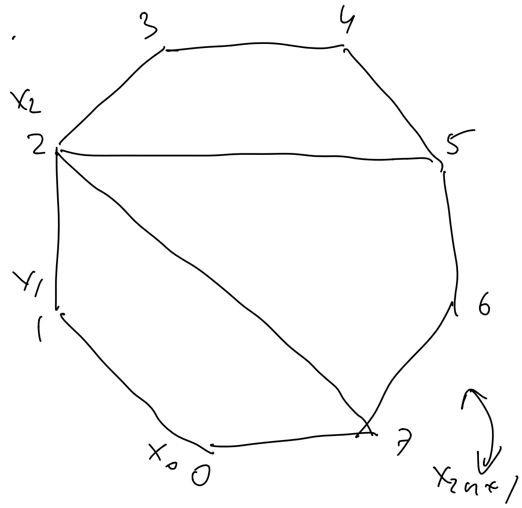
$$- Li_{1,1}([x_0,x_1,x_4,x_5],[x_1,x_2,x_3,x_4]^{-1})$$

$$+ Li_{1,1}([x_0,x_1,x_2,x_5],[x_2,x_3,x_4,x_5]).$$



In general, the formula for $QLi_{n,k}$ involves a summation over all "quadrangulations" of a polygon $x_0, x_1, \ldots x_{2n+1}$ of multiple polylogarithms, evaluated at cross-ratios of points in the squares.

$$\frac{(3n)!}{n!\,(2n+1)!}$$



$$Li_{1,1,1}\big(\underline{[0,1,2,7]}, \underline{[2,5,6,7]}, \underline{[2,3,4,5]}\big)$$

Goal: explain where it comes from.

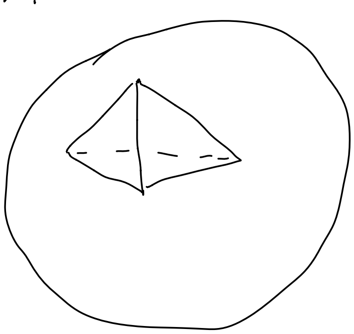
Orthoschemes

Origin: work of Gauss, Lobachevsky and Coxeter.

Volumes of non-Euclidean polytopes are examples of mixed Tate motives.

$\mathbb{H}^{2n-1} \supseteq T \leftarrow$ polytope in Klein's model.

$\mathbb{H}^3$ =



абсолют

Let $\underbrace{\mathbb{P}^{2n-1}}_{}$ = complexification of $\mathbb{R}^{2n-1} \supseteq \mathbb{H}^{2n-1}$

$\underbrace{Q}_{}$ = complexification of $\partial \mathbb{H}^{2n-1}$.

$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0.$

$\underbrace{H_0, \dots, H_{2n}}_{}$ = complexification of faces.

Easy to see:

Vol(T) is a period of

$$\underbrace{H^{2n-1}\left(\mathbb{P}^{2n-1} \setminus Q\,;\,(H_0\cup\dots\cup H_{2n+1})\setminus Q\right)}_{\substack{\| \\ H(T)}}$$

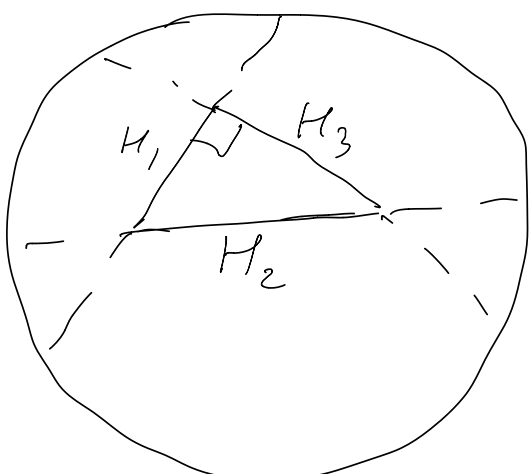
H(T) is a mixed Hodge structure of mixed Tate type (Goncharov).

$\Rightarrow$ Vol(T) should be a linear combination of multiple polylogs.

This was shown by Lobachevsky for $T \subseteq \mathbb{H}^3$.

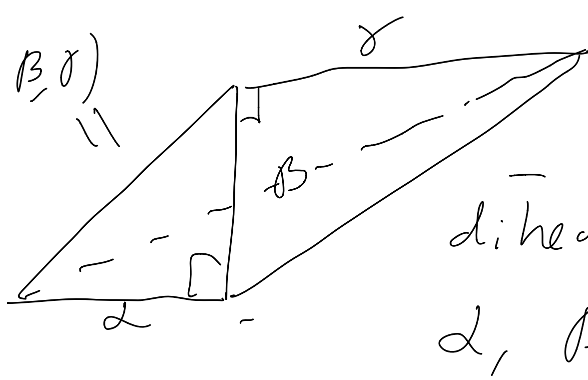
Orthoscheme: is a simplex in $\mathbb{H}^{n-1}$ with faces $H_1, \dots, H_n$ such that $H_i \perp H_j$ for $|i-j|>1$.

Examples: 1) Right triangle in $\mathbb{H}^2$



2) in $\mathbb{H}^3$

$T(\alpha, \beta, \gamma) =$



has three dihedral angles $(\neq \frac{\pi}{2})$: α, β, γ.

Formula of Lobachevsky:

Let $\delta = \operatorname{arctg}\left(\dfrac{\sqrt{\cos^2\beta - \sin^2\alpha\,\sin^2\gamma}}{\cos\alpha\,\cos\gamma}\right)$.

$$\operatorname{Vol}(T(\alpha,\beta,\gamma)) =$$

$$= \frac{1}{8}\Big[Li_2\left(e^{2i(\alpha+\delta)}\right) - Li_2\left(e^{2i(\alpha-\delta)}\right) + Li_2\left(e^{2i(\gamma+\delta)}\right) - Li_2\left(e^{2i(\gamma-\delta)}\right) - Li_2\left(-e^{2i(\delta-\beta)}\right) + Li_2\left(-e^{2i(\delta+\beta)}\right) + 2Li_2\left(-e^{-2i\delta}\right) \Big].$$

Every polytope in $\mathbb{H}^3$ can be disected into orthoschemes $\Rightarrow$ volume of a hyperbolic polytope is a sum of dilogarithms.

What happens in higher dimensions?

(Forgotten) theorem of Böhm

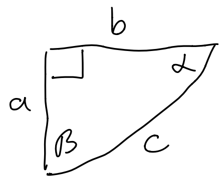
One can extract from works of Böhm that volumes of hyperbolic polytopes are linear combinations of multiple polylogarithms, but his proof is not very explicit (and > 70 pages).

My goal is to give an explicit formula for volume of an orthoscheme in an arbitrary dimension.

A slight modification of it gives a formula for $QLi_{n,k}$.

Orthoschemes and configurations of points.

Consider a right triangle in S^2



with angles α, β, and sides a, b, c.

Thm. (Gauss) $\exists$ a configuration of 5 points on $\mathbb{P}^1$ $(x_1, \dots, x_5)$ such that:

$$\underline{ctg^2(a)} = [x_1\ x_2\ x_3\ x_4]$$
$$ctg^2\left(\tfrac{\pi}{2} - \beta\right) = [x_2\ x_3\ x_4\ x_5]$$
$$ctg^2\left(\tfrac{\pi}{2} - c\right) = [x_3\ x_4\ x_5\ x_1]$$
$$ctg^2\left(\tfrac{\pi}{2} - \alpha\right) = [x_4\ x_5\ x_1\ x_2]$$
$$ctg^2(b) = [x_5\ x_1\ x_2\ x_3]$$

|| $ctg^2(a)$, $tg^2(a)$, $\cos^2(a)$, $\sin^2(a)$, $\sec^2(a)$, $\operatorname{cosec}^2(a)$,

$x_1 \dots x_5 \in \mathbb{P}^1$

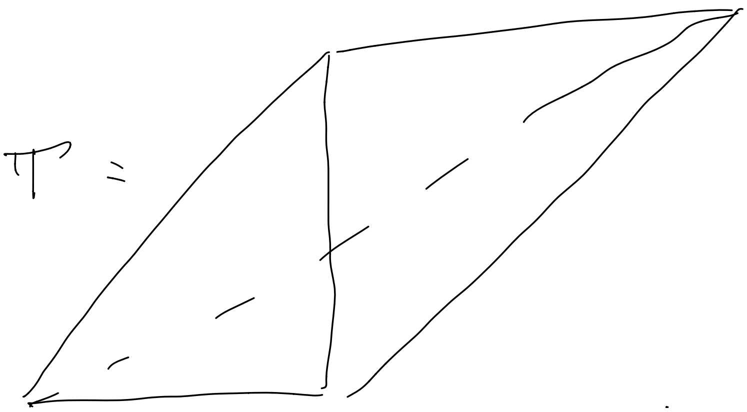
$$[a, b, c, d] = z$$

$$\boxed{z,\ \tfrac{1}{z},\ 1-z,\ \tfrac{1}{1-z},\ \tfrac{z}{z-1},\ 1-\tfrac{1}{z}.}$$

So we have a "triangle" over each point of $M_{0,5}$.

Coxeter generalized it to an arbitrary dimension.

Faces of an orthoscheme are also orthoschemes.



T =

Orthoscheme T has $\binom{n+2}{4}$ interesting "parameters": lengths of edges, dihedral angles, other angles: dihedral angles of its faces.

These are periods of $H(T)$ coming from extension

$$0 \to \mathbb{Q}(-k) \to ? \to \mathbb{Q}(-k-1) \to 0.$$

Theorem (Coxeter)

For every orthoscheme there exists a unique configuration of points $(x_0 \dots x_{n+1}) \in \mathcal{M}_{0,n+2}$ such that for every parameter α_i = there exist four points such that $i \in \{1 \dots \binom{n+2}{4}\}$

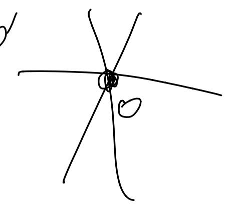
$$dg^2 \alpha_i = [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}]$$

So, we get a variation of mixed Tate motives over $\mathcal{M}_{0,n+2}$.

Construction: (Maslov index).

Consider $L_0, \dots, L_{n+1}$ – lines in $\mathbb{C}^2 = V$.

$L_0 = x_0, \dots, L_{n+1} = x_{n+1} \in \mathbb{CP}^1$



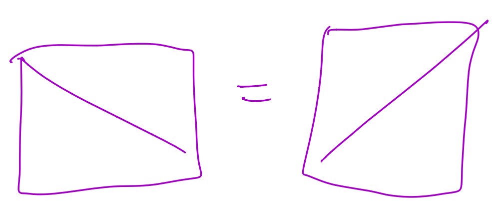
Let

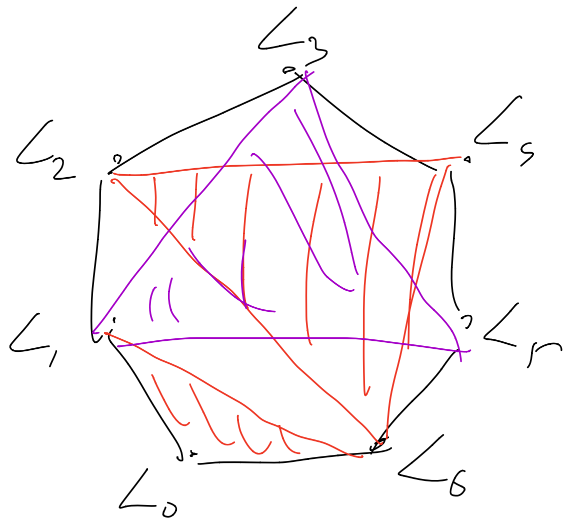
$$\underset{\boxed{n}}{E} = \operatorname{Ker}\Big(\underbrace{\bigoplus_{i=0}^{n+1} L_i \to V}_{\text{space of linear dependencies.}} \Big) -$$

Choose an "area" form $\omega \in \Lambda^2 V^*$.

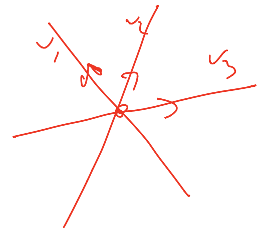
Then E carries a quadratic form:

$$q\big(\underset{\overset{\cap}{E}}{(v_0, \dots, v_{n+1})}\big) = \sum_{0 \le i < j \le n+1}' \omega(v_i, v_j).$$





E.



$v_1+v_2+v_3=0$

Every "triangle" $L_iL_jL_k$ defines a line $\underbrace{E_{ijk}}_{=\text{"linear dependence between } v_i, v_j, v_k\text{"}} \subseteq E$.

Lines E_{ijk} are orthogonal $\iff$ triangles don't intersect.

Every triangulation of a polygon $\longrightarrow$ an orthogonal decomposition of E_0

Configurations of points

$\rightsquigarrow$ ort.

Orthoschemes

$(x_0, \ldots, x_{n+1})$

$\rightsquigarrow$

$(L_0, \ldots, L_{n+1})$

$\rightsquigarrow$

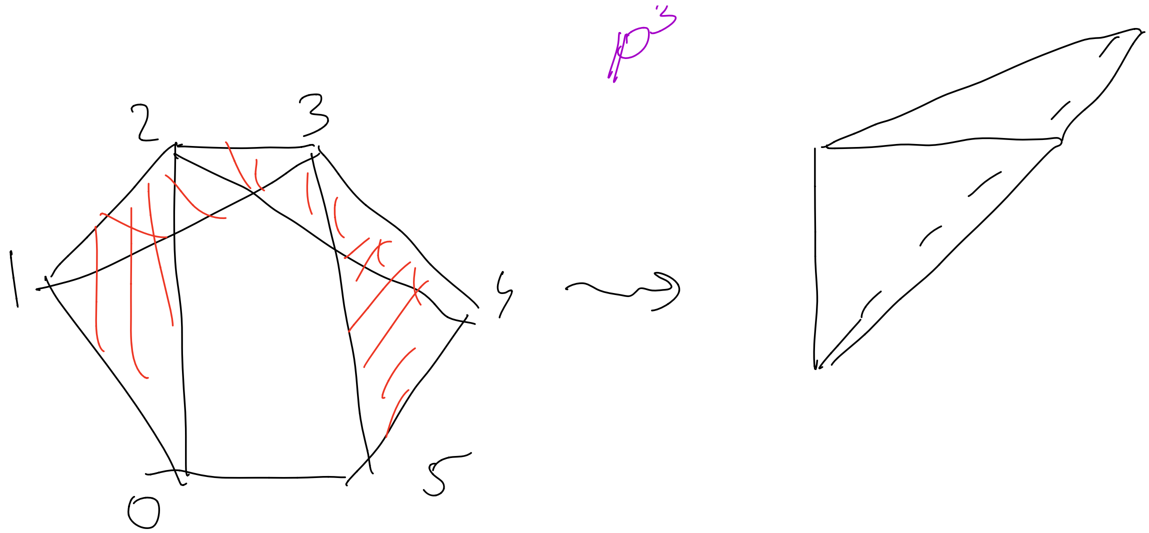
triple: $\underline{\underline{\mathbb{P}[E^\vee]}} = \mathbb{P}^{n-1}$, $\underline{\underline{\text{quadric } Q}} = \{q = 0\}$

hyperplanes

$H_1 = E^\vee_{012}$, $H_2 = E^\vee_{123}, \ldots, H_n = E^\vee_{n-1,n,n+1}$.

$\uparrow$

"ears"



Our goal is to understand motive

$H(T)$, where T is an orthoscheme

$$\underbrace{\mathrm{ort}((x_0, \dots, x_{2n+1}))}_{\mathcal{M}_{0,2n+2}}$$

Example: $(x_0, x_1, x_2, x_3) \in \mathcal{M}_{0,4}$

$$\mathrm{Vol}\left(\mathrm{ort}\left(x_0, x_1, x_2, x_3\right)\right) =$$

$$= \log\left(\frac{1+\sqrt{[x_0,x_1,x_2,x_3]}}{1-\sqrt{[x_0,x_1,x_2,x_3]}}\right)$$

Correction: $\mathrm{Vol}(\mathrm{ort}(T))$ is a period of a variation of mixed Tate motives over a finite cover of $\mathcal{M}_{0,2n+2}$: $\mathcal{M}^{s}_{0,2n+2}$, where square roots of cross-ratios are defined.

Consider (finally!) the abelian category of mixed Tate motives over F.

|| For F - number field it exists and for other fields is conjectured.

$\mathcal{M}_T^F$ is a Tannakian category,

$\mathcal{H}_\bullet^F$ - (graded) Hopf algebra of framed objects,

$\mathcal{L}_\bullet^F$ = Lie coalgebra of indecomposibles:

$$\mathcal{L} = \mathcal{H} / \mathcal{H}_{>0} \cdot \mathcal{H}_{>0}$$

· Elements of $\mathcal{H}_F$ are

"periods of motives of mixed Tate type".

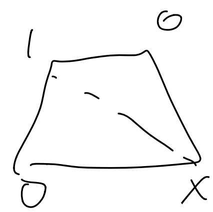
The main ingredient we need is the coproduct.

$$\Delta : \mathcal{L} \to \Lambda^2 \mathcal{L}.$$

Examples:

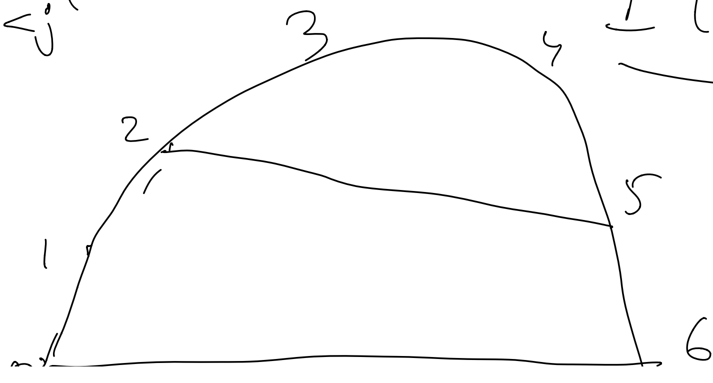
1) $$\Delta\big(Li_2(x)\big) = \log(x) \wedge \log(1-x),$$

$$\| \qquad I(0;1,0;x)$$



2) $$\Delta\Big(I(a_0;a_1,\dots,a_m;a_{m+1})\Big) =$$

$$= \sum_{i<j} I(a_0,a_1,\dots,a_i,a_j,\dots,a_{m+1}) \wedge I(a_i;a_{i+1},\dots,a_j)$$



$$\boxed{\begin{array}{c} i=2 \\ j=5 \end{array}}$$

Quite surprisingly,
the coproduct for
$H(\mathrm{ort}(x_0, \ldots, x_{2n+1}))$ is given
by the same formula!

But the motives are very different:

$$\underline{\underline{I(x_0, x_1, x_2) = \log\left(\frac{x_1 - x_2}{x_1 - x_0}\right)}}$$

vs

$$\underline{\underline{\mathrm{ort}(x_0, x_1, x_2) = 0}},$$

$$\underline{\underline{I(x_0, x_1, x_2, x_3) = Li_{1,1}\left(\frac{x_2 - x_0}{x_1 - x_0}, \frac{x_3 - x_0}{x_2 - x_0}\right)}}$$

vs

$$\underline{\underline{\mathrm{ort}(x_0, x_1, x_2, x_3) = \log\left(\frac{1 + \sqrt{[x_0 x_1 x_2 x_3]}}{1 - \sqrt{[x_0, x_1, x_2, x_3]}}\right)}}$$

And so on.

The goal is to find a linear combination of multiple polylogarithms with coproduct matching that for orthoschemes.

Example: ($\mathbb{H}^3$, compare to Lobachevsky)

$$\mathrm{Vol}(\mathrm{ort}(x_0, x_1, x_2, x_3, x_4, x_5)) =$$

$$= \sum_{\substack{\text{signs of} \\ \text{square} \\ \text{roots}}} \pm \Big(\angle_{i,1}\big(\sqrt{[0,3,4,5]}, \sqrt{[0,1,2,3]}\big) - \angle_{i,1}\big(\sqrt{[0,1,4,5]}, \sqrt{[0,2,3,4]}\big) + \angle_{i,1}\big(\sqrt{[0,1,2,5]}, \sqrt{[2,3,4,5]}\big) \Big).$$

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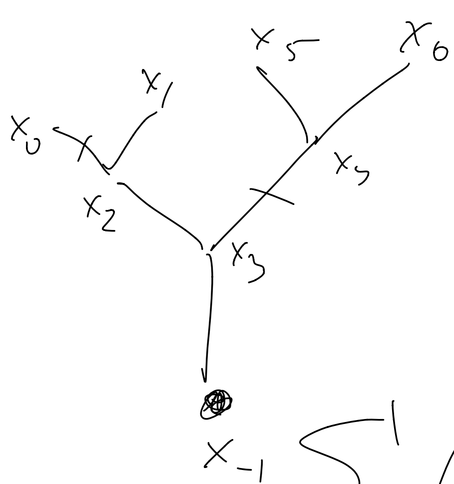
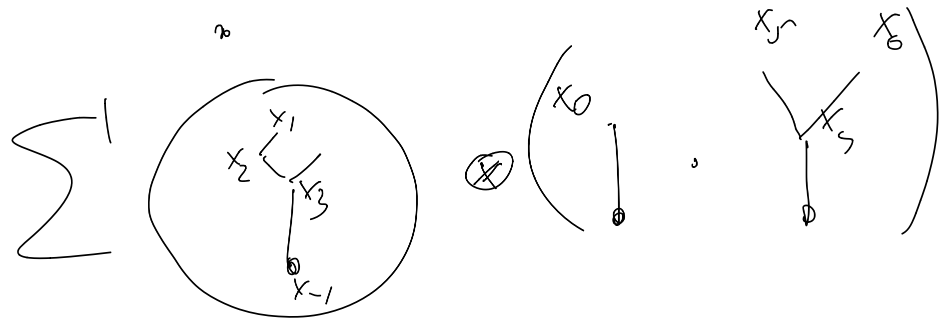
Formula for a general orthoscheme.

Consider <u>Connes-Kreimer Hopf algebra</u> of rooted trees $\mathcal{T}^F$ labelled by $x_i \in F$.



Coproduct is a sum over cuts.



One can define a map $\underline{Li}$

$\underline{\underline{\mathcal{T}^F}}$ to the $\underline{\text{Hopf algebra}}$ of $\underline{\text{mixed Tate motives}}$.

$$\overset{x}{\bullet} \xrightarrow{\underline{Li}} Li_1(x) = -\underline{\log(1-x)}.$$

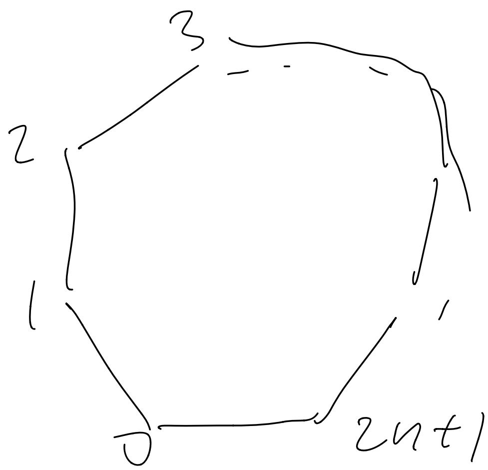
$$\begin{array}{c} y\,\bullet \\ | \\ x\,\bullet \end{array} \xrightarrow{\underline{Li}} Li_{1,1}(x,y)$$

etc., using the universal property of

$\mathcal{T}^F$.

$x_0, \ldots, x_{2n+1} \in \mathbb{P}^1$.

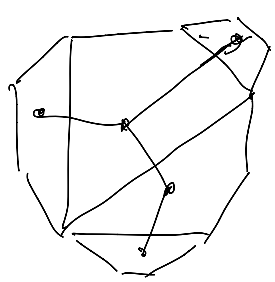
Consider a polygon



$= P$.

Let $Q(P)$ - set of quadrangulations.

$D \in Q(P) \rightsquigarrow t_D$ - dual tree of the quadrangulation



marked by
square roots
of cross ratios.

Then

$$\boxed{\mathrm{Vol}(z_D) = \sum_{\substack{\text{signs} \\ \text{of} \\ \text{square} \\ \text{roots}}} \pm \Big(\sum_{\underbrace{D \in Q(P)}} \angle_i (\underbrace{z_D}) \Big)}$$

If one takes away the signs and uses cross-ratios instead of their square roots, the same formula gives a formula for $QL_{n,k}$, which is used in the proof of the depth conjecture.