The KRW conjecture

Ivan Mihajlin

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Motivation

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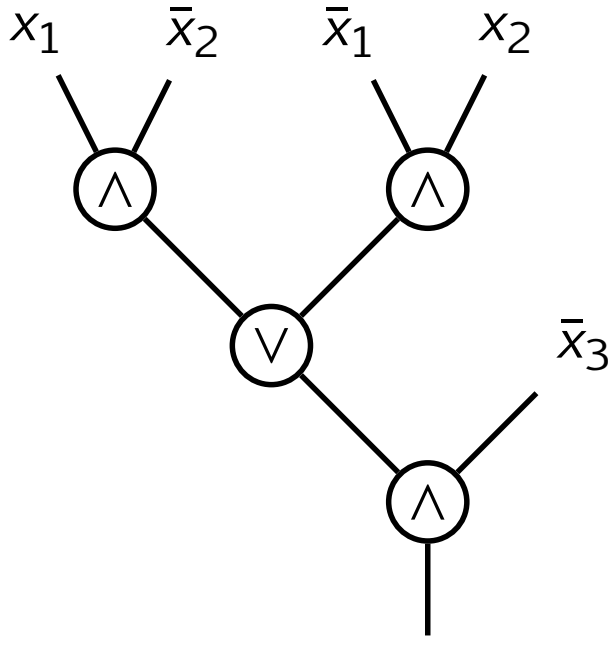
It would be amazing if any function computable in time $n^{O(1)}$ can be computed in time $O(\log n)$ on a parallel computer!

We want to prove that that it is not possible, in other words that:

$$\mathrm{P} \neq \mathrm{NC}^1$$

.

Formula



$$((x_1 \wedge \bar{x}_2) \vee (\bar{x}_1 \wedge x_2)) \wedge x_3$$

Formulas

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For $f : \{0,1\}^n \to \{0,1\}$: $D(f)$ is the minimal depth of a formula that computes f.

To prove $\mathrm{P} \neq \mathrm{NC}^1$ we need to show that for some $f \in \mathrm{P}$:

$$D(f) = \omega(\log n).$$

The KRW conjecture

Definition

For $f : \{0,1\}^m \to \{0,1\}$ and $g : \{0,1\}^n \to \{0,1\}$, *the block-composition* $f \diamond g : (\{0,1\}^n)^m \to \{0,1\}$ is defined by

$$(f \diamond g)(x_1, \ldots, x_m) = f(g(x_1), \ldots, g(x_m)),$$

where $x_1, \ldots, x_m \in \{0,1\}^n$.

Conjecture (The KRW conjecture)

Let $f : \{0,1\}^m \to \{0,1\}$ *and* $g : \{0,1\}^n \to \{0,1\}$ *be non-constant functions. Then*

$$\mathrm{D}(f \diamond g) \approx \mathrm{D}(f) + \mathrm{D}(g).$$

Theorem

KRW conjecture implies $\mathrm{P} \not\subseteq \mathrm{NC}^1$.

Karchmer-Wigderson games

Definition

The Karchmer-Wigderson game for $f : \{0,1\}^n \to \{0,1\}$:

- Alice gets $x \in \{0,1\}^n$ such that $f(x) = 0$.
- Bob gets $y \in \{0,1\}^n$ such that $f(y) = 1$.
- Their goal is to find $i \in [n]$ such that $x_i \neq y_i$.

The Karchmer-Wigderson relation for f:

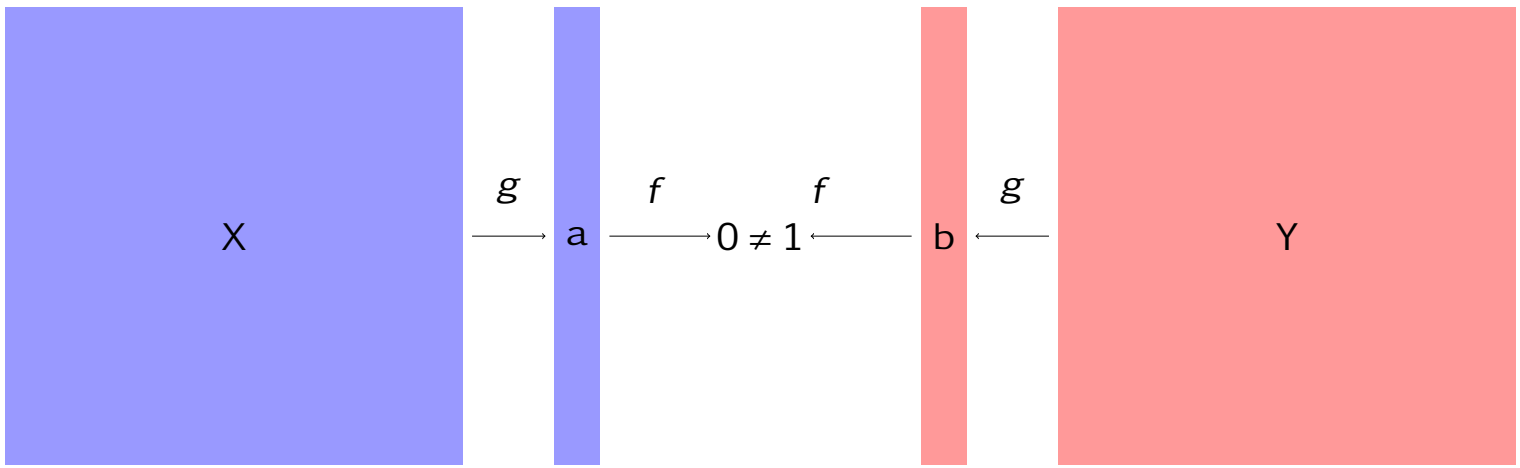
$$\mathrm{KW}_f = \{(x, y, i) \mid x, y \in \{0,1\}^n, i \in [n], f(x) = 0, f(y) = 1, x_i \neq y_i\}.$$

Conjecture (The KRW conjecture (reformulation))

Let $f, g : \{0,1\}^m \to \{0,1\}$ *be non-constant functions. Then*

$$\mathrm{CC}(\mathrm{KW}_f \diamond \mathrm{KW}_g) \approx \mathrm{CC}(\mathrm{KW}_f) + \mathrm{CC}(\mathrm{KW}_g).$$

Composition of KW games



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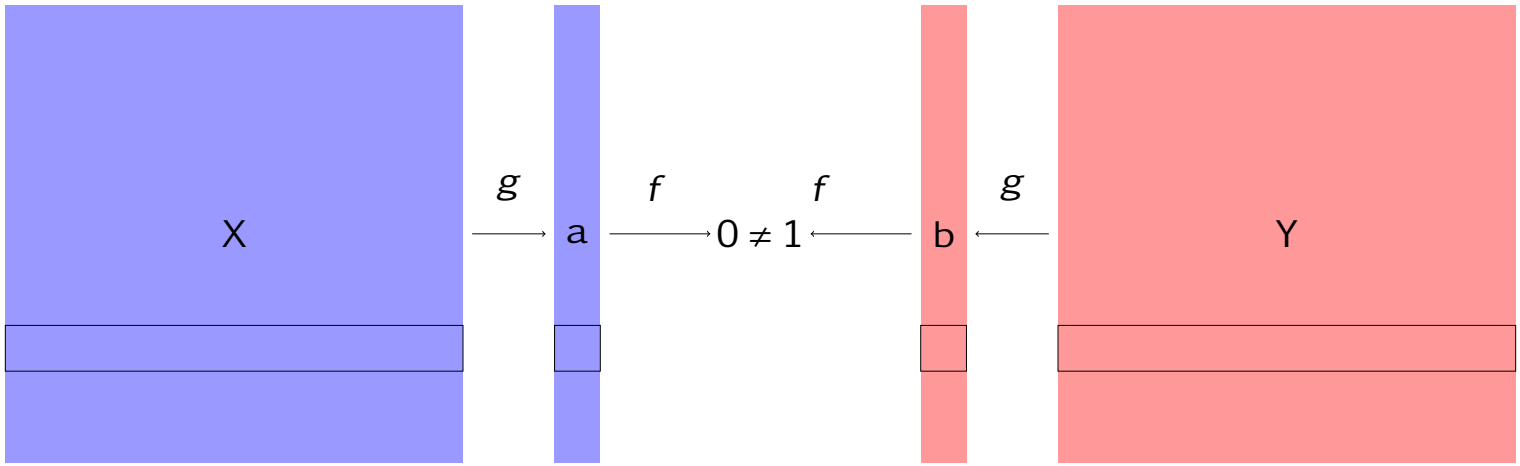

Solve KW_f on (a, b) first, then solve KW_g on (X_i, Y_i).

Universal relation

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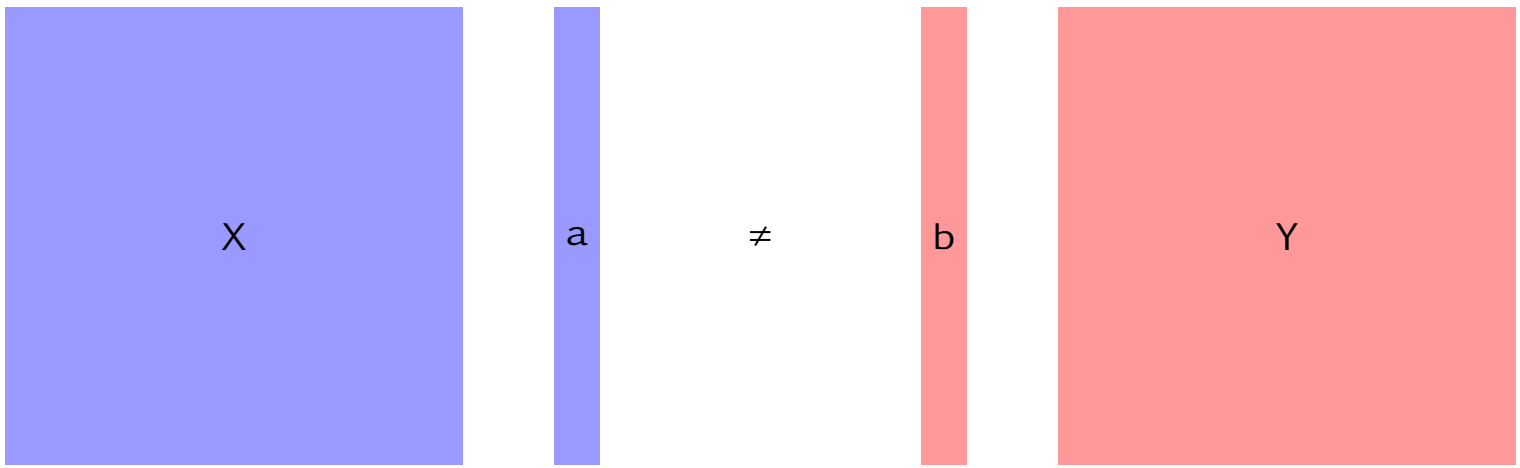
The composition of universal relations

$$\mathrm{U}_m \diamond \mathrm{U}_n = \Big\{\big((a, X), (b, Y), (i, j)\big) \mid a, b \in \{0,1\}^m, X, Y \in \{0,1\}^{m \times n},$$
$$i \in [m], j \in [n], a \neq b, \forall k \in [m] : a_k \neq b_k \implies X_k \neq Y_k\Big\}.$$

Composition of universal relations



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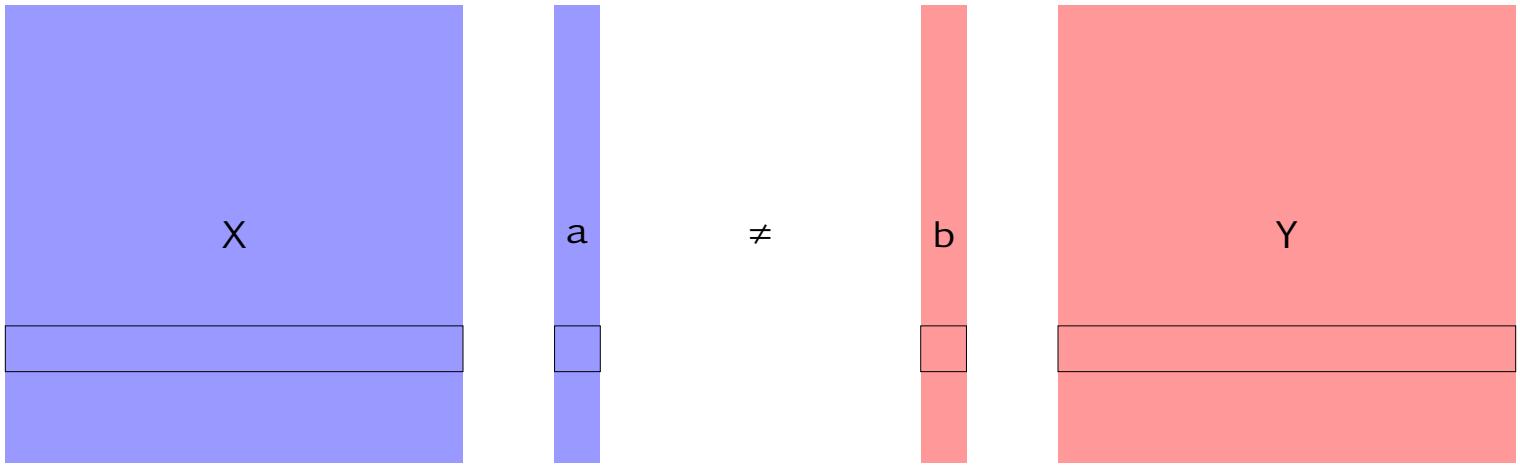

Use that $a_i \neq b_i$ implies that $X_i \neq Y_i$.
Solve U on (a, b) first, then solve U on rows (X_i, Y_i).

Known results

- [Edmonds, Impagliazzo, Rudich, Sgall, 01] and [Håstad, Wigderson, 98]:

$$\mathrm{CC}(\mathrm{U}_n \diamond \mathrm{U}_n) = 2n - o(n).$$

- [Gavinsky, Meir, Weinstein, Wigderson, 16], improved by [Meir, Koroth, 19]:

$$\mathrm{CC}(\mathrm{KW}_f \diamond \mathrm{U}_n) = \log \mathrm{L}(f) + n - O(\log^* n).$$

- [Mihajlin, Smal 21]]:

$$\exists g : \ \mathrm{CC}(\mathrm{U}_n \diamond g) \geq 1.5n - o(n).$$

Maintaining symmetry

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- Show that after $n - o(n)$ bits of communication there is still a row in A and B such that players don't know much about.
- Force players to find difference in that row.
- Show that it would take $n - o(n)$ bits of communication.

Measure argument

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- Show a lower bound on μ at the start of the protocol.
- Show an upper bound on μ at the end of the protocol.

Reasons to be optimistic

The lower bound for $\mathrm{U}_n \diamond g$ was proven by the symmetry argument!

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We hope that one can find a good measure to swap U_n for a function f.

Multiplexer relation

Definition

The same function multiplexer communication game MUX_n:

- Alice gets $g : \{0,1\}^n \to \{0,1\}$ and $x \in \{0,1\}^n$: $g(x) = 0$,
- Bob gets $g : \{0,1\}^n \to \{0,1\}$ and $y \in \{0,1\}^n$: $g(y) = 1$,
- Goal: find $i \in [n]$ such that $x_i \neq y_i$.

Definition

The non-promise version MUX'_n:

- Alice gets $f : \{0,1\}^n \to \{0,1\}$ and $x \in \{0,1\}^n$: $g(x) = 0$,
- Bob gets $g : \{0,1\}^n \to \{0,1\}$ and $y \in \{0,1\}^n$: $g(y) = 1$,
- Goal: find $i \in [n]$ such that $x_i \neq y_i$, or output $\perp$ if $f \neq g$.

Half-duplex communication complexity: definitions

- Alice and Bob communication over half-duplex channel.
- Every round each player chooses one of three actions: send 0, send 1, or receive. Three types of rounds:
  - **classical**: one sends, one receives
  - **wasted**: both sends
  - **silent**: both receives
- Protocol is a pair of rooted trees of arity 4 (send 0, send 1, receive 0, and receive 1).
- For all $f\{0,1\}^n \to \{0,1\}$,

$$\mathrm{CC}(f) \geq \mathrm{CC}^{hd}(f) \geq \mathrm{CC}(f)/2.$$

- $\mathrm{CC}^{hd}(\mathrm{EQ}_n) \geq n/\log 2.5$.

Half-duplex communication complexity: facts

Theorem

For any non-empty finite set S, $\mathrm{CC}^{hd}(\mathrm{EQ}_S) \geq \log|S|/\log 2.5$.

Lemma

For all $n \in \mathbb{N}$, there exist a function $f : \{0,1\}^n \to \{0,1\}$ such that

$$\mathrm{CC}(\mathrm{KW}_f) \geq \mathrm{CC}^{hd}(\mathrm{MUX}'_n) - O(\log n).$$

Proof.

- Suppose that $\mathrm{CC}(\mathrm{KW}_f) \leq d$ for all $f : \{0,1\}^n \to \{0,1\}$.
- The following protocol solves MUX'_n:
  - Alice follows the shortest protocol for f on x.
  - Bob follows the shortest protocol for g on y.
  - They check that the answer is correct using $O(\log n)$ bits.

Idea for the proof of $f \diamond g$

- Consider composition of multiplexor relations.

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- Show lower bound on this measure at the start and upper bound at the end of the protocol.
- Use it to prove lower bound and then use this lower bound to show existence of hard functions.
- Show That $\mathrm{P} \neq \mathrm{NC}^1$.