Data Structures Meet Circuits and Cryptography

Alexander Golovnev, Georgetown University

SACC 2021

Motivating Question

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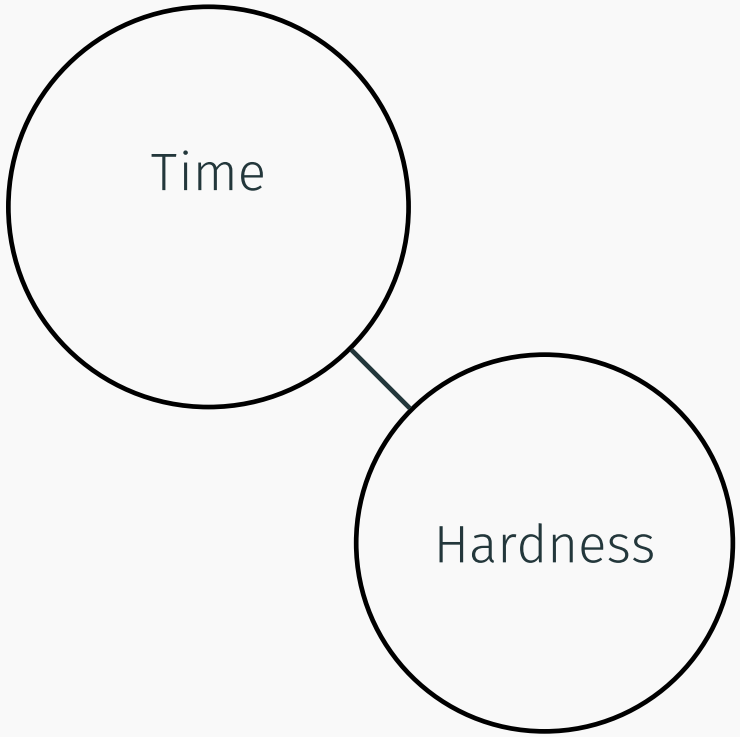

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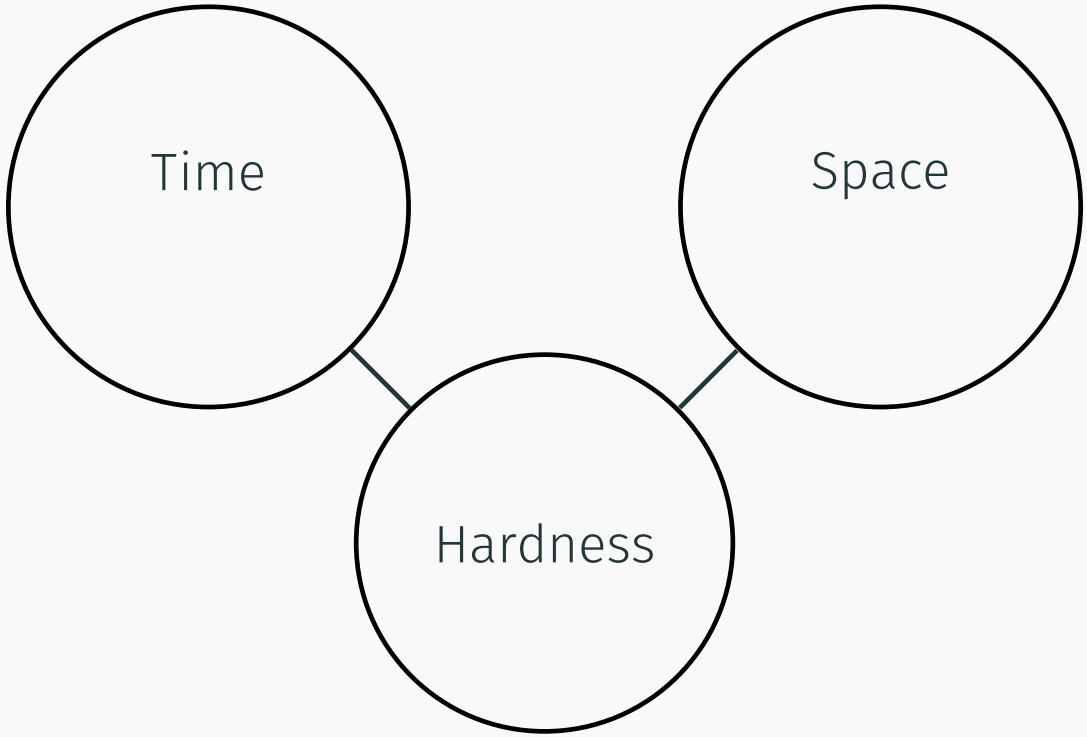

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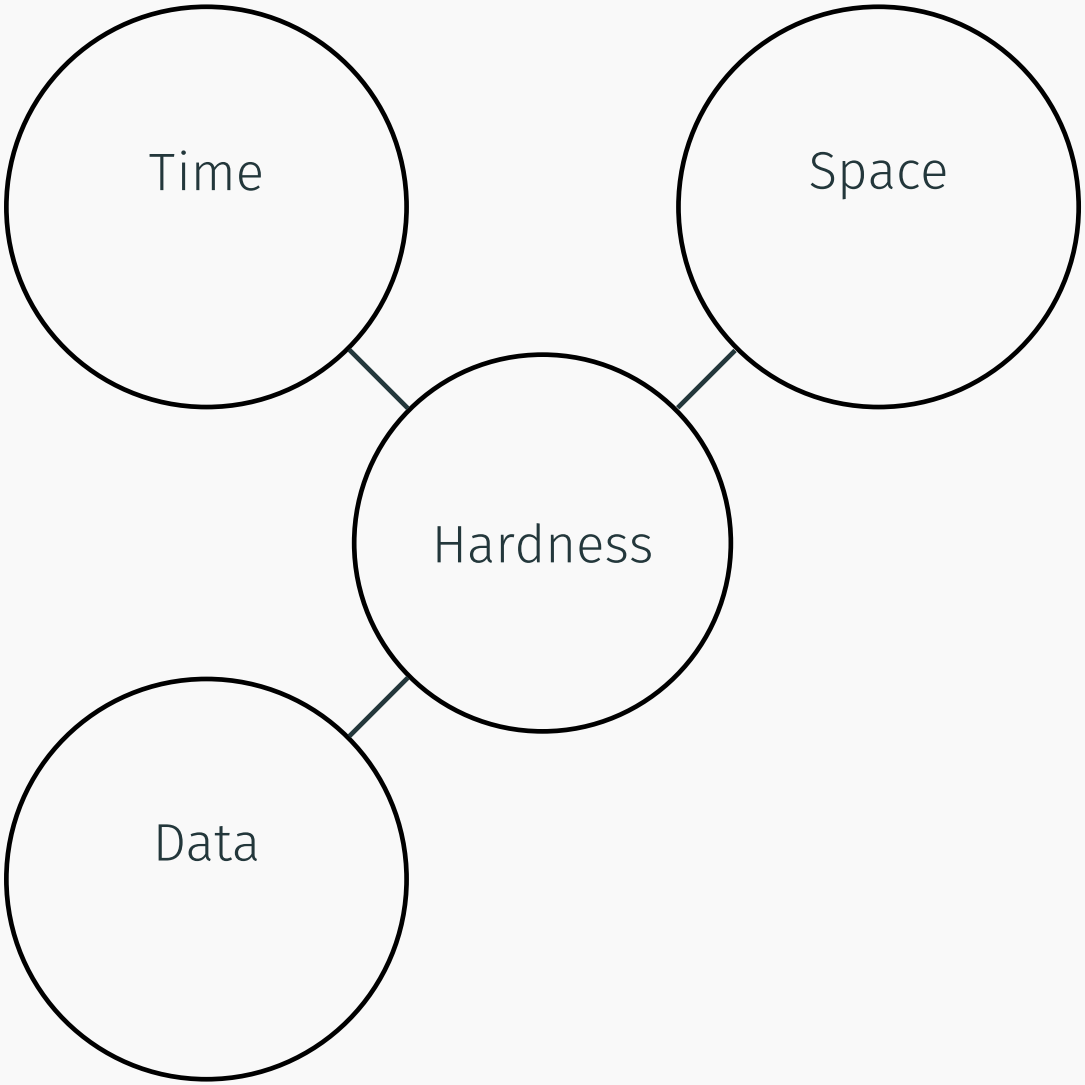

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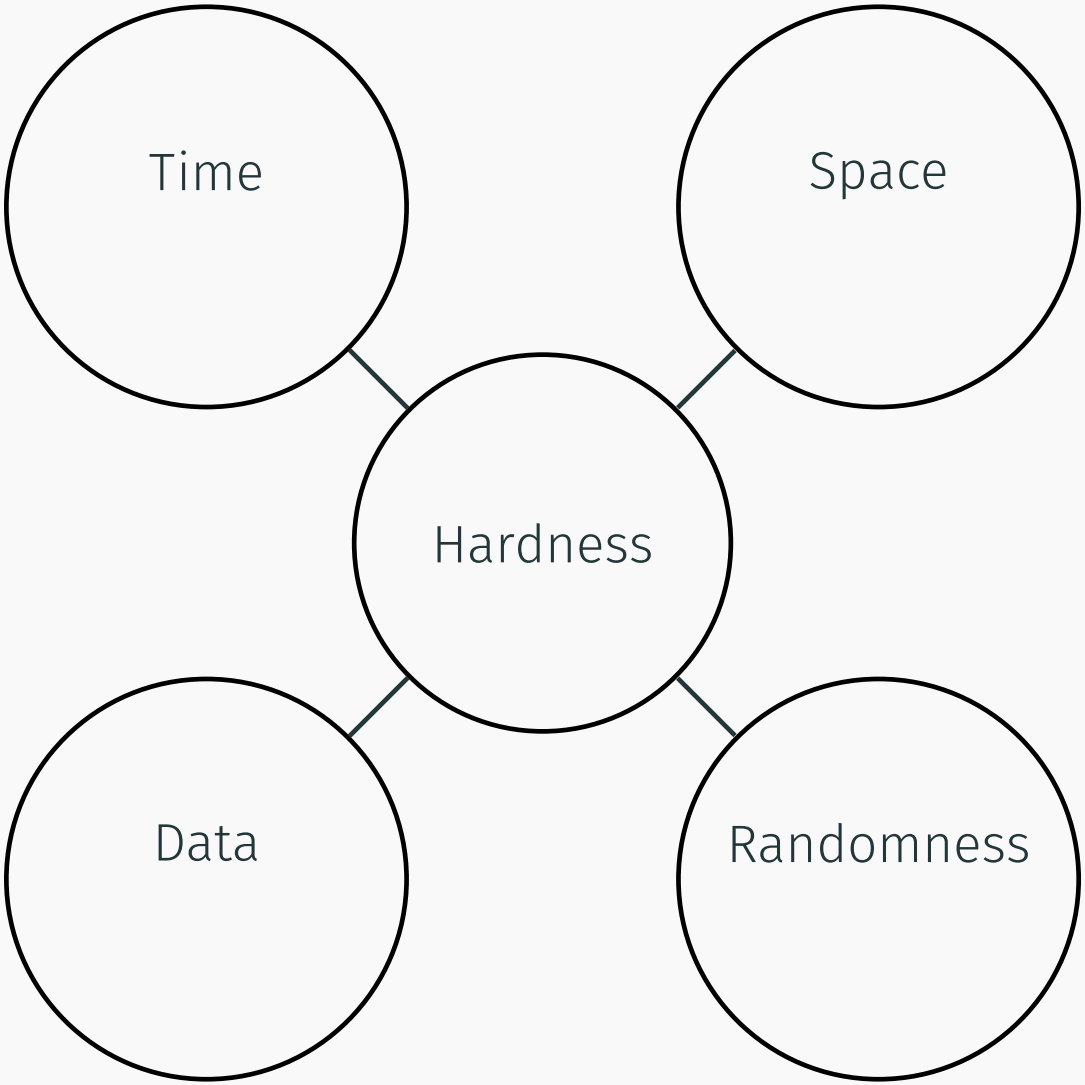

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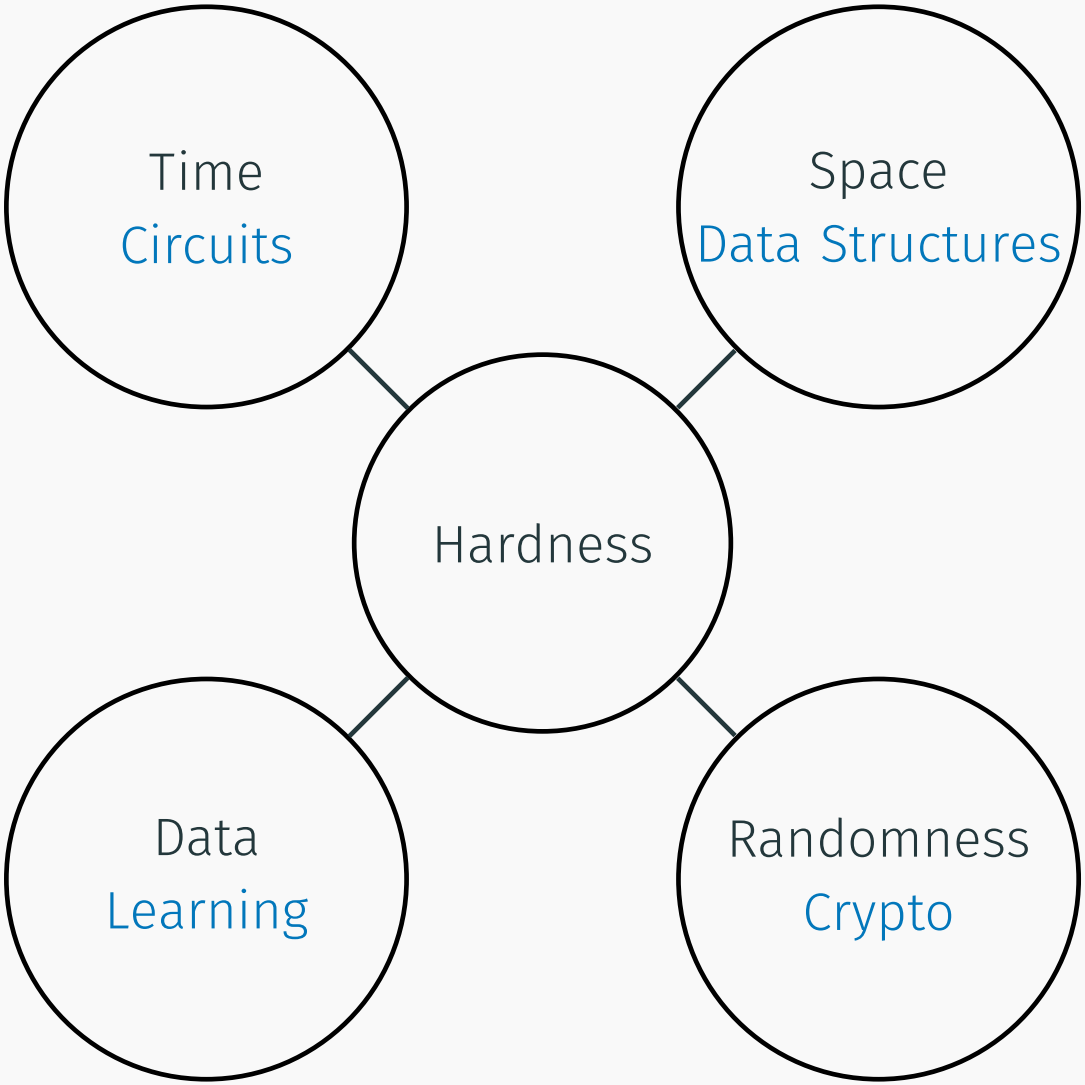

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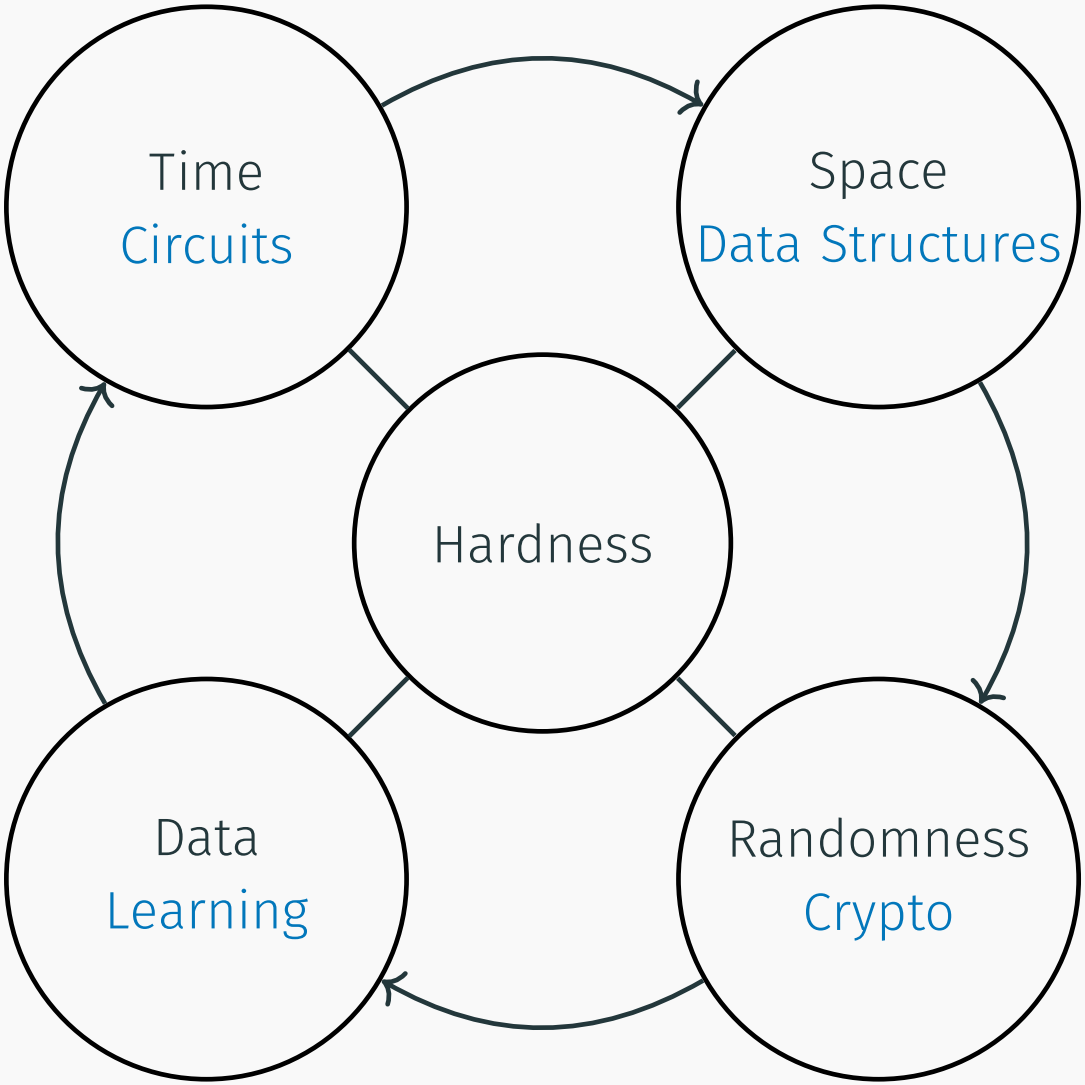

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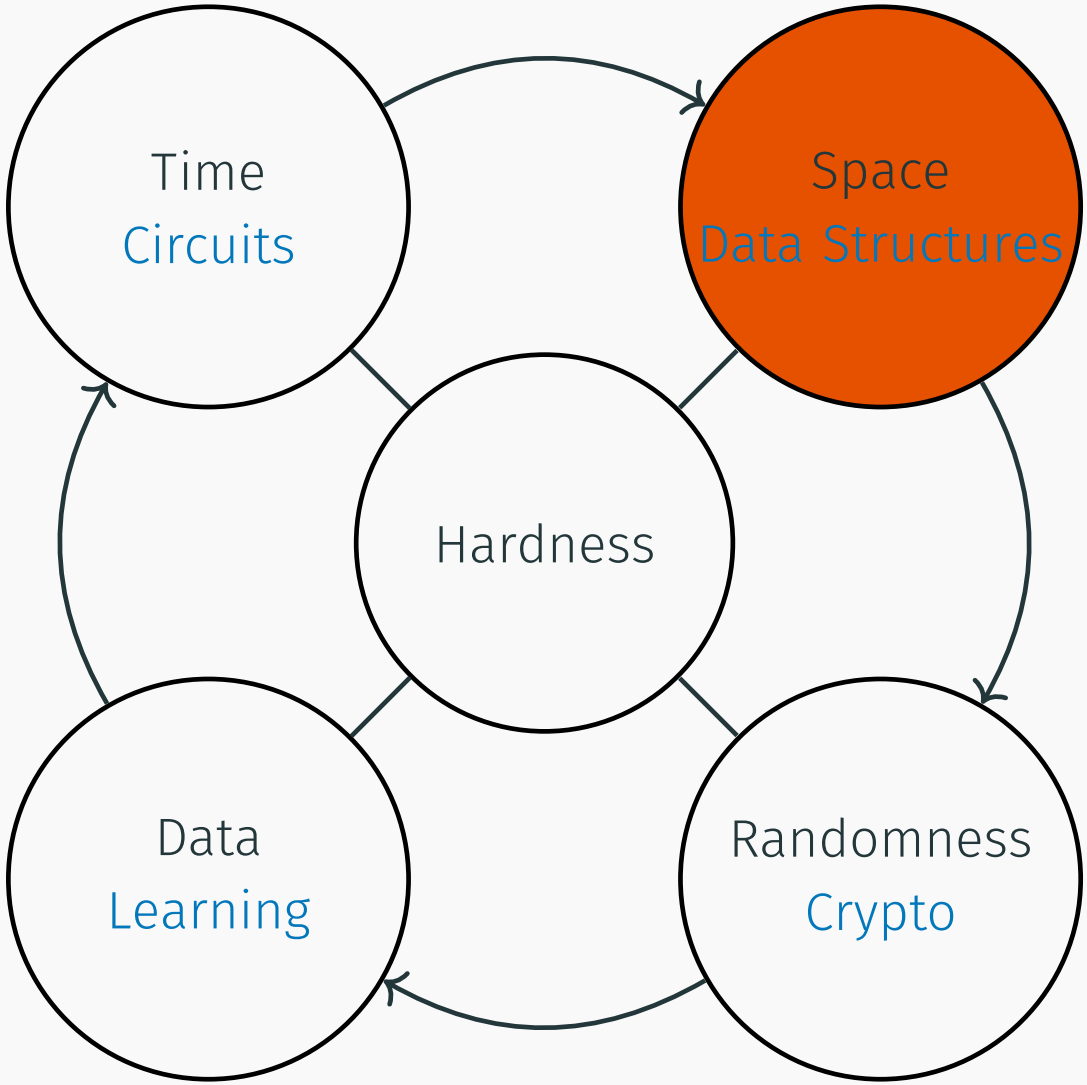

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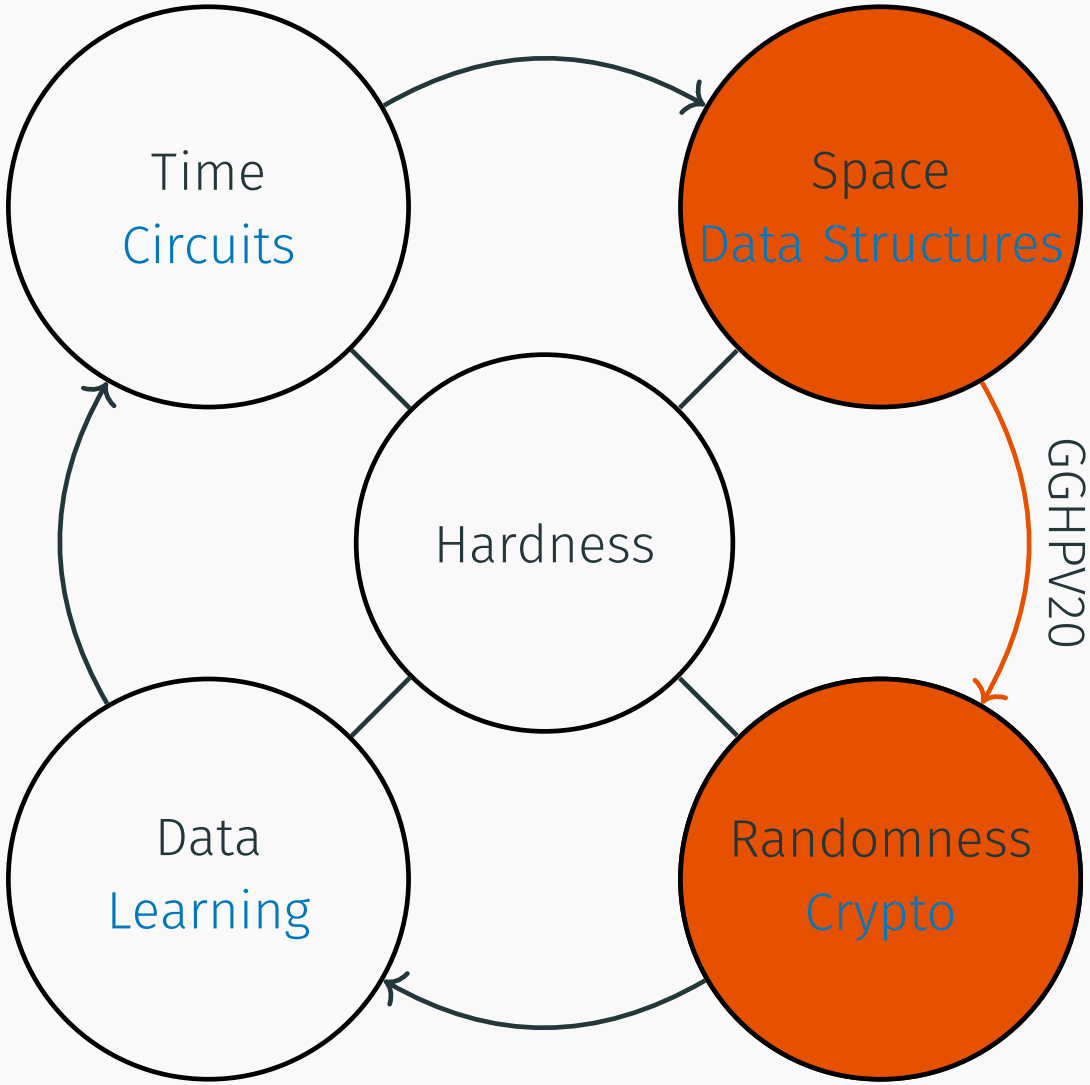

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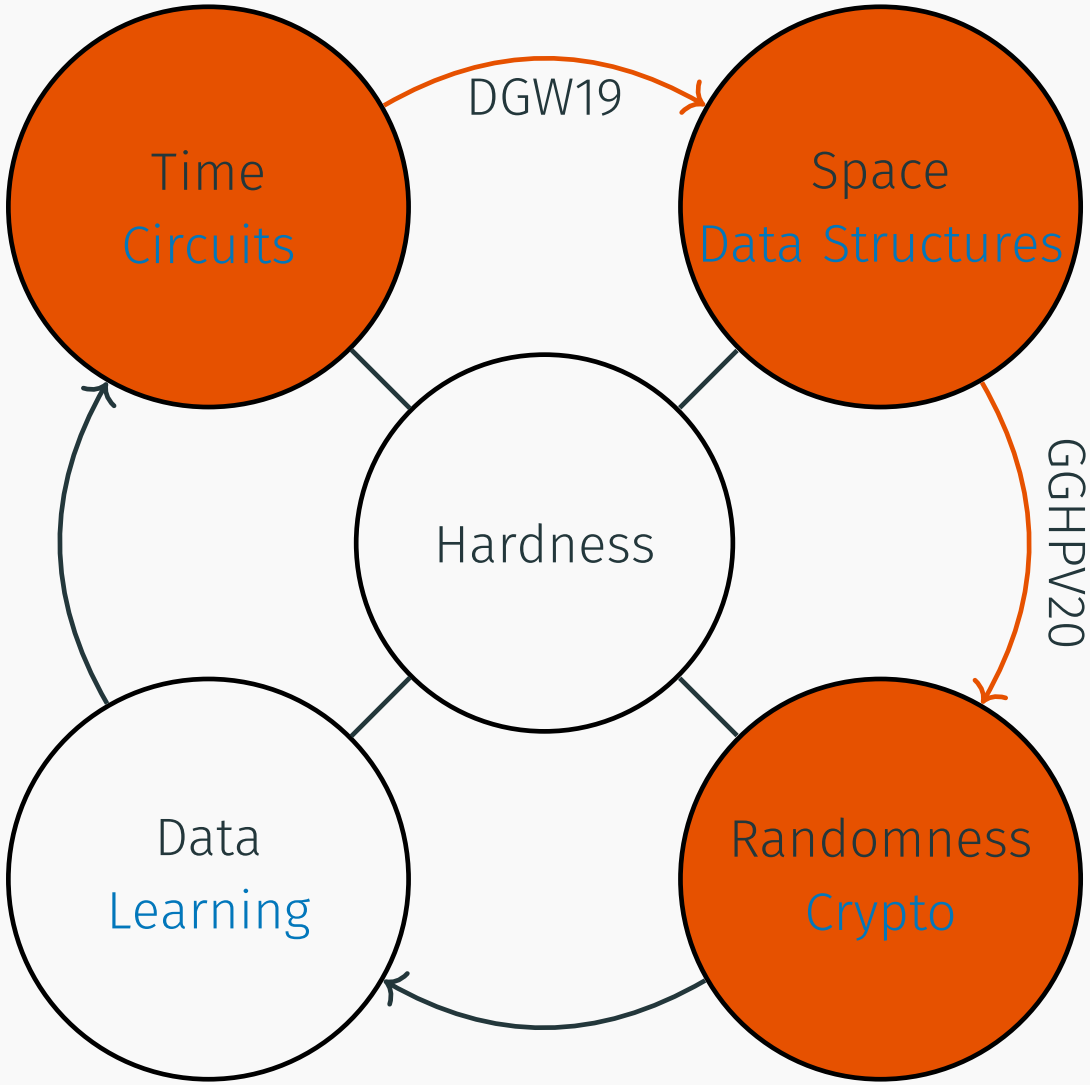
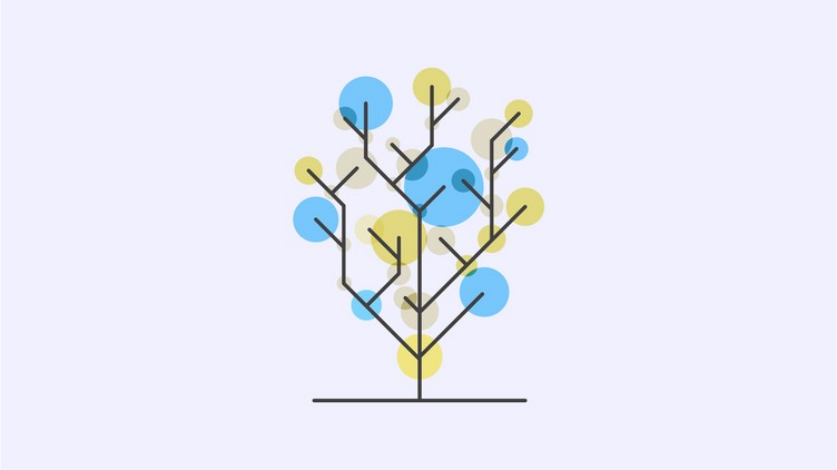
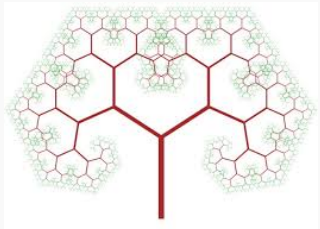


DATA STRUCTURES



Examples

Stack, Queue, List, Heap



Search Trees

```
hash(unsigned x) {
   x ^= x >> (w-m);
   return (a*x) >> (w-m);
}
```

Hash Tables

Static Data Structures. Examples

- Graph Distances: Preprocess a road network in order to efficiently compute distances between cities (Google Maps)

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  (Google Maps)
- Nearest Neighbors: Preprocess a set of points in order to efficiently find closest point to a query point
  (Netflix recommendations)
- Range Counting: Preprocess a set of points in order to efficiently compute the number of points in a given rectangle
  (Amazon market size estimation)

Static Data Structures

Preprocessing

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Queries

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Washington — St. Petersburg

Preprocessing

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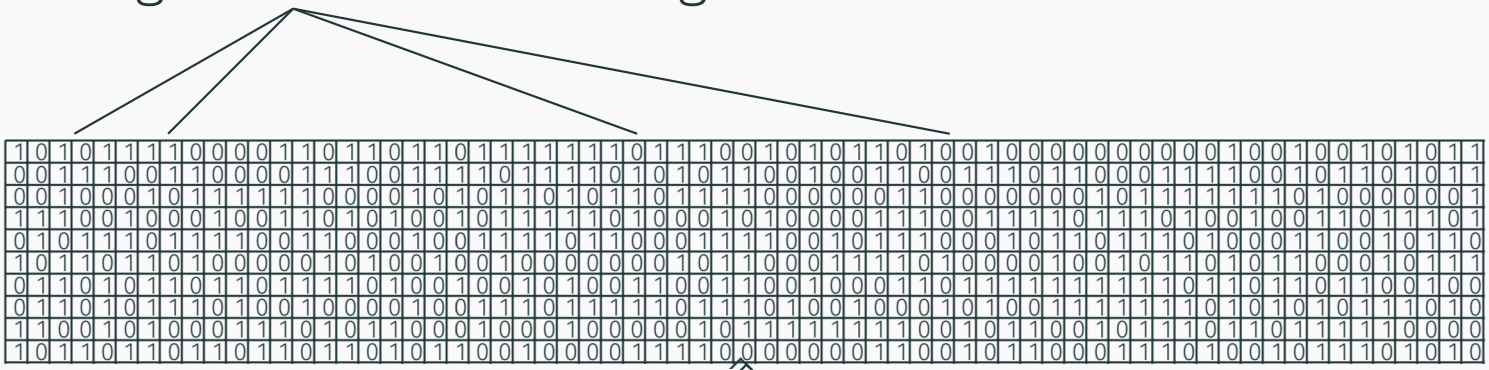

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Washington — St. Petersburg

Athens — Vilnius

Preprocessing

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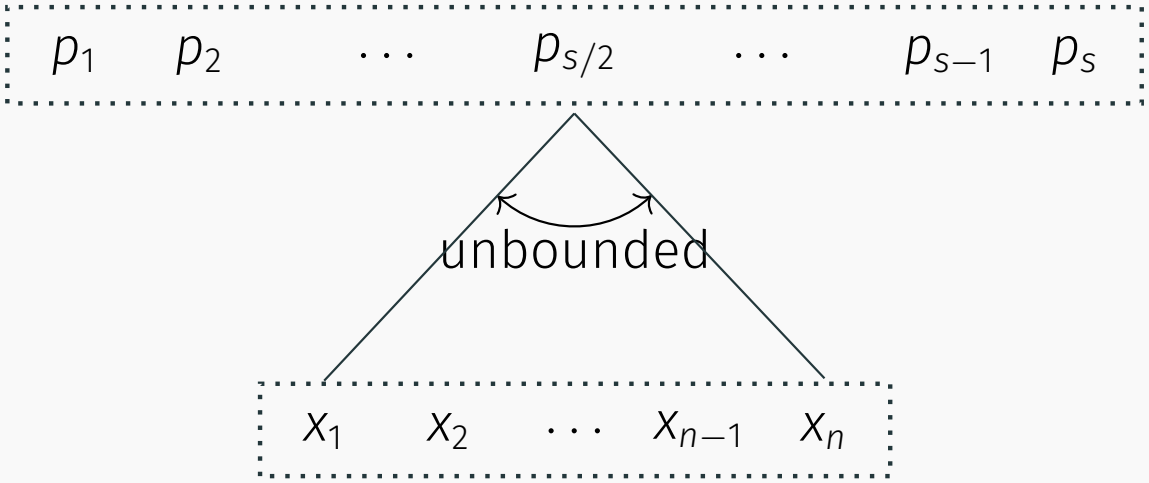

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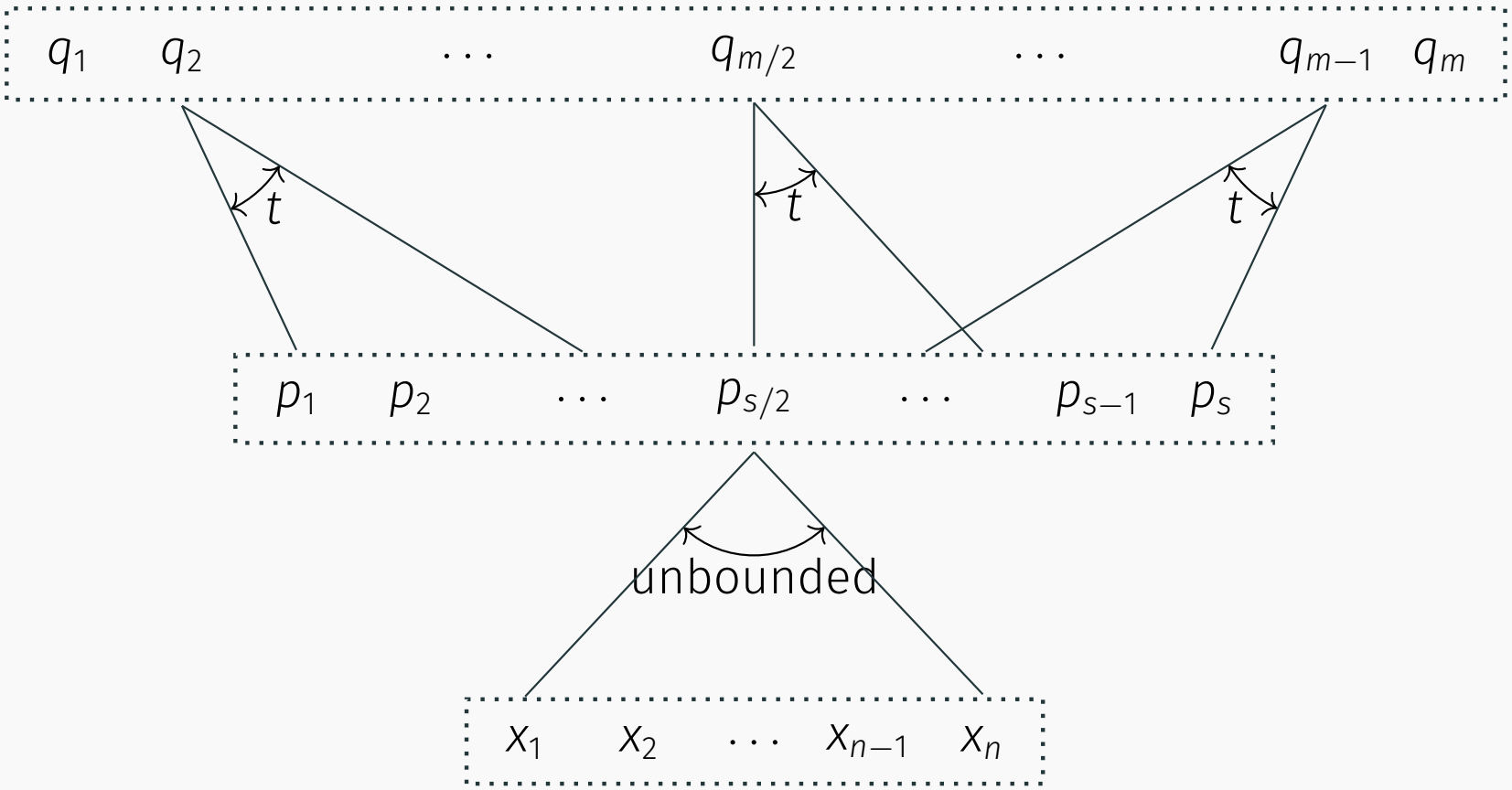

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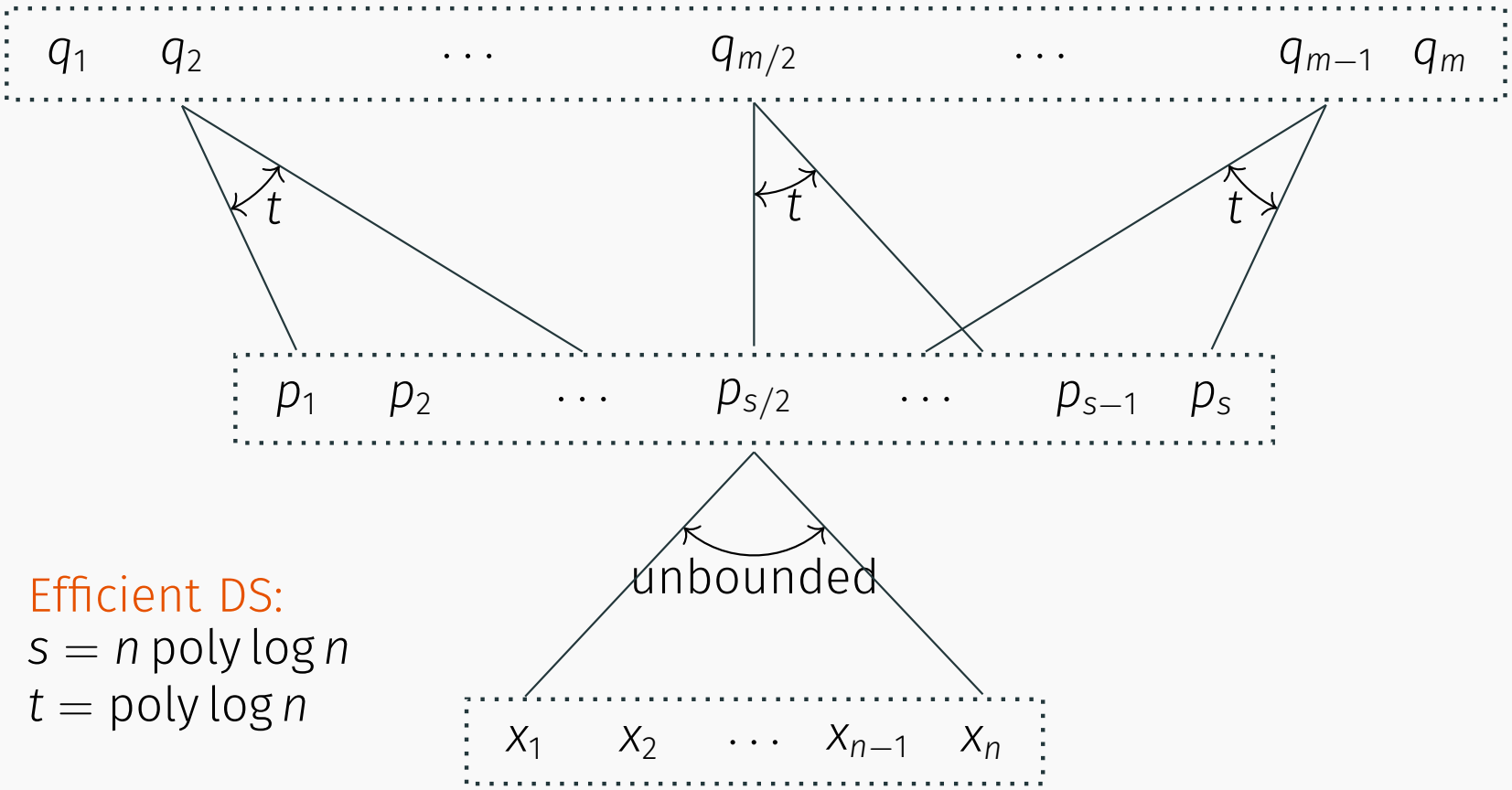

Efficient DS:
$s = n \operatorname{poly}\log n$
$t = \operatorname{poly}\log n$

DS Lower Bounds

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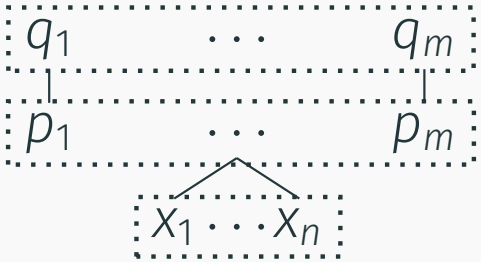

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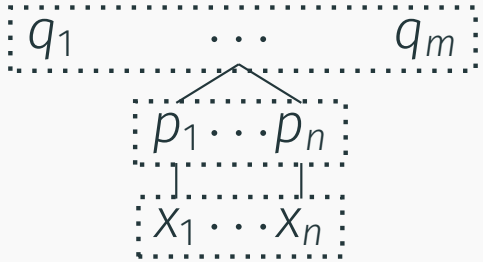

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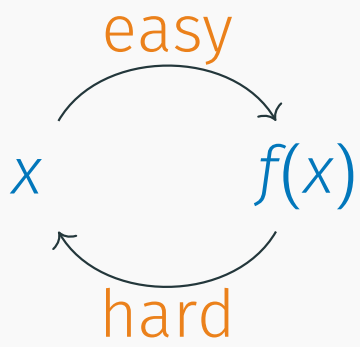
- $s = n^{1+\varepsilon} \implies t \geq \Omega(1)$

CRYPTO WITH PREPROCESSING

Cryptography with Preprocessing



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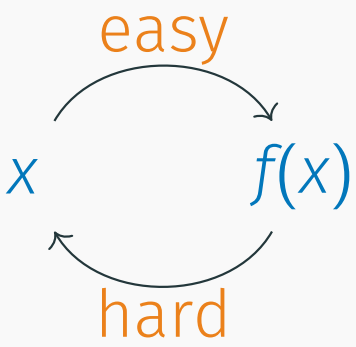

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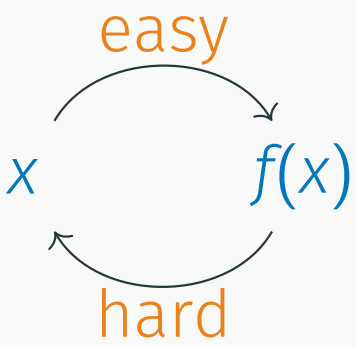

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Can we immunize a function against large preprocessing? Design $g\colon \{0,1\}^{5n} \to \{0,1\}^{5n}$ which is still secure?

Data Structures and Cryptography

Equivalence

Theorem (GGHPV20)

Cryptography with preprocessing is equivalent to data structure lower bounds (for problems with efficient query generation).

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Modern 3SUM Conjecture

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Theorem [GGHPV20]

3SUM-Indexing can be solved with $S = n^{1.9}$ and $T = n^{0.3}$

Inverting a Function

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- Special case: Invert a bijection with $S = T = \widetilde{O}(\sqrt{N})$

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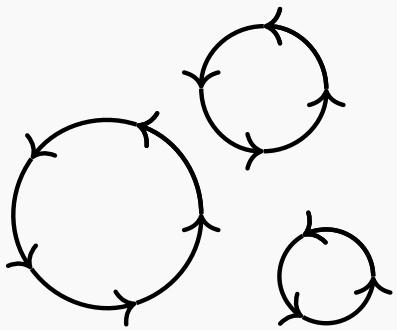
- In- and out-degrees of all vertices are 1
- Thus, this graph is a union of cycles



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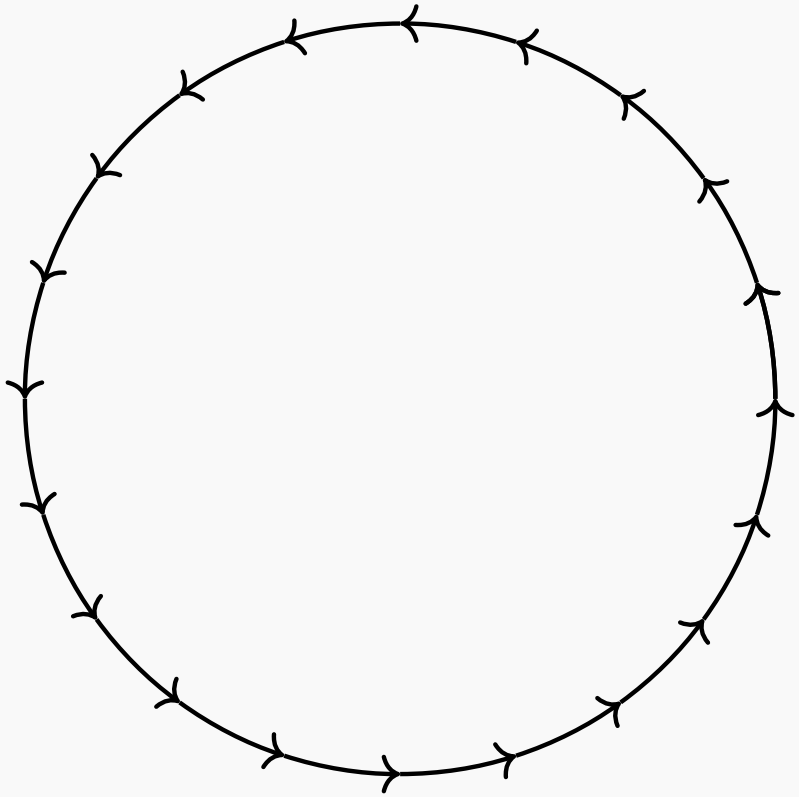

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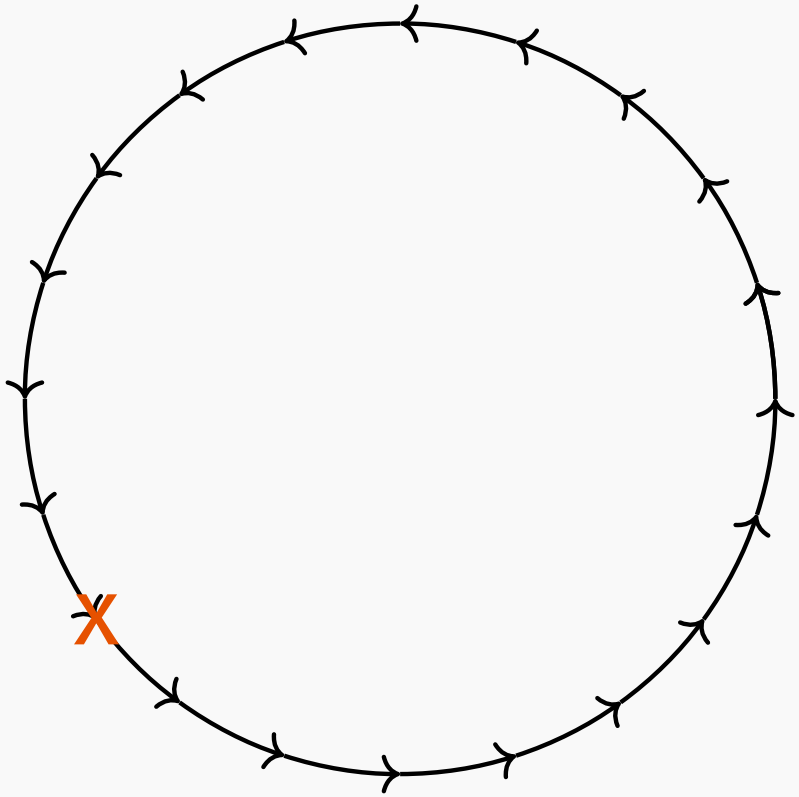

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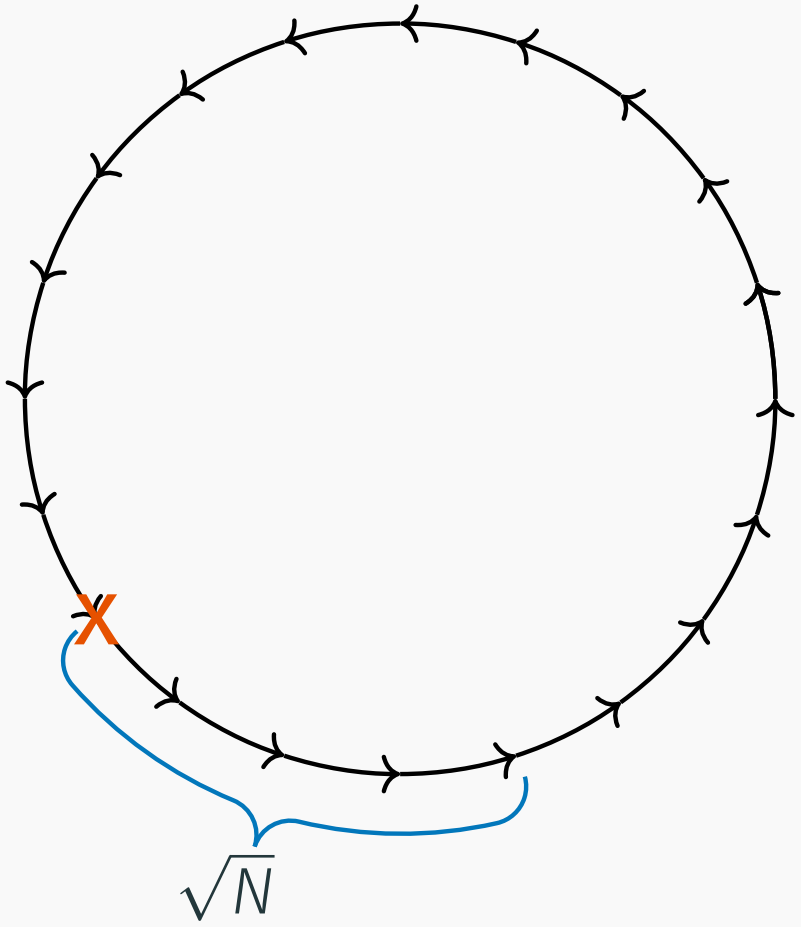

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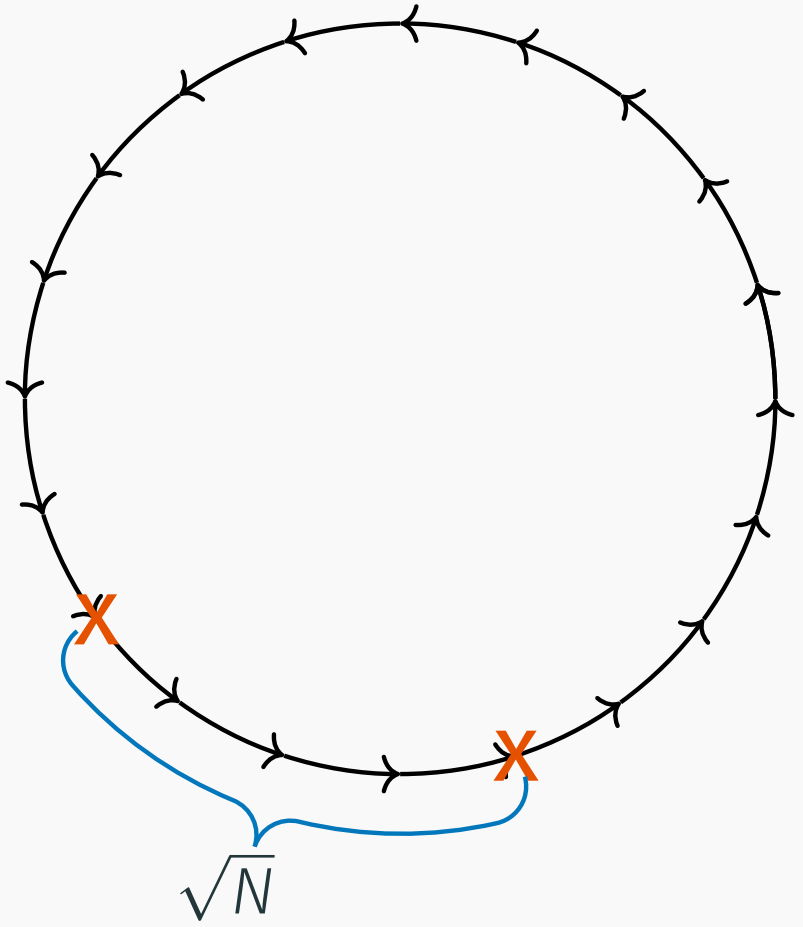

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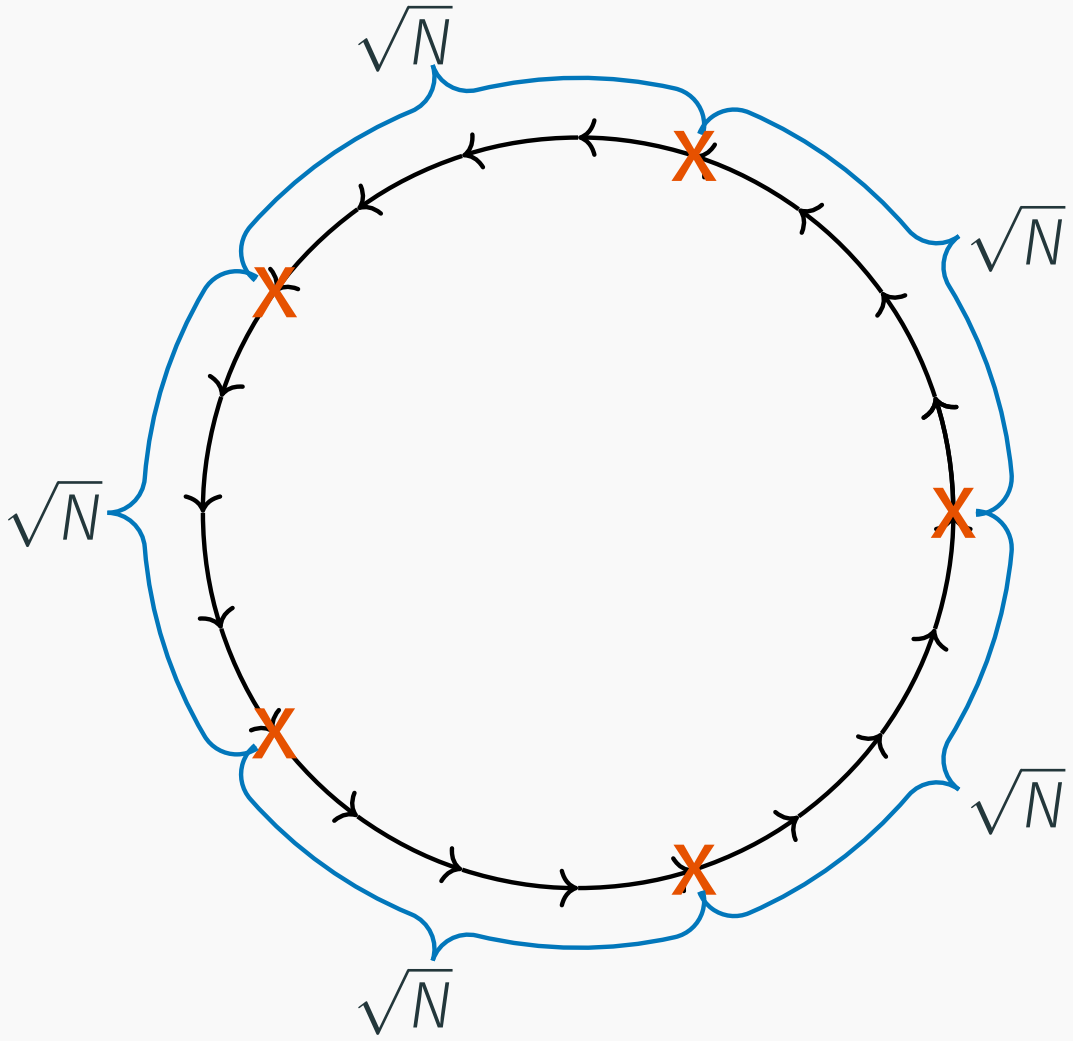
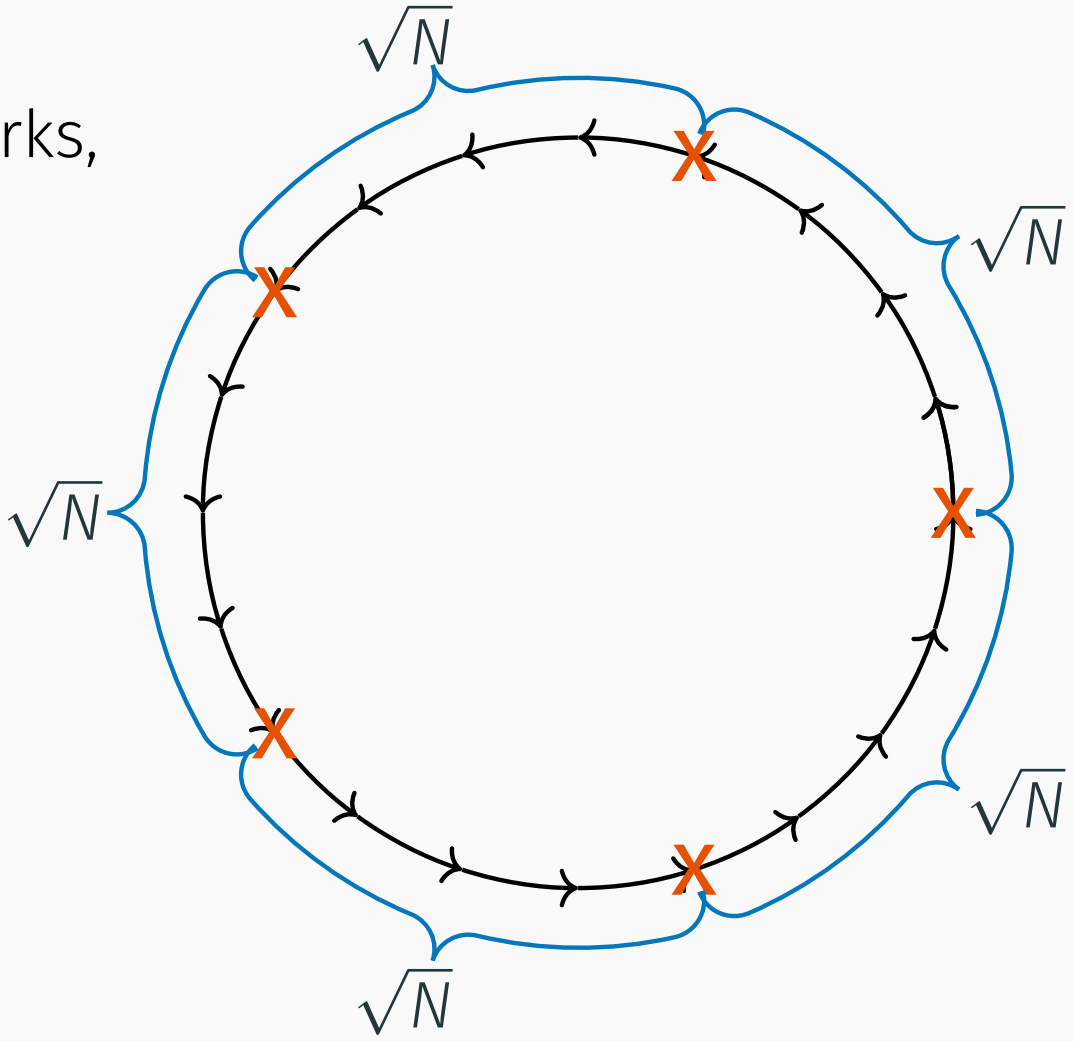


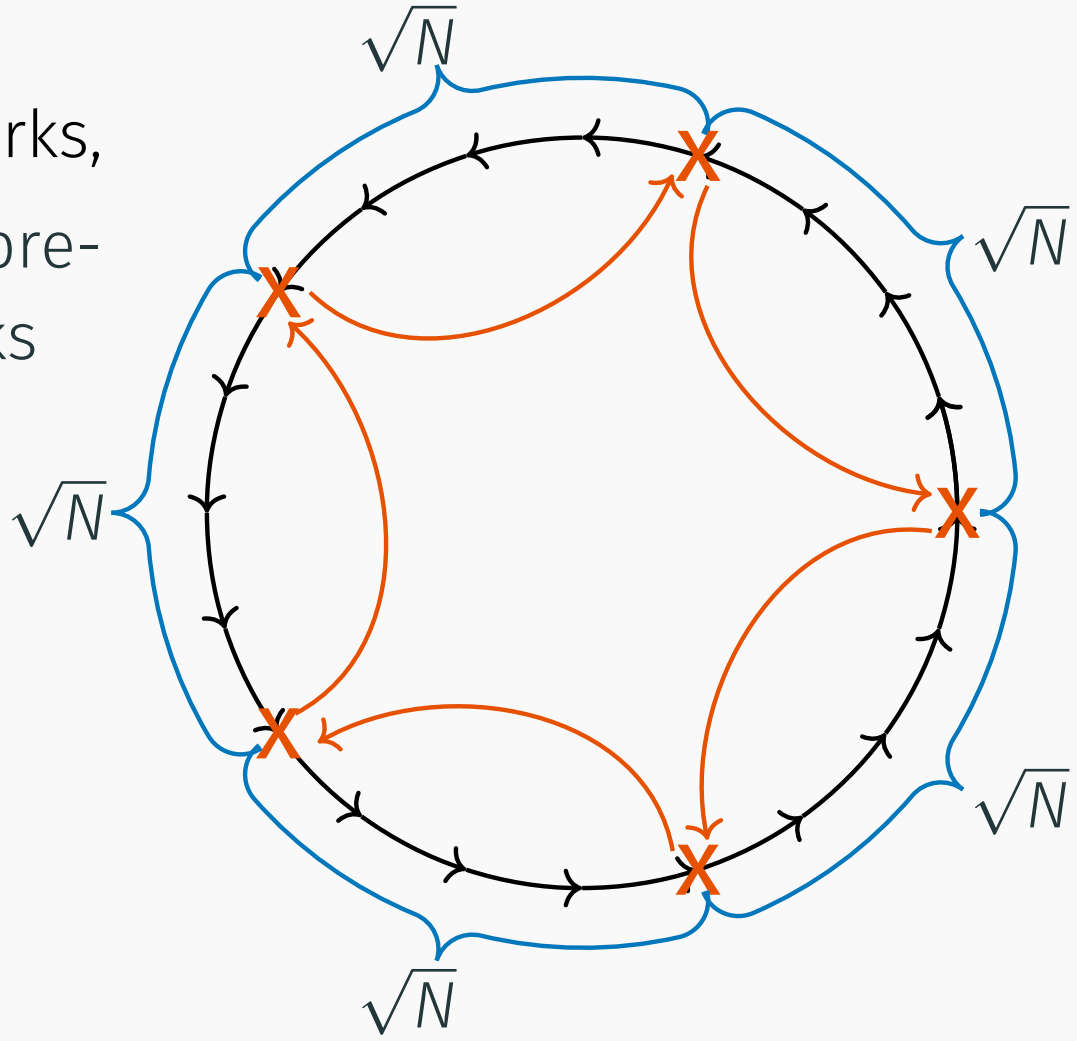
Inverting a Bijection

Store x landmarks,



Inverting a Bijection

Store x landmarks, and links $\curvearrowright$ to previous landmarks



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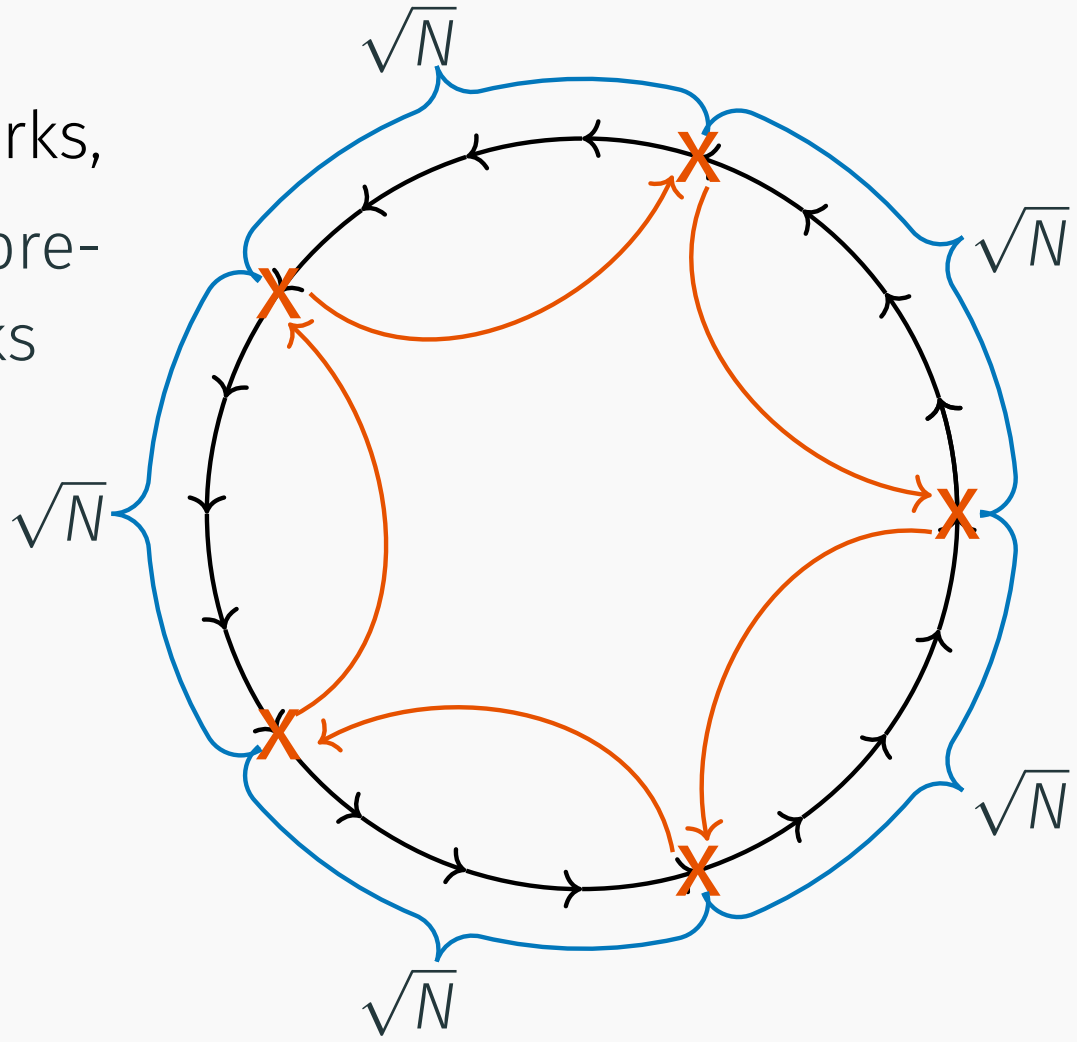

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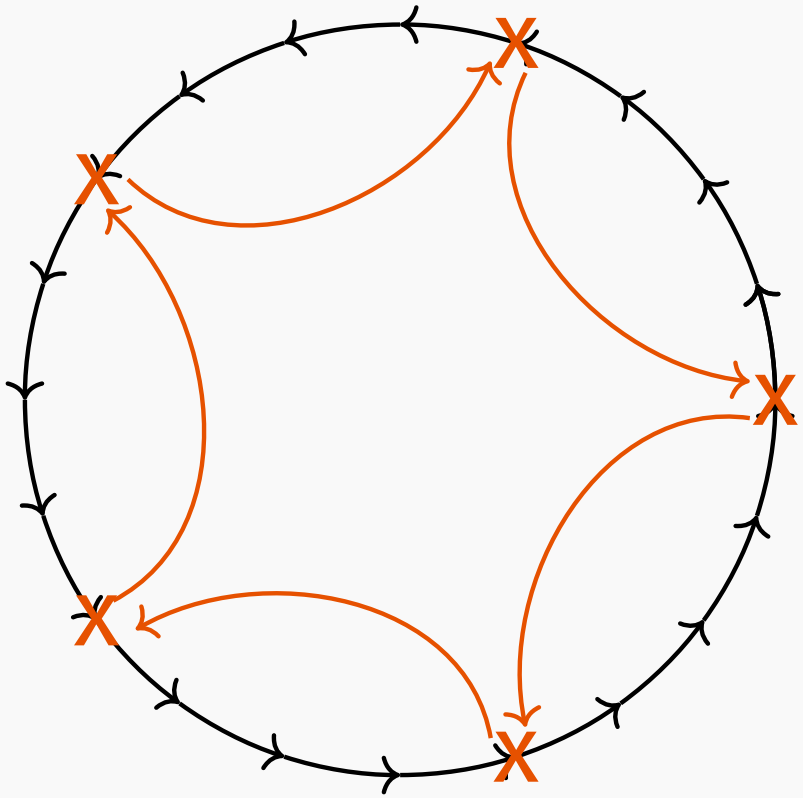

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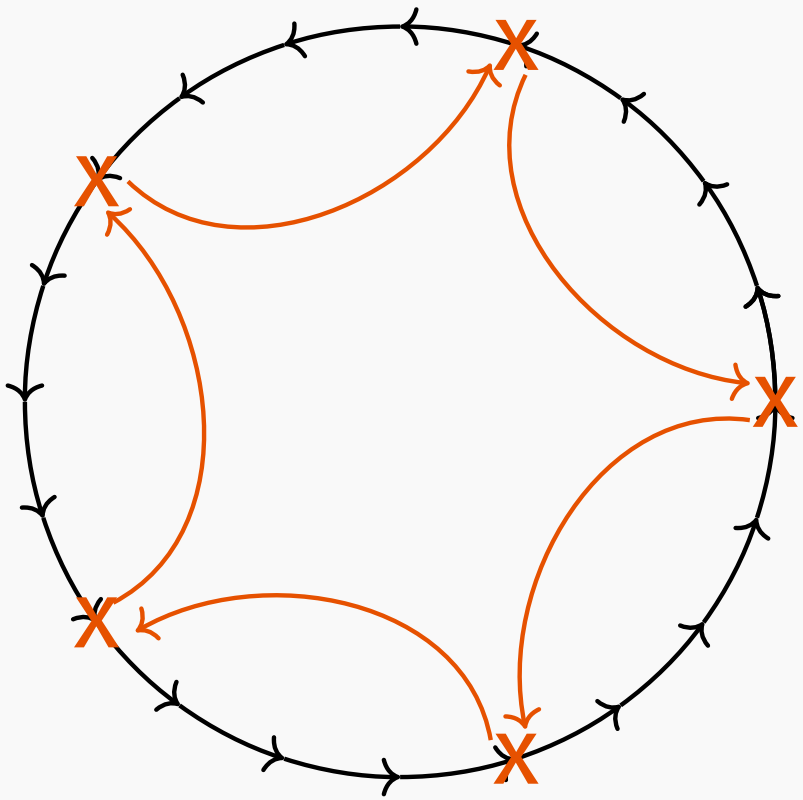

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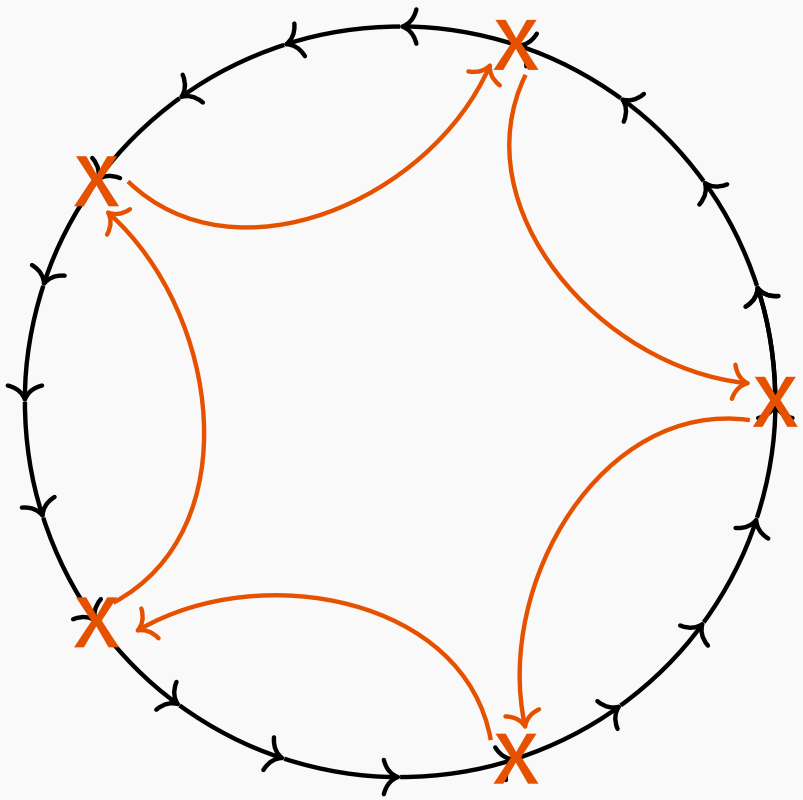

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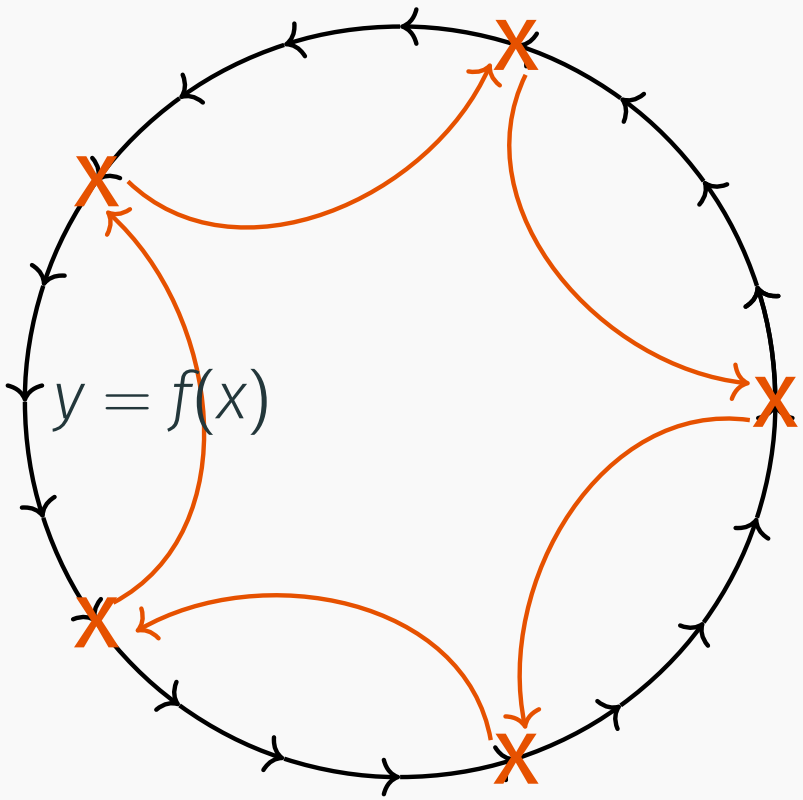

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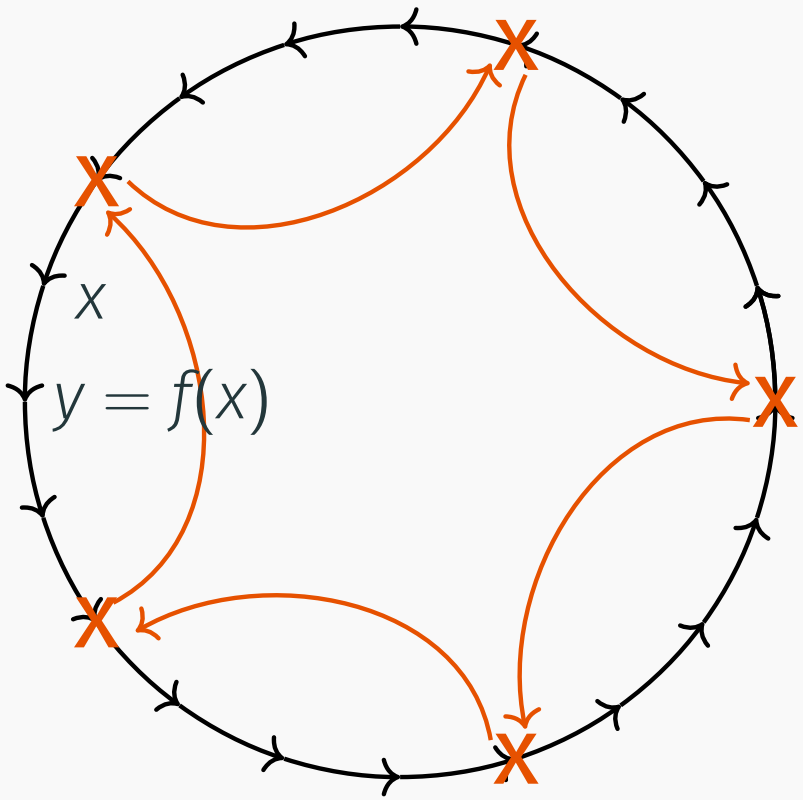

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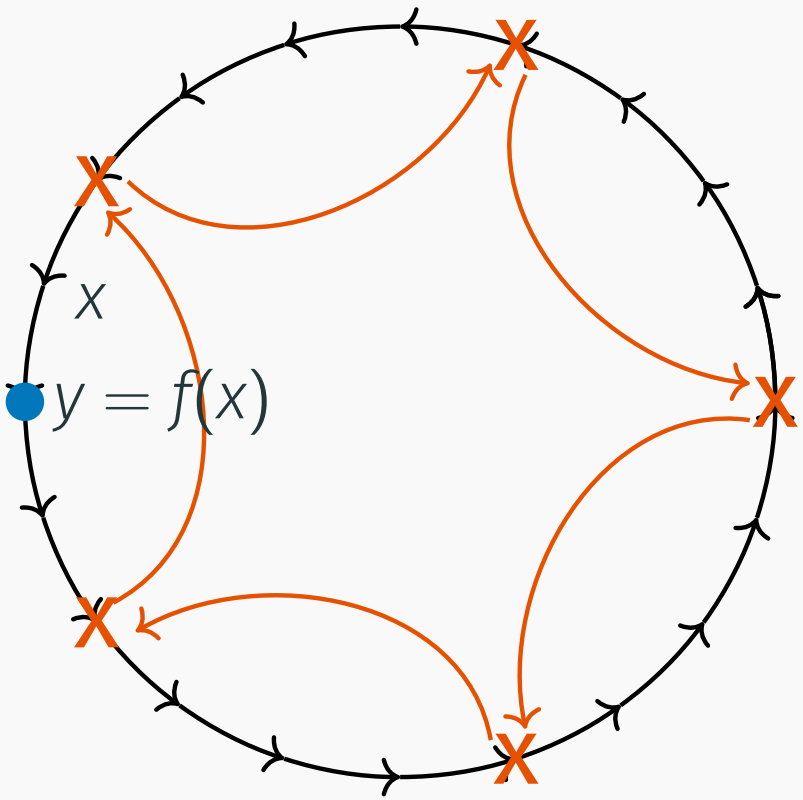

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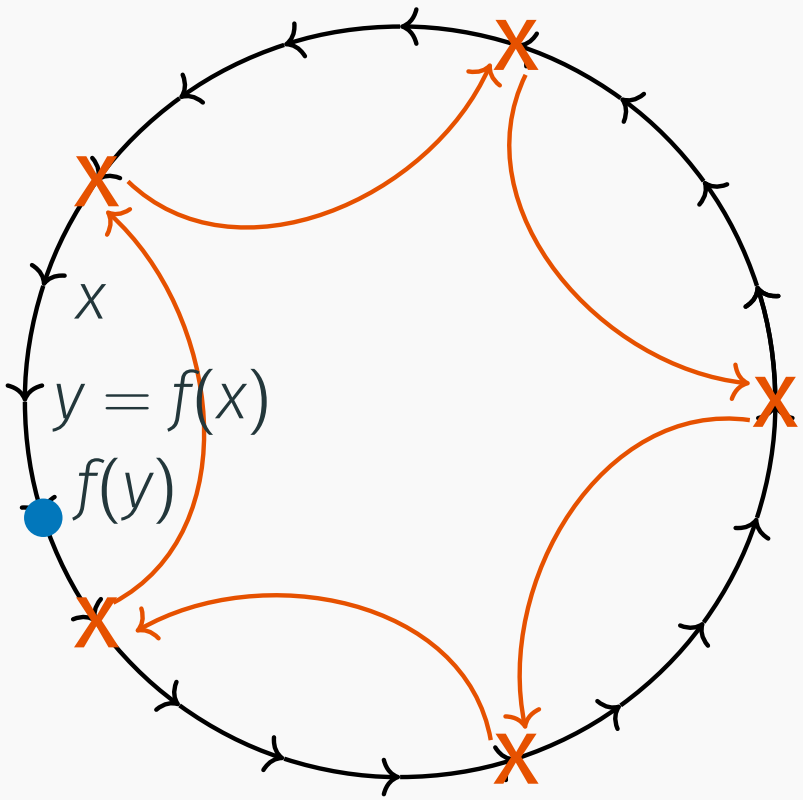

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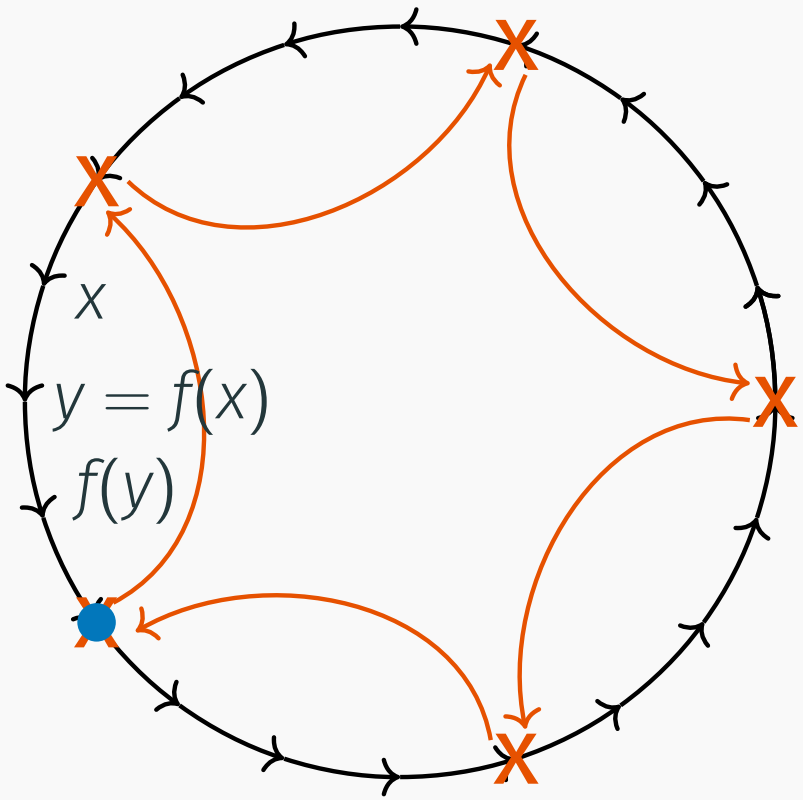

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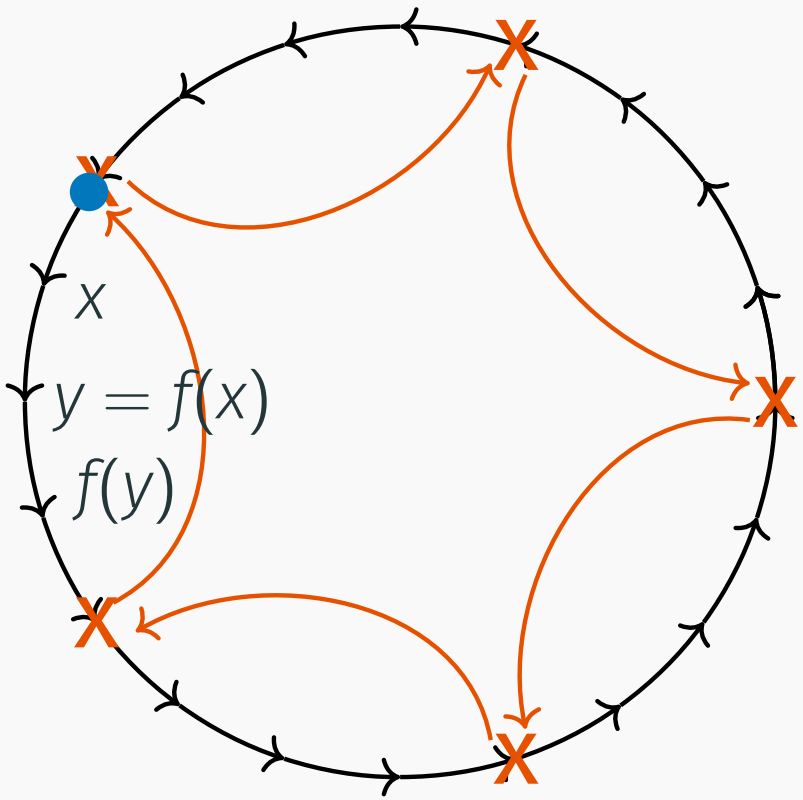

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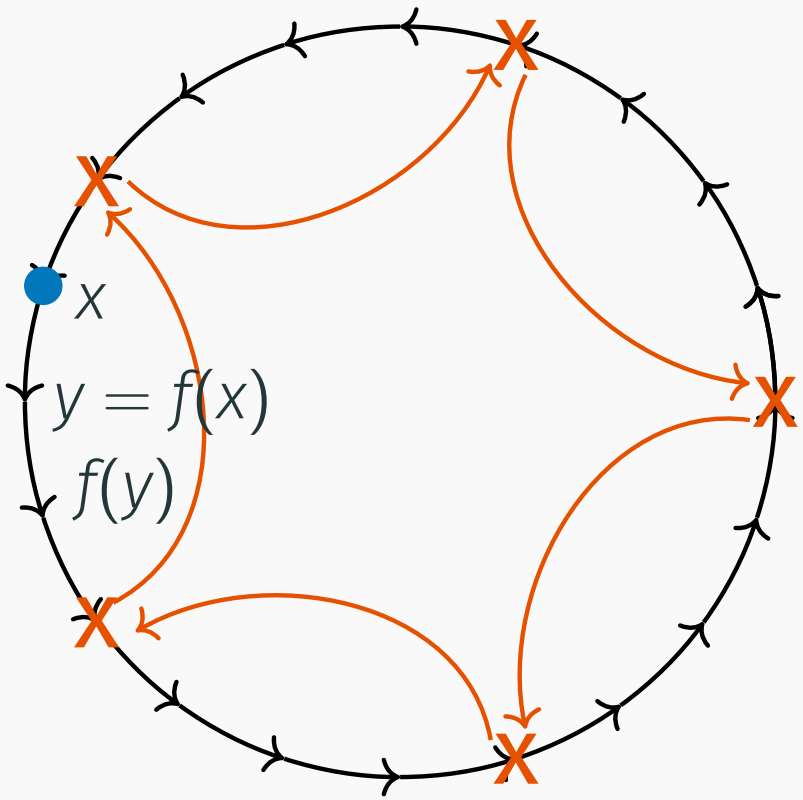

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- [Hel80, FN00] gives a way to invert any f which can be computed efficiently and has small domain and range
- Can solve 3SUM-Indexing in time T and space S for any $S^3T = n^6$. E.g., $S = n^{1.9}$ and $T = n^{0.3}$

Equivalence

Theorem (GGHPV20)

Cryptography with preprocessing is equivalent to data structure lower bounds for problems with efficient query generation.

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- Now $G\colon \{0,1\}^{2n-2} \to \{0,1\}^{2n}$:

$$G(x,y) = F'(x) + F'(y)$$

New DS Lower Bounds

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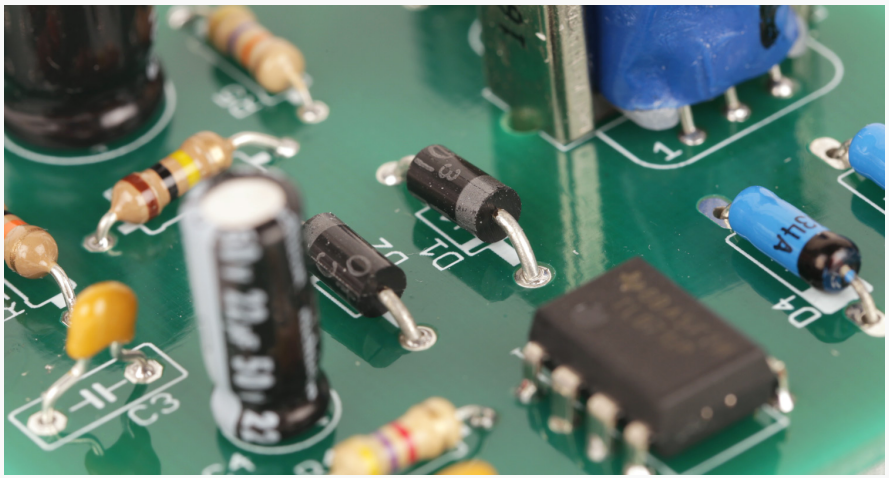
CIRCUIT COMPLEXITY



Boolean Circuits

$$f\colon \{0,1\}^n \to \{0,1\}^n$$

$$\begin{aligned} g_1 &= x_1 \oplus x_2 \\ g_2 &= x_2 \wedge x_3 \\ g_3 &= g_1 \vee g_2 \\ g_4 &= g_2 \vee 1 \\ g_5 &= g_3 \equiv g_4 \end{aligned}$$

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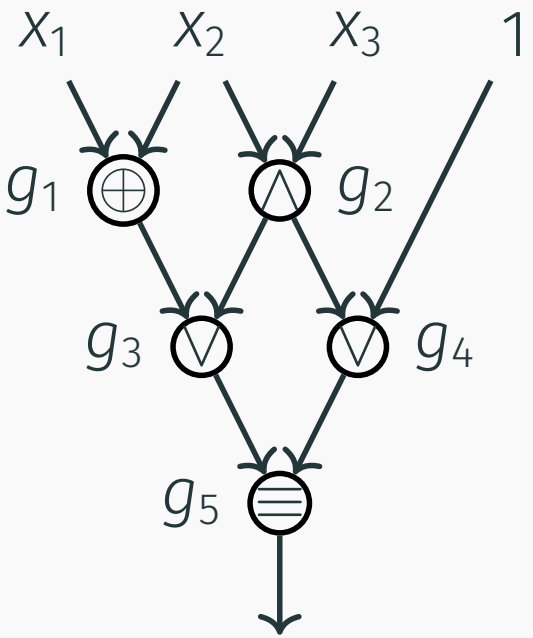

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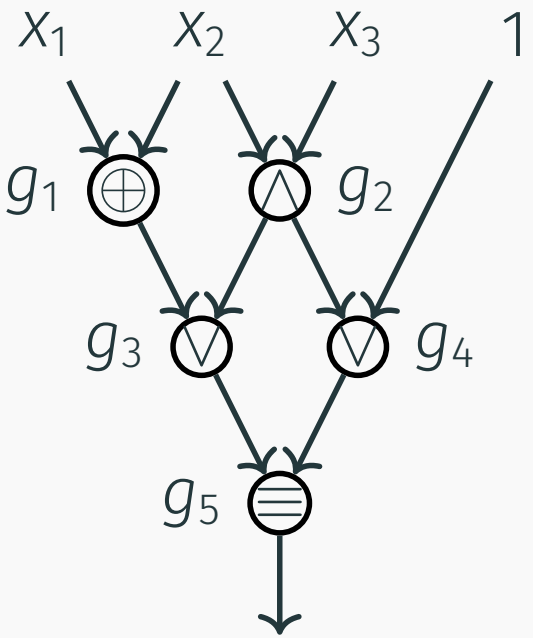

Inputs:

$x_1, \ldots, x_n, 0, 1$

Gates:

binary functions

Fan-out:

unbounded

Depth:

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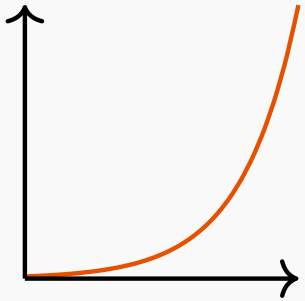
Explicit Bounds



Most functions have exponential circuit complexity

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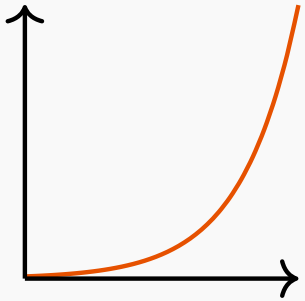

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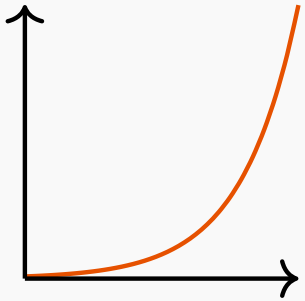

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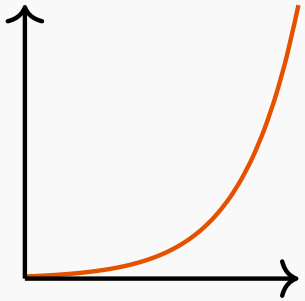

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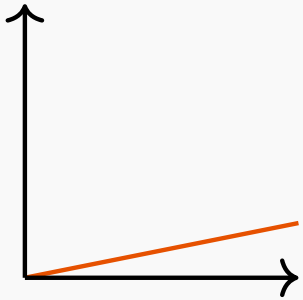

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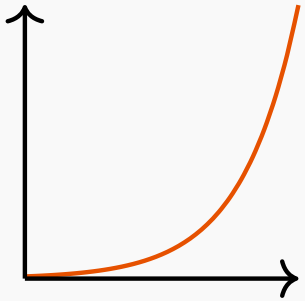

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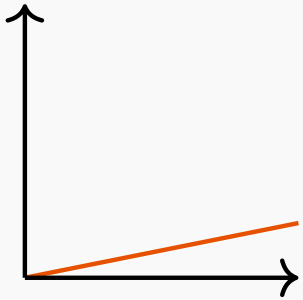

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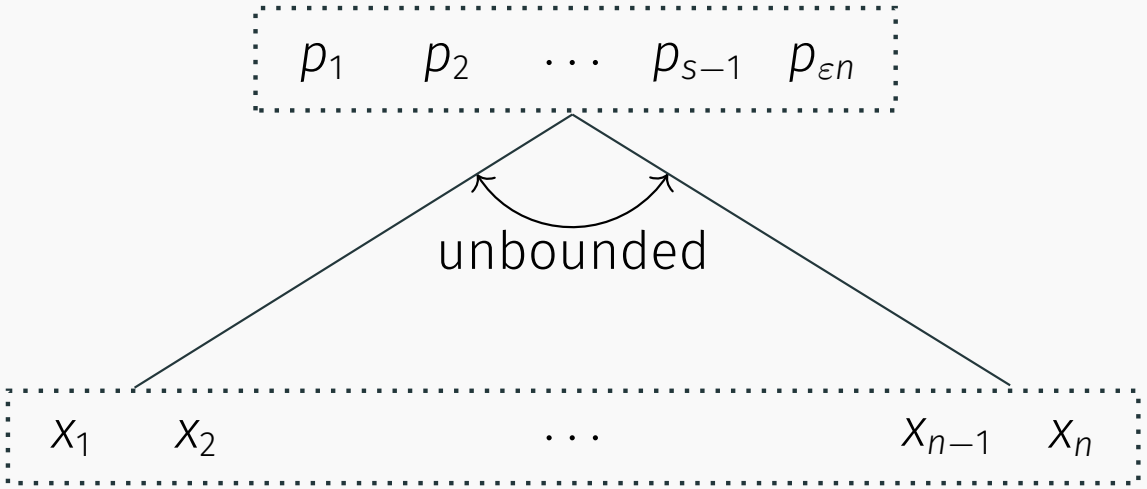

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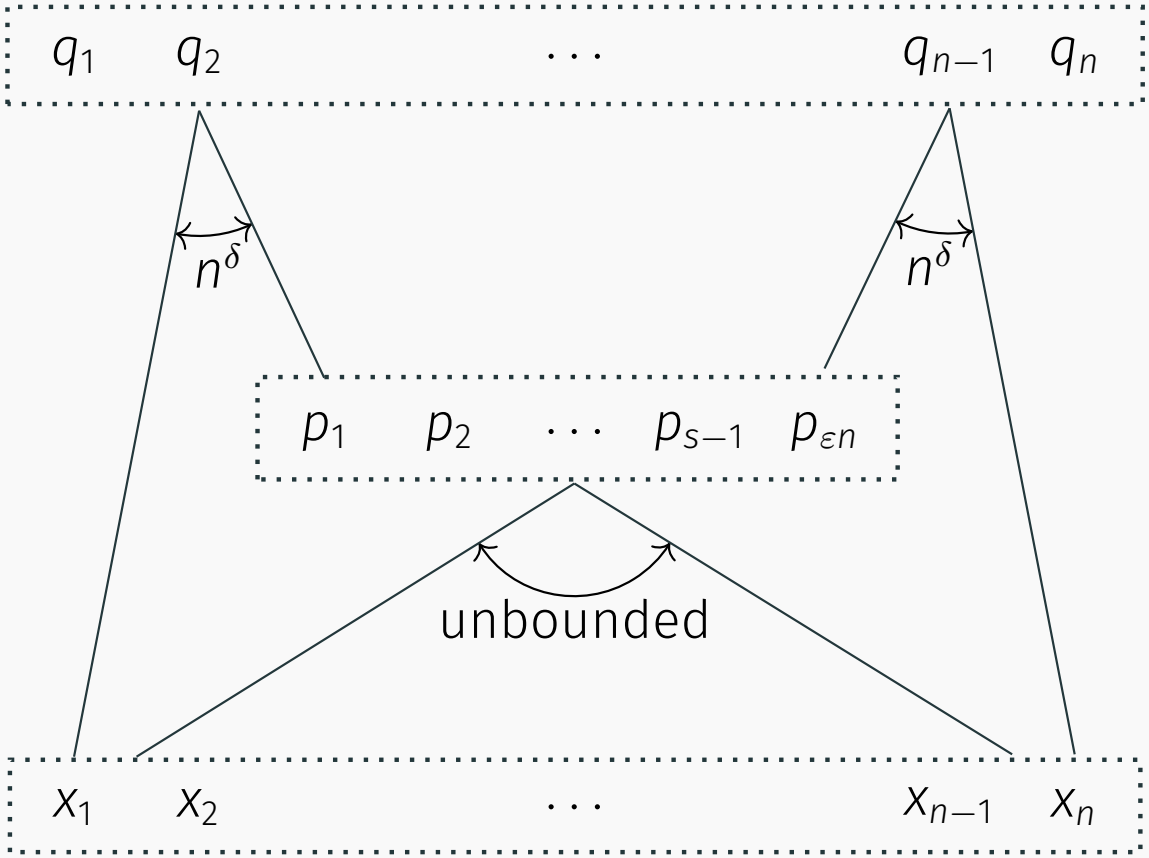
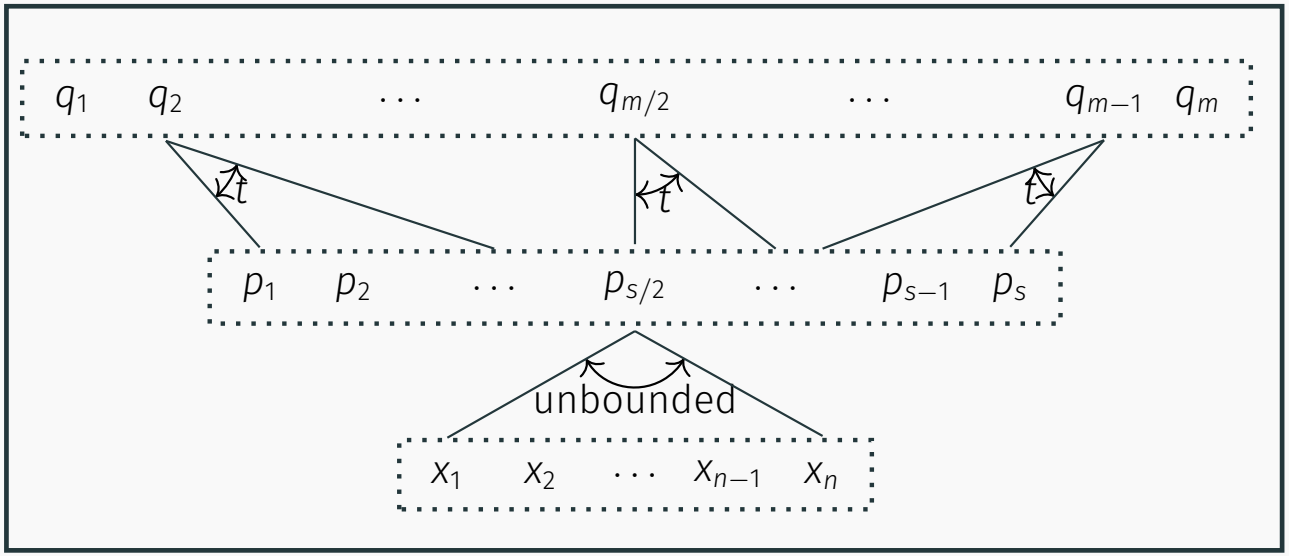
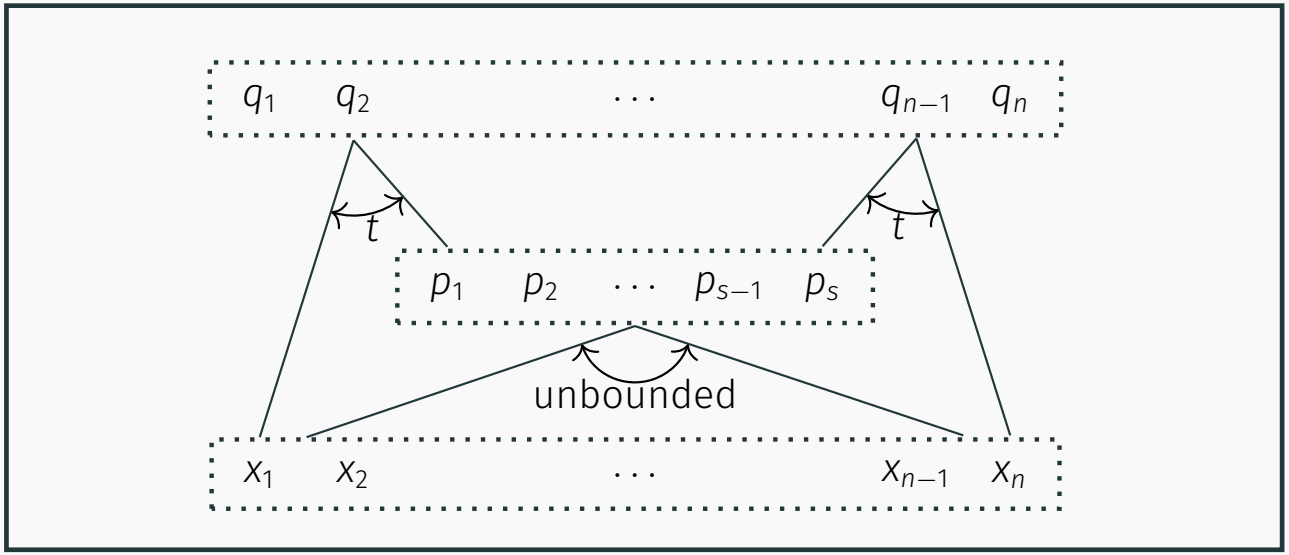


Comparison

Efficient Data Structures



Small Circuits



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Examples: Polynomial evaluation, Matrix-vector multiplication, Range counting, Partial sums, ...

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Small circuit / Non-rigid

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A: t-sparse; B: $\mathrm{rk} \leq \varepsilon n$

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Efficient Data Structure

$$\overset{m \times n}{M} = \overset{m \times s}{A} \cdot \overset{s \times n}{B}$$

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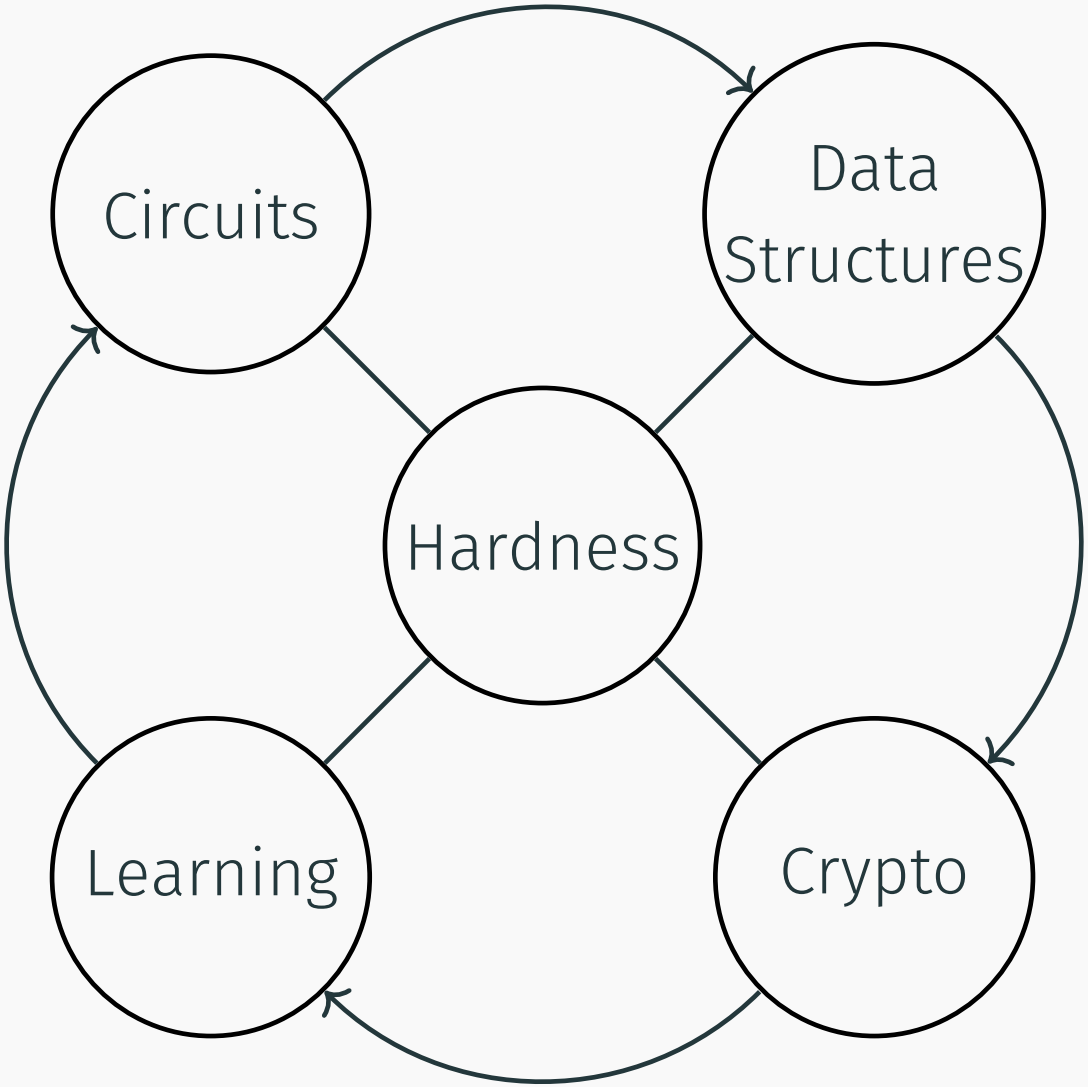

Thank you for your attention!