Mixed Volumes of Convex Hulls of Random Processes

Artyom Bolotin

Leonhard Euler International Mathematical Institute
St. Petersburg Department of Steklov Mathematical Institute

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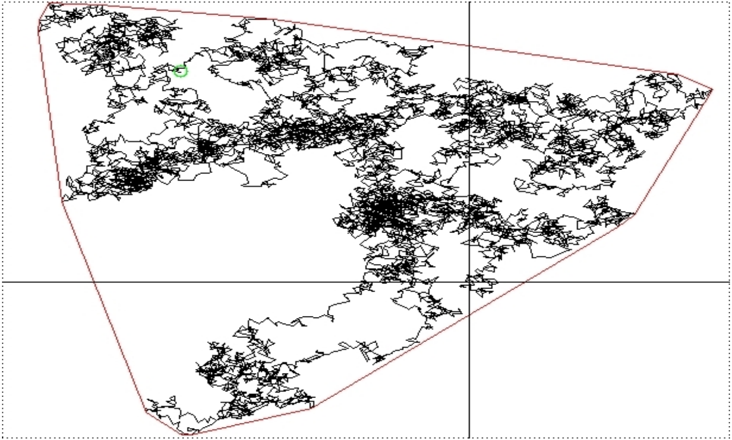
Problem Statement



From: Satya N. Majumdar, Alain Comtet, Julien Randon-Furling. "Random Convex Hulls and Extreme Value Statistics".

Intrinsic Volumes

Let $K \subset \mathbb{R}^d$ be a convex compact set. We denote the d-dimensional volume of the convex compact set K as $\mathrm{Vol}_d(K)$.

Theorem (Steiner)

Let B^d be the d-dimensional unit ball, $\lambda \geq 0$. Then

$$\mathrm{Vol}_d(K + \lambda B^d) = \sum_{k=0}^{d} \kappa_{d-k} V_k(K) \lambda^{d-k},$$

where $\kappa_k = \mathrm{Vol}_k(B^k) = \frac{\pi^{k/2}}{\Gamma(k/2+1)}$.

Definition

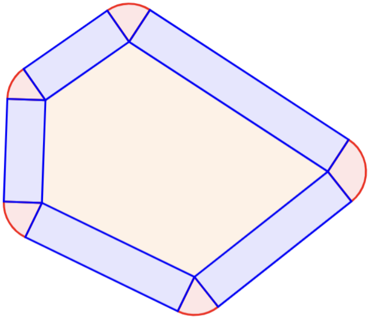
The coefficients $V_k(K)$ are called the *intrinsic volumes* of the convex compact set K.

For example, in the two-dimensional case, when K is a polygon, we have

$$\mathrm{Vol}_2(K + \lambda B^2) = \pi V_0(K)\lambda^2 + 2V_1(K)\lambda + V_2(K).$$

Hence $V_0(K) = 1$, $V_1(K)$ is half the perimeter of K, $V_2(K) = \mathrm{Vol}_2(K)$.

In the general case, it can be shown that $V_0(K) = 1$, $V_1(K)$ is the mean width of K up to a constant factor, $V_{d-1}(K)$ is half the surface area, $V_d(K) = \mathrm{Vol}_d(K)$.



Mixed Volumes

Theorem (Minkowski)

Let $K_1, K_2, ..., K_s$ be convex compact sets in $\mathbb{R}^d$. Then for $\lambda_1, \lambda_2, ..., \lambda_s \geq 0$, the function $\mathrm{Vol}_d(\lambda_1 K_1 + \lambda_2 K_2 + ... + \lambda_s K_s)$ is a homogeneous polynomial of degree d:

$$\mathrm{Vol}_d(\lambda_1 K_1 + \lambda_2 K_2 + ... + \lambda_s K_s) = \sum_{i_1=1}^{s} ... \sum_{i_d=1}^{s} \lambda_{i_1} ... \lambda_{i_d} V_d(K_{i_1}, ..., K_{i_d}),$$

where the functions $V_d(K_{i_1}, ..., K_{i_d})$ are symmetric.

Definition

The coefficient $V_d(K_{i_1}, ..., K_{i_d})$ is called the *mixed volume* of the convex compact sets $K_{i_1}, ..., K_{i_d}$.

For brevity, $V_d(K_1[m_1], ..., K_k[m_k]) := V_d(\underbrace{K_1, ..., K_1}_{m_1 \text{ times}}, ..., \underbrace{K_k, ..., K_k}_{m_k \text{ times}})$.

Relation between Mixed and Intrinsic Volumes

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$$\mathrm{Vol}_d(K + \lambda B^d) = \sum_{k=0}^{d} \binom{d}{k} V_d(K[k], B^d[d-k]) \lambda^{d-k}.$$

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Comparing the coefficients in the polynomials, we obtain

$$V_k(K) = \frac{\binom{d}{k}}{\kappa_{d-k}} V_d(K[k], B^d[d-k]).$$

Random Walks

Let $X_1, X_2, ...$ be random vectors in $\mathbb{R}^d$, $S_n := X_1 + ... + X_n$ be a random walk in $\mathbb{R}^d$. We will consider walks satisfying the next assumption.

The increments are exchangeable, i.e., for any permutation σ we have

$$(X_1, X_2, ..., X_n) \stackrel{d}{=} (X_{\sigma(1)}, X_{\sigma(2)}, ..., X_{\sigma(n)}).$$

Theorem (Vysotsky, Zaporozhets, 2017)

Let S_n be a random walk with exchangeable increments and $\mathbb{P}(X_1 \in h) = 0$ for any affine hyperplane h. Denote by $C_n := \operatorname{conv}\{S_0, S_1, \dots, S_n\}$ the convex hull of the first n steps, including the origin $S_0 := 0$. Then

$$\mathbb{E} V_k(C_n) = \frac{1}{k!} \sum_{\substack{j_1 + \dots + j_k \le n \\ j_1, \dots, j_k \ge 1}} \frac{\mathbb{E} \det^{1/2} \left(\langle S^{(m)}_{j_m}, S^{(l)}_{j_l} \rangle \right)^k_{m,l=1}}{j_1 \cdot \dots \cdot j_k}, \quad k = 1, \dots, d,$$

$$\text{In particular, } \mathbb{E} V_d(C_n) = \frac{1}{d!} \sum_{\substack{j_1 + \dots + j_d \le n \\ j_1, \dots, j_d \ge 1}} \frac{\mathbb{E} \left| \det \left[S^{(1)}_{j_1}, \dots, S^{(d)}_{j_d} \right] \right|}{j_1 \cdot \dots \cdot j_d},$$

where $S^{(1)}_n, \dots, S^{(d)}_n$ are independent copies of the walk S_n.

Theorem (B., 2025)

Let $\{S_{n_i,i}\}_{i=1}^k$ be independent random walks in $\mathbb{R}^d$ with exchangeable increments, $C_i = \mathrm{conv}\{S_{0,i}, S_{1,i}, ..., S_{n_i,i}\}$ be their convex hulls. Then

$$\mathbb{E} V_d\left(C_1[m_1], ..., C_k[m_k]\right) =$$

$$= \frac{1}{d!} \sum \frac{\mathbb{E}\left|\det\left[..., S_{j_1^{(i)},i}, S_{j_2^{(i)},i} - S_{j_1^{(i)},i}, ..., S_{j_{m_i}^{(i)},i} - S_{j_{m_i-1}^{(i)},i}, ...\right]\right|}{\prod_{i=1}^k j_1^{(i)} \cdot \left(j_2^{(i)} - j_1^{(i)}\right) \cdot ... \cdot \left(j_{m_i}^{(i)} - j_{m_i-1}^{(i)}\right)},$$

where $m_1 + ... + m_k = d$ and the summation is over all sets of indices

$$1 \le j_1^{(i)} < ... < j_{m_i}^{(i)} \le n_i.$$

Lévy Processes

Recall that a random process $X(t)$, $t \geq 0$ is called a Lévy process if

- $X(0) = 0$ almost surely;
- for any $0 \leq t_1 < t_2 < \ldots < t_n$ the increments $X(t_2) - X(t_1)$, $X(t_3) - X(t_2)$, ..., $X(t_n) - X(t_{n-1})$ are independent;
- $X(t+s) - X(t) \stackrel{d}{=} X(s)$ for any $t, s \geq 0$;
- $\forall \delta > 0$ and $t \geq 0$, $\lim\limits_{\varepsilon \to 0} \mathbb{P}(\|X(t+\varepsilon) - X(t)\| > \delta) = 0$ holds.

A process is symmetric if $X(t) \stackrel{d}{=} -X(t)$ for any $t > 0$.

A symmetric Lévy process is called α-stable if $X(t) \stackrel{d}{=} t^{1/\alpha} X(1)$ for any $t > 0$.

Theorem (Molchanov, Wespi, 2016)

Let $X = X(t)$, $t \geq 0$ be a symmetric α-stable Lévy process in $\mathbb{R}^d$, where $\alpha > 1$. Denote the closure of the convex hull of the process by

$$Z \coloneqq \mathrm{cl}\,\mathrm{conv}\{X(t), 0 \leq t \leq 1\}.$$

Then

$$\mathbb{E} V_k(Z) = \frac{\Gamma(1/\alpha)^k \Gamma(1-1/\alpha)^k}{\pi^k \Gamma(k/\alpha+1)} V_k(K), \quad k = 1, ..., d,$$

where K is an associated zonoid of $X(1)$.

Lévy Processes, Mixed Volumes

Theorem (B., 2025)

Let $\{X_i\}_{i=1}^k$ be symmetric independent α_i-stable Lévy processes in $\mathbb{R}^d$, $\alpha_i > 1$, $Z_i = \text{cl conv}\{X_i(t), 0 \le t \le 1\}$ be the closures of their convex hulls. Then

$$\mathbb{E} V_d(Z_1[m_1], ..., Z_k[m_k]) =$$

$$= \frac{V_d(K_1[m_1], ..., K_k[m_k])}{\pi^d} \prod_{i=1}^{k} \frac{\Gamma(1/\alpha_i)^{m_i} \Gamma(1 - 1/\alpha_i)^{m_i}}{\Gamma(m_i/\alpha_i + 1)},$$

where K_i are associated zonoids of $X_i(1)$ respectively, $m_1 + ... + m_k = d$.

Example. Let $\{X_i\}_{i=1}^{k}$ be independent Brownian motions. Then $\alpha_i = 2$ and $K_i = \frac{1}{\sqrt{2}} B^d$. Thus,

$$\mathbb{E} V_d(Z_1[m_1], ..., Z_k[m_k]) = \frac{\kappa_d}{2^{d/2}} \prod_{i=1}^{k} \frac{1}{\Gamma(m_i/2+1)}.$$

Wiener Spirals, Mixed Volumes

Theorem (Dospolova, 2022)

Let $S_i = \{\mathbb{1}_{[i-1,i-1+t]}(\cdot) : t \in [0,1]\}$, $i = 1, ..., k$ be k orthogonal Wiener spirals in $L^2[0,k]$. Define $\bar{S}_i = \text{cl conv } S_i$, then

$$V_d(\bar{S}_1[m_1], ..., \bar{S}_k[m_k]) = \frac{(2\pi)^{d/2}}{d!\kappa_d}\mathbb{E}V_d(Z_1[m_1], ..., Z_k[m_k]).$$

Here $Z_1, ..., Z_k$ are closed convex hulls of independent Brownian motions.

Using example from previous slide, one can obtain

$$V_d(\bar{S}_1[m_1], ..., \bar{S}_k[m_k]) = \frac{\pi^{d/2}}{d!}\prod_{i=1}^{k}\frac{1}{\Gamma(m_i/2+1)}.$$

Properties of Mixed Volumes

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- Additivity: if $A, B, A \cup B$ are nonempty convex compact sets, then

$$V_d(A \cup B, ..., A \cup B, K_{i+1}, ..., K_d) = V_d(A, ..., A, K_{i+1}, ..., K_d) +$$
$$+ V_d(B, ..., B, K_{i+1}, ..., K_d) - V_d(A \cap B, ..., A \cap B, K_{i+1}, ..., K_d).$$

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- Multilinearity:

$$V_d(A + B, K_2, ..., K_d) = V_d(A, K_2, ..., K_d) + V_d(B, K_2, ..., K_d).$$

Proof Sketch

We will assume that the number of steps in each of the walks is not greater than the dimension of the space and that steps are linearly independent with probability one. The general case reduces to this one.

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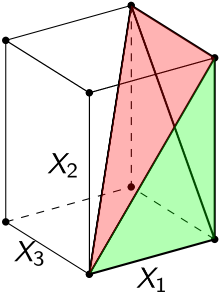
Let $X_1, ..., X_n$ be the steps of the k-th walk. Consider $P = [0, X_1] + ... + [0, X_n]$. Then $P = \bigcup_{\sigma \in S_n} K_\sigma$, where

$K_\sigma = \text{conv}\{0, X_{\sigma(1)}, X_{\sigma(1)} + X_{\sigma(2)}, ..., X_{\sigma(1)} + ... + X_{\sigma(n)}\}$.



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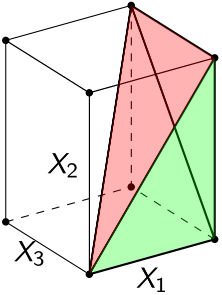

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$$K_\sigma = \text{conv}\{0, X_{\sigma(1)}, X_{\sigma(1)} + X_{\sigma(2)}, ..., X_{\sigma(1)} + ... + X_{\sigma(n)}\}.$$

Note that $K_{id} = C_k$ is the convex hull of the walk. Due to the exchangeability of increments, the mathematical expectations of the mixed volumes with K_σ coincide with the desired mixed volume.

Proof Sketch

To find the mixed volume

$$\mathbb{E}V_d(C_1[m_1], ..., C_{k-1}[m_{k-1}], P[m_k]), \tag{1}$$

we use the fact that $P = [0, X_1] + ... + [0, X_n]$ and the multilinearity of the mixed volume. Thus, computing (1) reduces to computing the mixed volume with segments.

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On the other hand, using the additivity of the mixed volume, we can write an inclusion-exclusion formula and express (1) in terms of functions of the form

$$\mathbb{E}V_d\left(C_1[m_1], ..., C_{k-1}[m_{k-1}], \bigcap_{\sigma \in I} K_\sigma[m_k]\right).$$

Proof Sketch

Let $\pi = \{B_1, ..., B_r\} \in \Pi_n$ be some partition of $\{1, ..., n\}$.
Say $X_{B_i} = \sum_{j \in B_i} X_j$ and consider

$$P_\pi = [0, X_{B_1}] + ... + [0, X_{B_r}] .$$

Introduce

$$E(K) := \mathbb{E} V_d \left(C_1[m_1], ..., C_{k-1}[m_{k-1}], K[m_k] \right);$$
$$E(\pi) := E(P_\pi);$$
$$s(\pi) := \sum_{C \in \mathcal{C}_\pi} E(C),$$

where $\mathcal{C}_\pi$ is the set of chambers of P_π, i.e., sets of the form

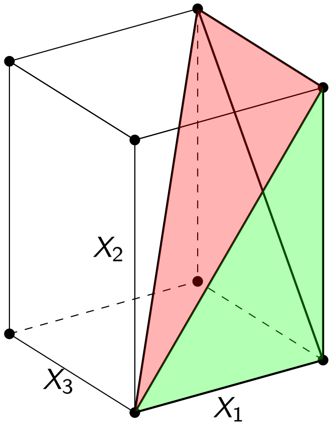
$$\mathrm{conv}\{0, X_{B_{\sigma(1)}}, ..., X_{B_{\sigma(1)}} + ... + X_{B_{\sigma(r)}}\}.$$

Proof Sketch

In particular, due to the exchangeability of increments

$$s(\pi_0) = n!\mathbb{E}V_d(C_1[m_1], ..., C_{k-1}[m_{k-1}], C_k[m_k]) = n!E(C_k),$$

where $\pi_0 = \{\{1\}, ..., \{n\}\}$ is the finest partition.



Proof Sketch

It can be shown that

$$E(\pi) = \sum_{\rho \geq \pi} (-1)^{|\pi| - |\rho|} s(\rho).$$

And vice versa, s can be expressed in terms of E:

$$n! E(C_k) = s(\pi_0) = \sum_{\rho \in \Pi_n} E(\rho) \prod_{B \in \rho} (|B| - 1)!.$$

From this follows the expression for $E(C_k)$:

$$\sum_{1 \leq j_1 < \ldots < j_m \leq n} \frac{\mathbb{E} V_d \left(C_1[m_1], \ldots, C_{k-1}[m_{k-1}], [0, S_{j_1}], \ldots, [0, S_{j_m} - S_{j_{m-1}}] \right)}{j_1 \cdot (j_2 - j_1) \cdot \ldots \cdot (j_m - j_{m-1})}.$$

Carrying out similar reasoning for $C_1, \ldots, C_{k-1}$, we obtain the statement of the theorem.

Thank you for your attention!