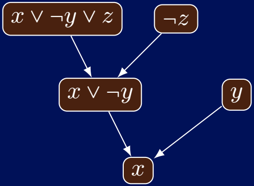
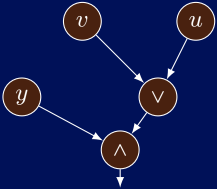
A Few Words About the Proof Complexity





Dmitry Sokolov

SACC 2021
May 27

St Petersburg University

PDMI
RAS

Proof Systems

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Definition[Cook, Reckhow 79]

Proof system for $L \Leftrightarrow$ poly-time algorithm Π: $\{0,1\}^* \times \{0,1\}^* \to \{0,1\}$:

- (completeness) $x \in L \Rightarrow \exists w\ \Pi(x,w) = 1$;
- (soundness) $\exists w\ \Pi(x,w) = 1 \Rightarrow x \in L$.

Length of $|w|$ is the complexity measure.

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Cook's Program

Prove superpolynomial lower bounds for stronger and stronger proof systems until the techniques are developed to do it in a general case.
Goal: $\mathbf{NP} \neq \mathbf{coNP}$.

Proof Systems

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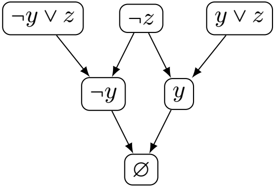

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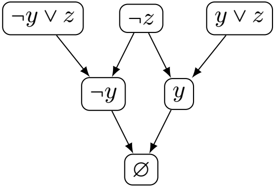

Cutting Planes: proof is a sequence of inequalities over $\mathbb{Z}$ $(p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, \ldots, p_\ell \geq 0)$:

- p_i is an encoding of $C \in \varphi$, $x_k \geq 0$ or $-x_k + 1 \geq 0$;
- $\frac{p_i \quad p_j}{p_k}$, $(p_i \geq 0) \wedge (p_j \geq 0)$ imply $(p_k \geq 0)$ over $\mathbb{Z}^n$;
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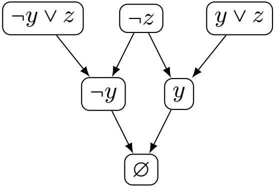

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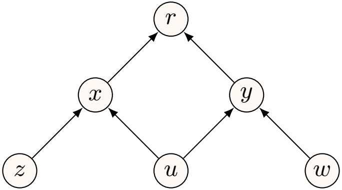
Nullstellensatz: proof of a system of polynomial equalities $f_1 = 0, f_2 = 0, \dots$:

$$\sum_{u=1}^{a} p_u f_u = 1.$$

Pebbing



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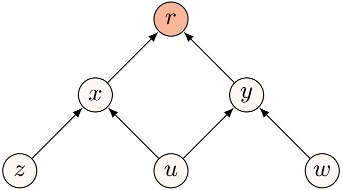

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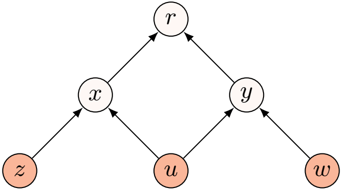

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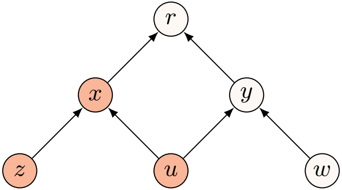

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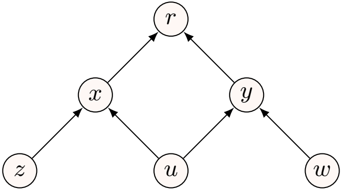

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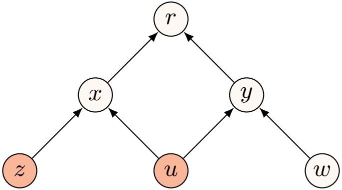

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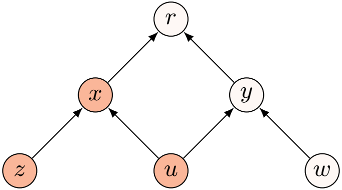

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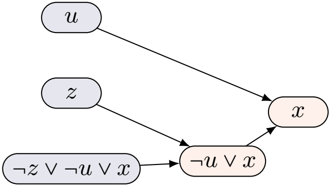

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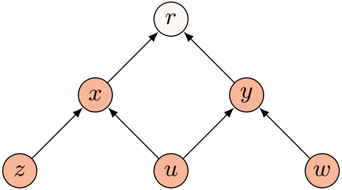

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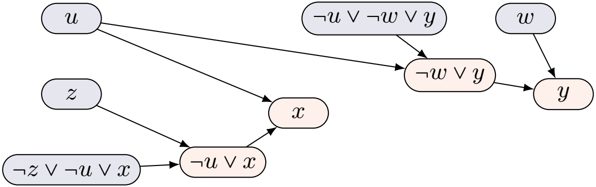

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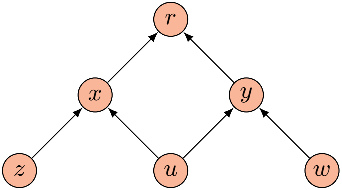

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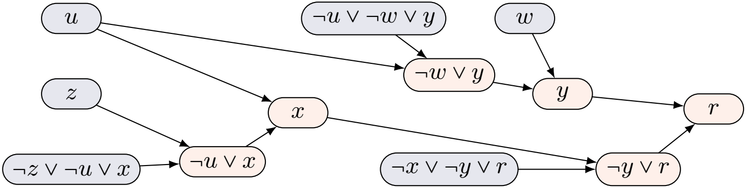

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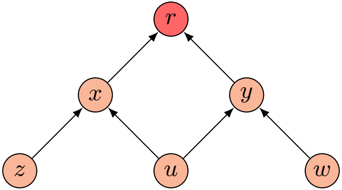

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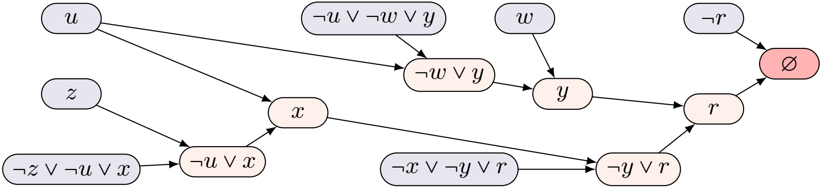

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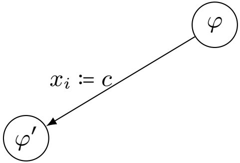

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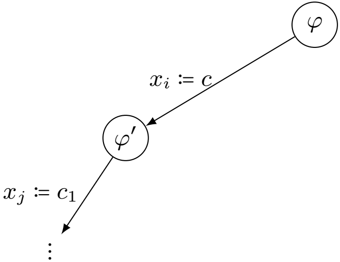

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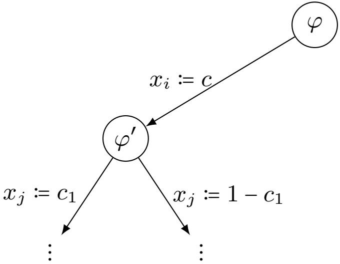

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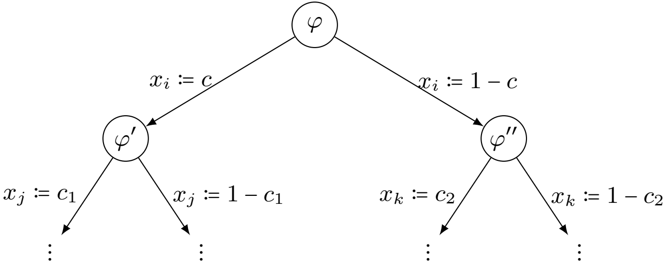

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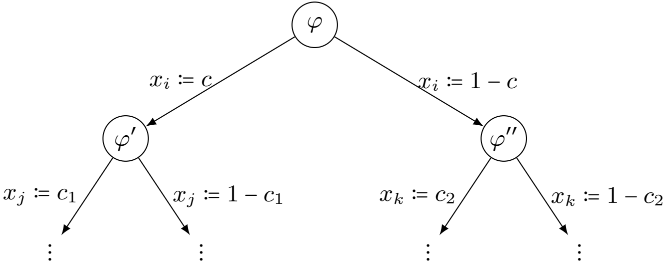

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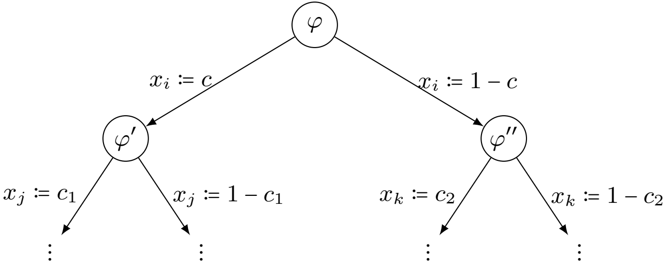

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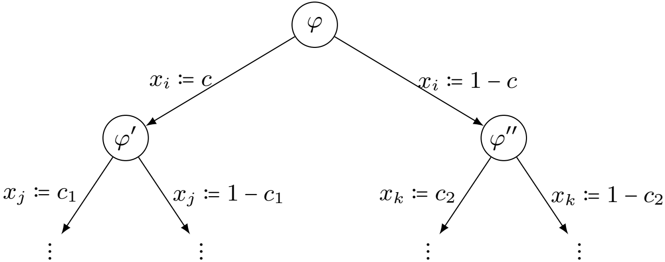

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- Simplification rules: no simplifications!

DPLL and Resolution

Theorem

DPLL algoritm makes t splitting on unsatisfiable CNF formula

$$\varphi \coloneqq \bigwedge_i C_i$$

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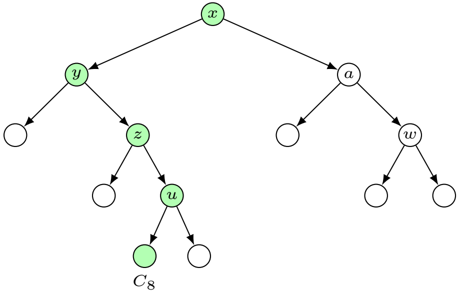

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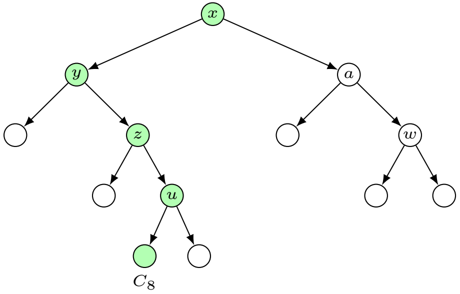
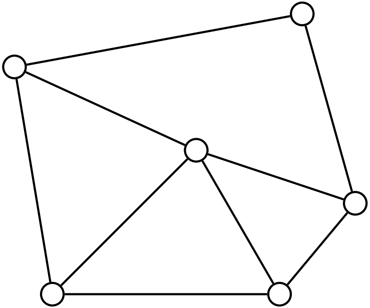
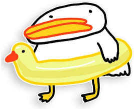


$$\frac{A \vee x \quad B \vee \neg x}{A \vee B} \qquad \frac{A}{A \vee z}$$

- Node $\Rightarrow$ disjunction of negations of queries.
- $(x \vee \neg y \vee \neg z \vee u)$.

Flow formulas





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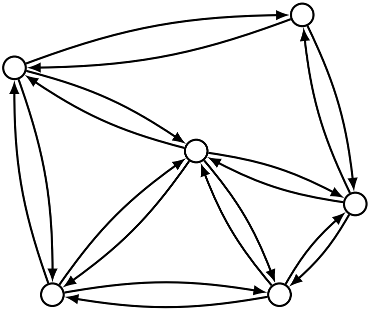

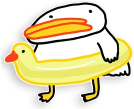

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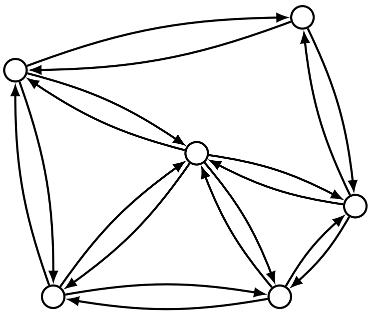

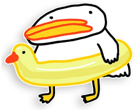

- v: $\sum_{e \in E_v^{\text{in}}} x_e - \sum_{e \in E_v^{\text{out}}} x_e = c(v)$ $(\mathbb{R})$;
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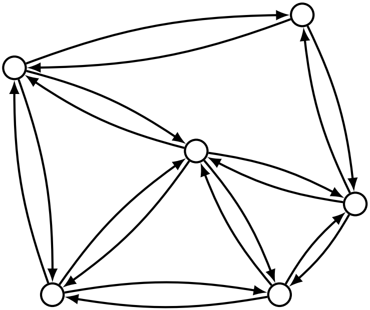

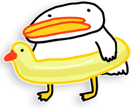

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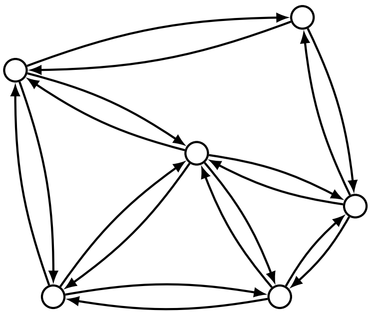

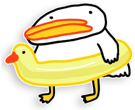

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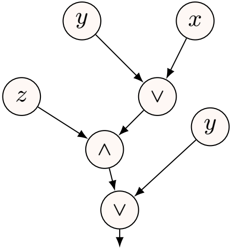
Corollary[Göös, Kamath, Robere, S 19]

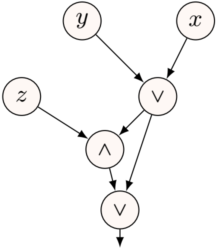
There is a monotone function in $\mathbf{NC}_2$ that cannot be computed by subexponential monotone circuits.

Monotone Computations

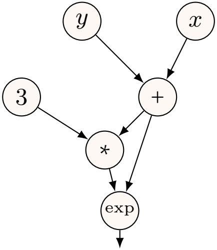
Formulas



Circuits



More circuits



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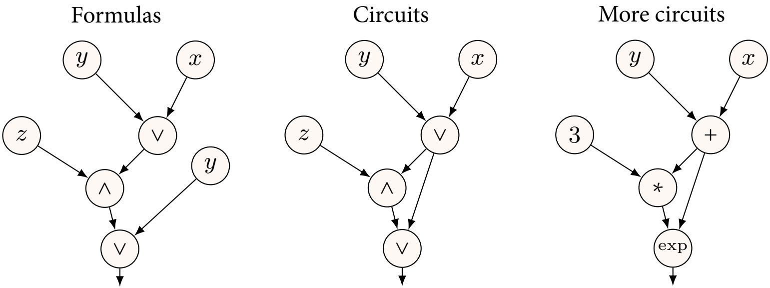
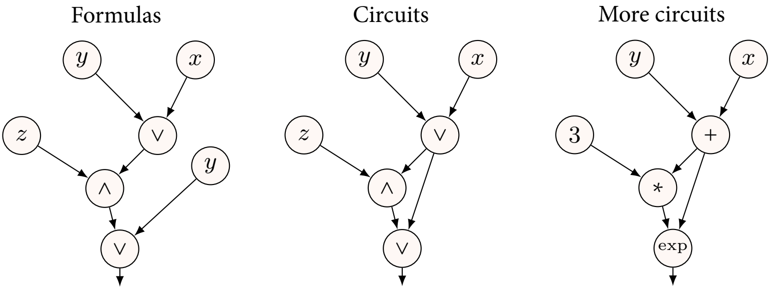


Why do we care about lower bounds on monotone computations?

- We can proof something!

Monotone Computations



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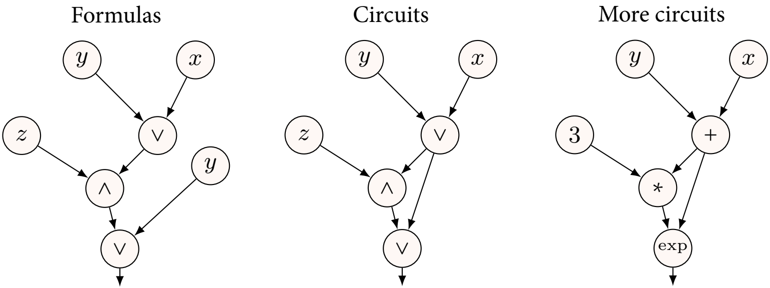

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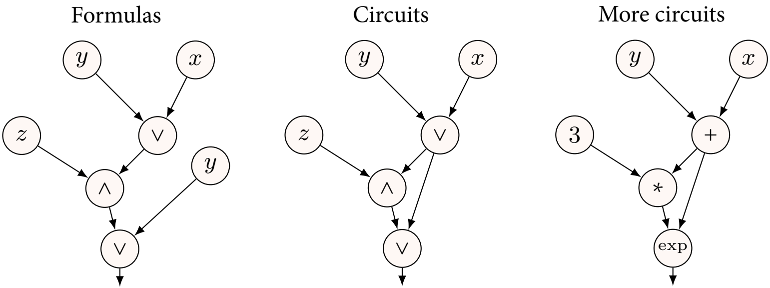

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- We can proof something!
- We can control relative error.
- Strong enough lower bounds on monotone circuits $\Rightarrow$ lower bounds on general circuits.
- Secret sharing/cryptography.

Communication Protocols. $f\colon U \times V \to T$

$$f(x, y) = ?$$

$x \in U$

$y \in V$

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$r_1 = a(x)$ →

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Communication Protocols. $f\colon U \times V \to T$

$$f(x, y) = ?$$

$x \in U$ $y \in V$

$r_1 = a(x)$ →

$r_2 = b(y, r_1)$ ←

Communication Protocols. $f\colon U \times V \to T$

$$f(x, y) = ?$$

$x \in U$ $\qquad\qquad$ $y \in V$

$$r_1 = a(x) \longrightarrow$$

$$\longleftarrow r_2 = b(y, r_1)$$

$$\longrightarrow$$

$$\longrightarrow$$

$$\vdots$$

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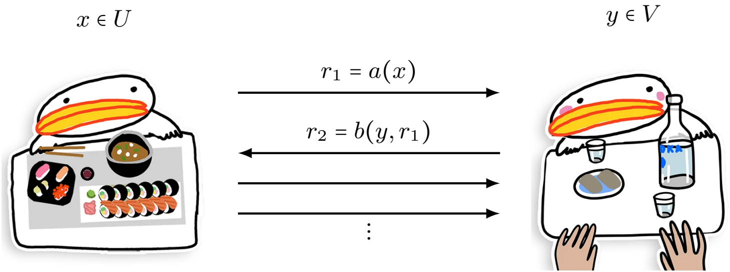
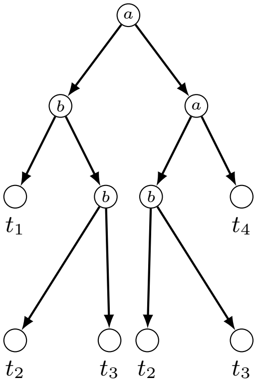


- Depth is the number of rounds (in the worst case).
- $\mathrm{D}(f) = \min_{P \in \mathcal{P}} \mathrm{depth}(P)$, where $\mathcal{P}$ is a set of protocols for f.

Protocols and Trees

Alice gets $u \in U$, Bob gets $v \in V$. Protocol is a tree:



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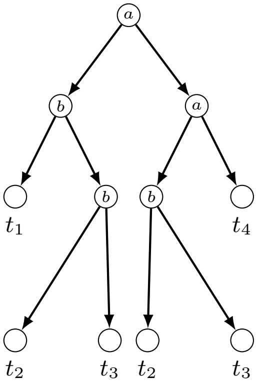

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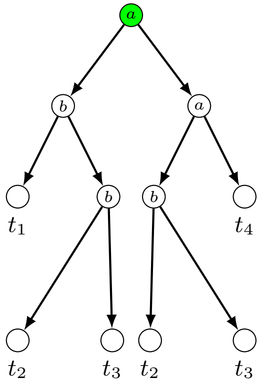

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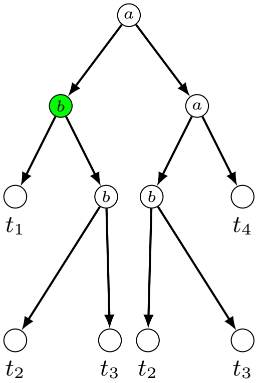

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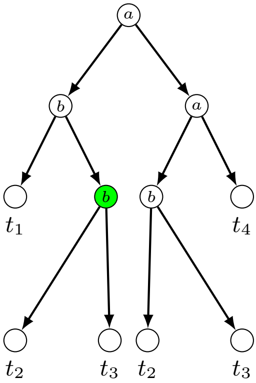

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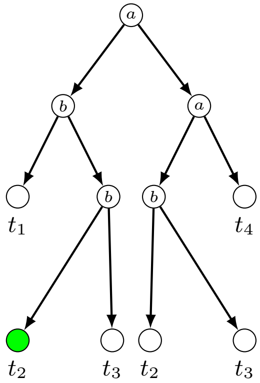

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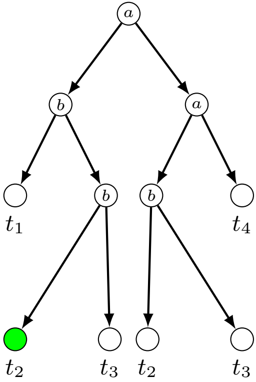

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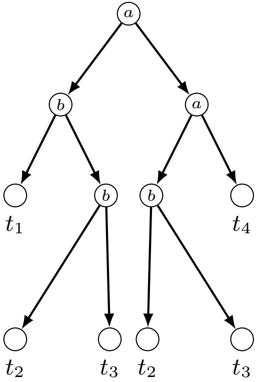

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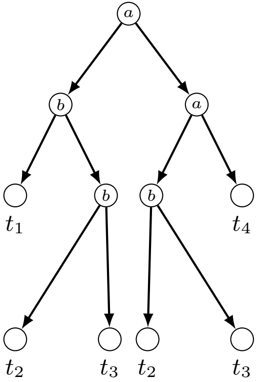
Lemma

$\mathrm{D}(f) = \Omega(\log(\mathrm{Size}(f)))$.



KW Relation [Karchmer, Wigderson 90]

Let $U, V \subseteq \{0,1\}^n$ and $U \cap V = \varnothing$.

KW:

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Theorem[Karchmer, Wigderson 90]

Monotone formula for a function f of size S $\Leftrightarrow$ communication protocol for KW^{m} KW of size S, where $U := f^{-1}(1), V := f^{-1}(0)$.

KW^{m} is a "Complete Relation"

- $\mathcal{S} \subseteq U \times V \times \mathcal{O}$;
- define $F_{\mathcal{S}}\colon \{0,1\}^m \to \{0,1\}$ such that $\mathrm{D}(\mathsf{KW}^{\mathsf{m}}_{F_{\mathcal{S}}}) = \mathrm{D}(S)$.

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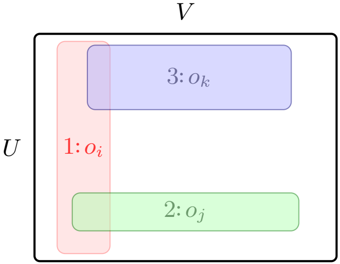

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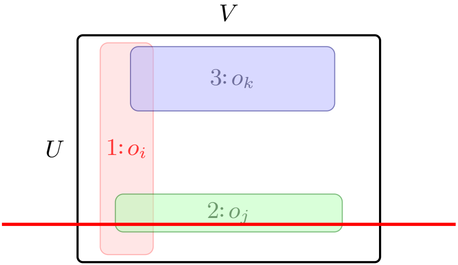

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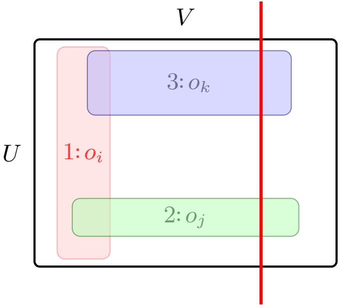

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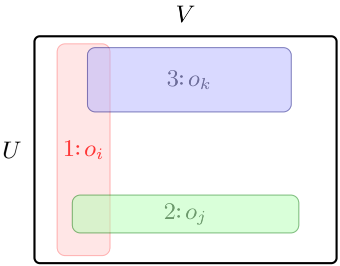

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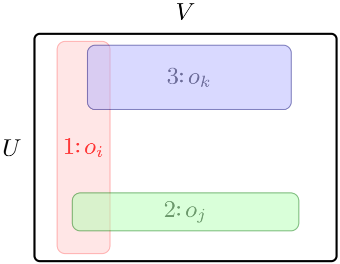

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Search$_\varphi$ [Lovász, Naor, Newman, Wigderson et al. 94]

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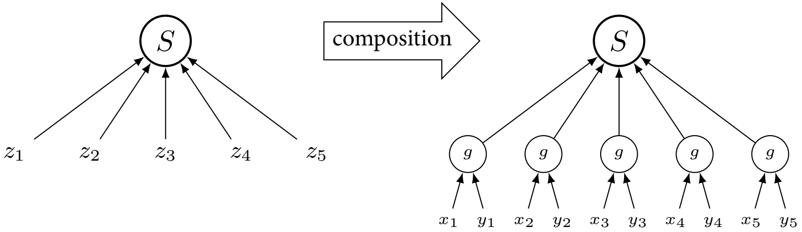

$\mathsf{Search}_\varphi \circ g \equiv \mathsf{Search}_{\varphi \circ g}$.

Lifting 

Theorem[Raz, McKenzie 99; Göös, Pitassi, Watson 16]

Resolution depth of φ is at least $d \Rightarrow \mathrm{D}(\mathsf{Search}_\varphi \circ \mathsf{Ind}_m) \geq n^{\mathcal{O}(d)}$, where $m := \mathsf{poly}(n)$. $\mathrm{D}(\mathsf{Search}_\varphi \circ \mathsf{Ind}_m) \approx \mathrm{D}(\mathsf{Ind}) \cdot \text{res-depth}(\varphi)$.

Corollary: lower bound on monotone formulas $2^{n^{\varepsilon}}$.

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Theorem[Pitassi, Robere 16; Robere, Pitassi 18, informal]

Nullstellensatz $\Leftrightarrow$ algebraic tiling for $\mathsf{Search}_\varphi \circ g$.

Easy Function?

$f\colon \{0,1\}^{2n^3} \to \{0,1\}$

- Enumerate equalities $z_i \oplus z_j \oplus z_k = c$ (at most $2n^3$);
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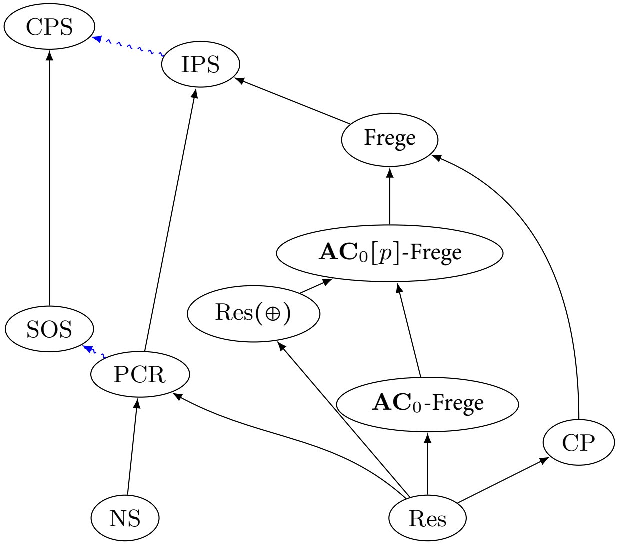
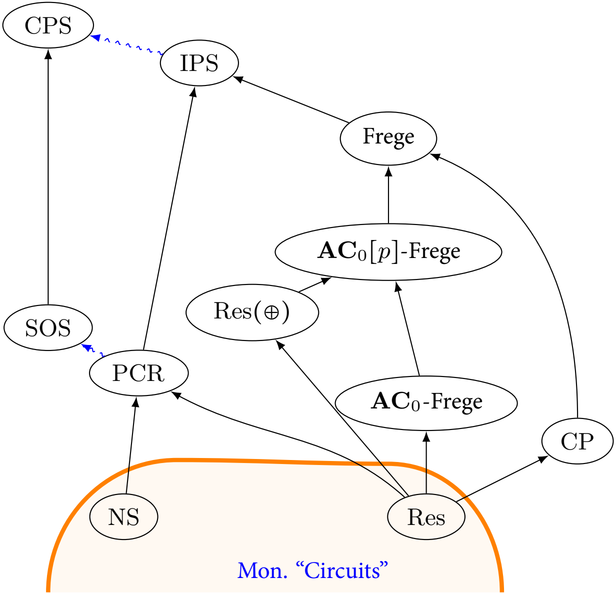
Facts about f:

- $f \in \mathbf{NC}^2$;
- F_{Flow} can be embedded into f (since there is an efficient NS proof of Flow!);
- there is no small monotone circuit for f (since there is no efficient proofs in resolution of Flow + lifting Theorem).

Hierarchy


CPS
IPS
Frege
$\mathbf{AC}_0[p]$-Frege
Res(⊕)
SOS
PCR
$\mathbf{AC}_0$-Frege
CP
NS
Res

Hierarchy


CPS
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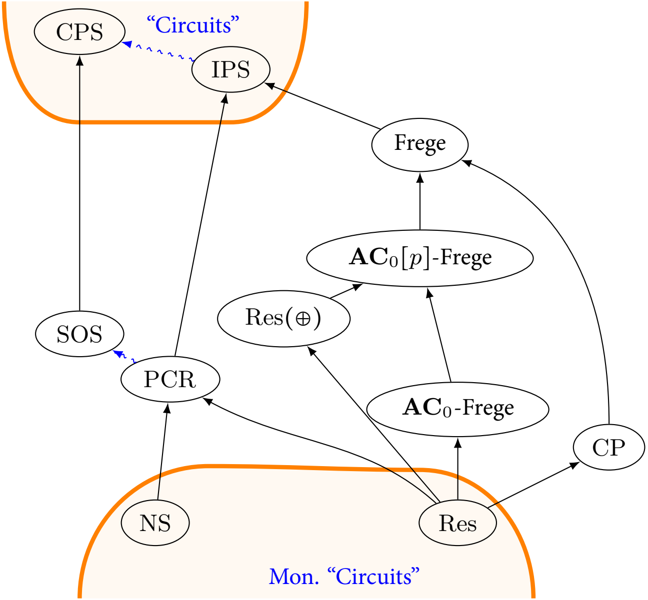
Mon. "Circuits"

Hierarchy



"Circuits"
CPS
IPS
Frege
$\mathbf{AC}_0[p]$-Frege
Res(⊕)
SOS
PCR
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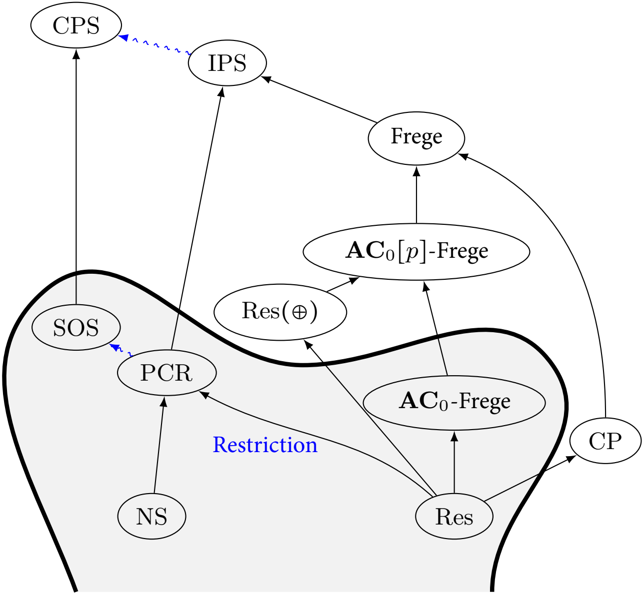
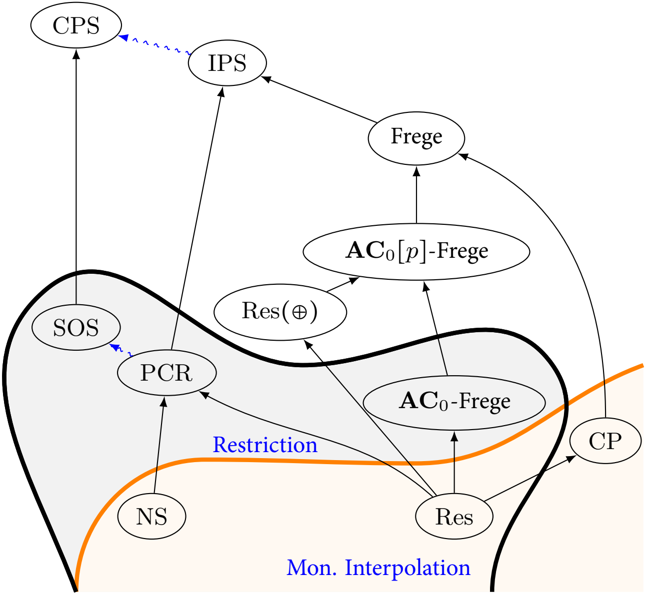
NS
Res
Mon. "Circuits"

Hierarchy


CPS
IPS
Frege
$\mathbf{AC}_0[p]$-Frege
Res(⊕)
SOS
PCR
$\mathbf{AC}_0$-Frege
Restriction
CP
NS
Res

Hierarchy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Frege
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CP
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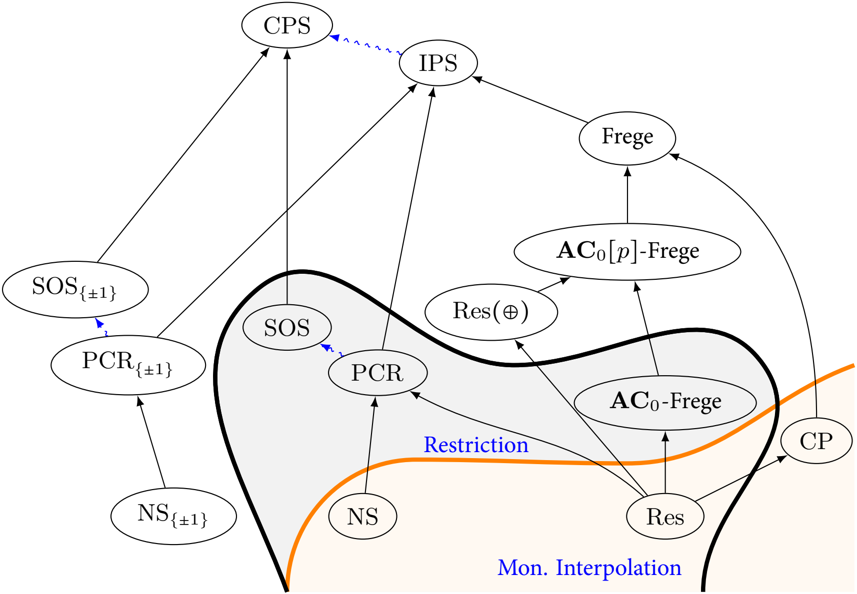
Mon. Interpolation

Hierarchy


CPS
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Frege
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$\text{SOS}_{\{\pm1\}}$
$\text{Res}(\oplus)$
SOS
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CP
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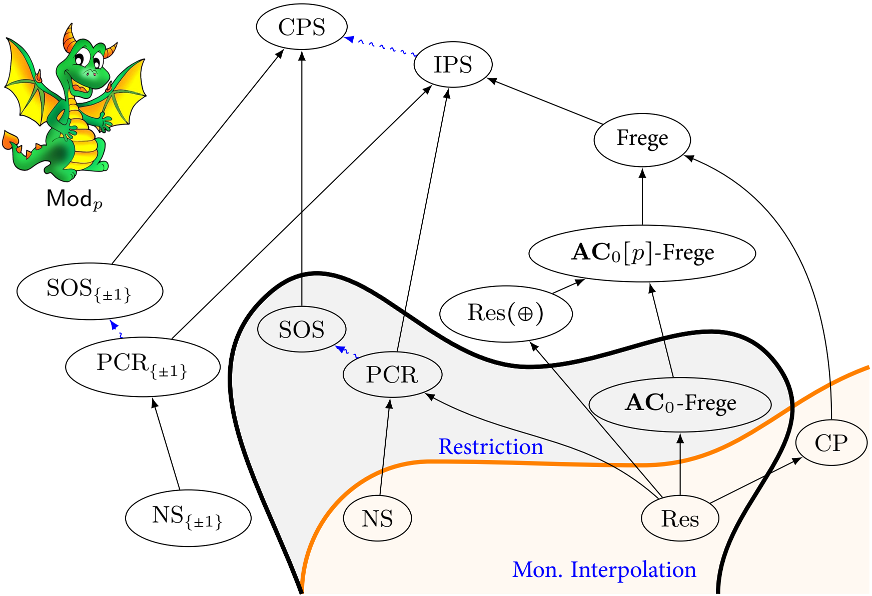
NS
Res
Mon. Interpolation

Hierarchy



Mod$_p$
CPS
IPS
Frege
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SOS$_{\{\pm1\}}$
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SOS
PCR$_{\{\pm1\}}$
PCR
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NS
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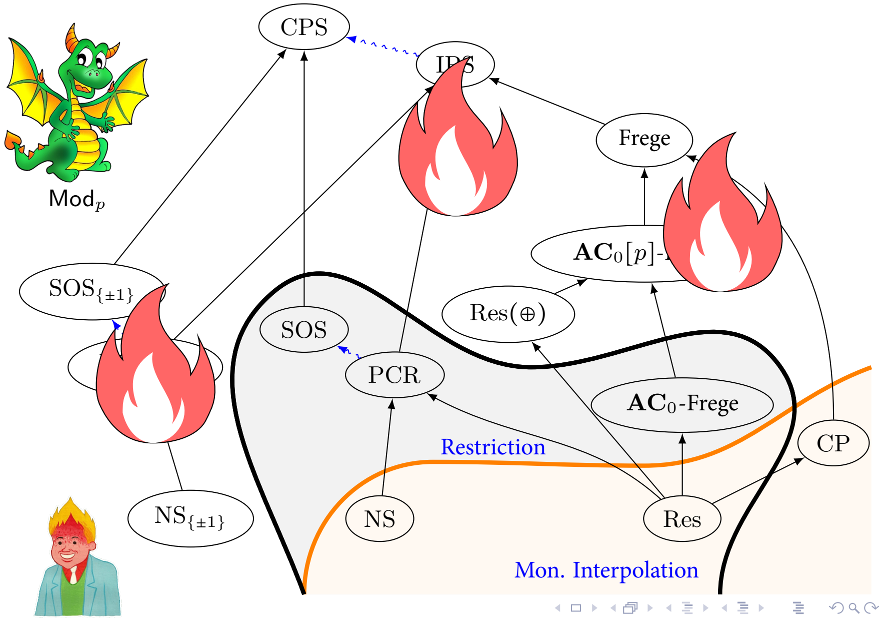

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