The generalized Vaserstein symbol

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For inductive purposes, one may just consider the following question:

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Main goal: Describe the fibers $\phi_r^{-1}([P \oplus R])$.

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We can identify $Um(R^{r+1})$ and $Aut(R^{r+1})$ with

$Um_{r+1}(R)$ = set of unimodular rows of length $r+1$ over R, i.e. row vectors $(a_1, ..., a_{r+1})$ with $a_i \in R$, $1 \leq i \leq r+1$, and $\langle a_1, ..., a_{r+1} \rangle = R$

$GL_{r+1}(R)$ = group of invertible matrices of rank $r+1$ over R

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Description in case $P = R^r$

$Um_{r+1}(R)/GL_{r+1}(R) \cong \phi_r^{-1}([R^{r+1}])$

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Theorem (Suslin, 1977)

If $r = d$ and R is an affine algebra over an algebraically closed field, then P is cancellative.

Theorem (Bhatwadekar, 2003)

If $r = d$ and R is an affine algebra over an infinite perfect field k with $c.d.(k) \leq 1$ and $d! \in k^{\times}$, then P is cancellative.

Case $r = d - 1 = 2$ with P free

Theorem (Fasel, 2011)

If R is a smooth affine algebra of dimension 3 over an algebraically closed field k with $char(k) \neq 2$, then R^2 is cancellative, i.e. stably free R-modules of rank 2 are free.

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Rao-van-der-Kallen proved that the Vaserstein symbol

$$V : Um_3(R)/E_3(R) \xrightarrow{\cong} W_E(R)$$

is a bijection, inducing a group structure on $Um_3(R)/E_3(R)$.

Case $r = d - 1 = 2$ with P free

Sketch of proof (continued).

One uses

- $n \cdot V(b_1, b_2, b_3) = V(b_1, b_2, b_3^n)$ for all $(b_1, b_2, b_3) \in Um_3(R)$
- $W_E(R) \cong GW^3_{1,red}(R)$ is 2-divisible
- Any unimodular row of the form (b_1, b_2, b_3^2) is the first row of an invertible 3×3-matrix of det. 1 (Swan-Towber theorem)

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Now let $(a_1, a_2, a_3) \in Um_3(R)$. Since

$$V : Um_3(R)/E_3(R) \xrightarrow{\cong} W_E(R)$$

is a bijection, this implies

$$(a_1, a_2, a_3) \sim_{E_3(R)} 2 \cdot (b_1, b_2, b_3) \sim_{E_3(R)} (b_1, b_2, b_3^2) \sim_{SL_3(R)} (1, 0, 0).$$

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Hence $Um_3(R)/GL_3(R)$ is trivial. □

Case $r = d - 1$ with P free or $r = 2$

Theorem (Fasel-Rao-Swan, 2012)

If R is a normal affine algebra of dimension $d \geq 4$ over an algebraically closed field k with $(d-1)! \in k^{\times}$, then R^{d-1} is cancellative, i.e. stably free modules of rank $d-1$ are free.

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Theorem (Asok-Fasel, 2014)

If R is a smooth affine algebra of dimension 3 over an algebraically closed field k with $char(k) \neq 2$ and P has rank 2, then P is cancellative.

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If R is a smooth affine algebra of dimension 3 over an algebraically closed field k with $char(k) \neq 2$ and P has rank 2, then P is cancellative.

In fact, the map $(c_1, c_2) : \mathcal{V}_2(R) \xrightarrow{\cong} \mathrm{CH}^1(X) \times \mathrm{CH}^2(X)$ is a bijection, where $X = Spec(R)$.

Case $r = d - 1$ with $\det(P)$ free

Conjecture (Asok-Fasel, 2013)

Let k be a perfect field with $char(k) \neq 2$ and $d \geq 4$. Then there is a sequence

$$\mathbf{K}^M_{d+2}/24 \to \pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0) \to \mathbf{GW}^d_{d+1} \to 0$$

of homomorphisms in $\mathrm{Ab}^{\mathbb{A}^1}_k$ which is exact at $\pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)$ and also becomes exact at $\mathbf{GW}^d_{d+1}$ after $d-3$-fold contraction.

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Theorem (Peng Du, 2020)

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Let k be an algebraically closed field. Then there is a smooth affine k-algebra R of dimension 4 with a non-free stably free R-module of rank 2 (N. Mohan Kumar).

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Approach: Define a generalized Vaserstein symbol

$$V_\theta : Um(P \oplus R)/E(P \oplus R) \to \tilde{V}(R) \cong W_E(R)$$

associated to P and any fixed isomorphism $\theta : R \xrightarrow{\cong} \det(P)$.

The set $A(R)$

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$A_{2n}(R)$ = the set of alternating invertible matrices of rank $2n$ over R, i.e. the set of matrices $M \in GL_{2n}(R)$ with $x^t Mx = 0$ for any column vector x of length $2n$ over R.

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We inductively define an element $\psi_{2n} \in A_{2n}(R)$ by setting

$$\psi_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \psi_{2n+2} = \psi_{2n} \perp \psi_2.$$

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We let $A(R) = \bigcup_{n \geq 1} A_{2n}(R)$.

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$M \in A_{2m}(R)$ and $N \in A_{2n}(R)$ are called G-equivalent, $M \sim_G N$, if there is an integer $s \in \mathbb{N}$ and a matrix $\varphi \in SL_{2n+2m+2s}(R) \cap G$ such that

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This defines an equivalence relation on $A(R)$. We denote the sets of equivalence classes by

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The operation $\perp$ equips these sets with the structure of an abelian group.

The groups $W_E(R)$ and $W_{SL}(R)$

Any alternating invertible matrix M has a so-called Pfaffian $Pf(M) \in R^{\times}$, which satisfies the following formulas:

- $Pf(M \perp N) = Pf(M)Pf(N)$ for $M \in A_{2m}(R), N \in A_{2n}(R)$
- $Pf(\varphi^t N \varphi) = \det(\varphi) Pf(N)$ for $\varphi \in GL_{2n}(R), N \in A_{2n}(R)$
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We define

$$W_E(R) := \ker(Pf) \qquad W_{SL}(R) := \ker(\overline{Pf}).$$

The Vaserstein symbol

Let $a = (a_1, a_2, a_3) \in Um_3(R)$. Then choose $b = (b_1, b_2, b_3)$ with $b_1, b_2, b_3 \in R$ and $\sum_{i=1}^{3} a_i b_i = 1$.

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The element of $W_E'(R)$ defined by the matrix

$$V(a,b) = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -b_3 & b_2 \\ a_2 & b_3 & 0 & -b_1 \\ a_3 & -b_2 & b_1 & 0 \end{pmatrix}$$

has Pfaffian 1 and does not depend on the choice of b_1, b_2, b_3.

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has Pfaffian 1 and does not depend on the choice of b_1, b_2, b_3. We obtain a well-defined *Vaserstein symbol*

$$V : Um_3(R)/E_3(R) \to W_E(R)$$

The Vaserstein symbol

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The element of $W'_E(R)$ defined by the matrix

$$V(a,b) = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -b_3 & b_2 \\ a_2 & b_3 & 0 & -b_1 \\ a_3 & -b_2 & b_1 & 0 \end{pmatrix}$$

has Pfaffian 1 and does not depend on the choice of b_1, b_2, b_3. We obtain a well-defined *Vaserstein symbol*

$$V : Um_3(R)/E_3(R) \to W_E(R)$$

and an induced map

$$V : Um_3(R)/SL_3(R) \to W_{SL}(R).$$

Reinterpretation of the Vaserstein symbol

Let $X = Spec(R)$ be a smooth affine scheme of finite type over a perfect base field k with $char(k) \neq 2$.

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such that the induced map $[X_+, BGL_r]_{\mathcal{H}_\bullet(k)} \to [X_+, BGL_{r+1}]_{\mathcal{H}_\bullet(k)}$ corresponds to the stabilization map

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The composite of natural morphisms

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Two such triples (P, f_0, f_1) and (P', f_0', f_1') are called isometric if there exists an isomorphism $h : P \to P'$ such that $f_i = h^t f_i' h$ for $i = 0, 1$.

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Any element $a : P \oplus R \to R$ of $Um(P \oplus R)$ gives rise to an exact sequence of the form

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$$V_\theta(a,s) = [P \oplus R^2, \chi \perp \psi_2, (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)]$$

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$$V_\theta : Um(P \oplus R)/SL(P \oplus R) \to \tilde{V}_{SL}(R).$$

Connection with the usual Vaserstein symbol

Remark

If we take $P = R^2$ and let $e_1 = (1,0), e_2 = (0,1) \in R^2$, then there is a canonical isomorphism $\theta : R \xrightarrow{\cong} \det(R^2)$ given by $1 \mapsto e_1 \wedge e_2$.

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The generalized Vaserstein symbol $V_{-\theta}$ associated to R^2 and $-\theta$ coincides with the usual Vaserstein symbol via the canonical isomorphism $\tilde{V}(R) \cong W_E(R)$.

Surjectivity

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Theorem (S., 2019)

Let R be a Noetherian ring of Krull dimension ≤ 4. Then $V_\theta : Um(P \oplus R)/SL(P \oplus R) \rightarrow \tilde{V}_{SL}(R)$ is surjective if $SL(P_5)$ acts transitively on $Um(P_5)$.

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Special case $P = R^2$ (Suslin-Vaserstein, 1976)

Let R be a Noetherian ring of Krull dimension ≤ 4. Then $V : Um_3(R)/SL_3(R) \to W_{SL}(R)$ is surjective if $SL_5(R)$ acts transitively on $Um_5(R)$.

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Special case $P = R^2$ (S., 2019)

Let R be a Noetherian ring of dimension ≤ 4 such that $SL_5(R)$ acts transitively on $Um_5(R)$. Then $V : Um_3(R)/SL_3(R) \to W_{SL}(R)$ is injective if and only if $SL_4(R)e_4 = Sp(M)e_4$ for all $M \in A_4(R)$ with Pfaffian 1, where $e_4 = (0,0,0,1)^t$.

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Theorem (S., 2020)

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Sketch of proof.

To show: $\pi Sp(M) = Um_d(R)$, where $\pi = (1, 0, ..., 0)$.

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Sketch of proof (continued).

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In particular, $(a_1, \ldots, a_d) = \pi\varphi_a = \pi\tau\psi$ and $\tau\psi \in Sp(M)$.

Transitivity of $Sp(M)$

Sketch of proof (continued).

Hence for any $a = (a_1, ..., a_d)$ it suffices to find $\varphi_a \in SL_d(R)$ as above.

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Hence it actually suffices to find $\varphi_{b'}$ for a row of the form $b' = (b_1, ..., b_d^{(d-1)!})$ as above.

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Idea: Use Suslin matrices!

Transitivity of $Sp(M)$

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Let $x = (x_1, \ldots, x_d) \in Um_d(R)$ and $y = (y_1, \ldots, y_d)$ s.t. $\sum_{i=1}^{d} x_i y_i = 1$.

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Bijectivity result

Theorem (S., 2020)

Let R be a smooth affine algebra of dimension 4 over an algebraically closed field k with $6 \in k^{\times}$. Then

$$V : Um_3(R)/SL_3(R) \xrightarrow{\cong} W_{SL}(R)$$

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Criterion

Theorem (S., 2019)

Let R be a smooth affine algebra of dimension $d = 4$ over an algebraically closed field k with $6 \in k^\times$. Then all stably free R-modules of rank 2 are free if and only if $W_{SL}(R)$ is trivial.

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By the previous results, the latter condition holds if and only if $W_{SL}(R)$ is trivial. □

Some further applications

As before, let P be a projective R-module of rank 2 with a fixed trivialization $\theta : R \xrightarrow{\cong} \det(P)$ of its determinant.

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The map $V_\theta : Um(P \oplus R)/E(P \oplus R) \to \tilde{V}(R)$ is a bijection if R is a 2-dimensional regular Noetherian ring or a 3-dimensional regular affine algebra over a perfect field k with $c.d.(k) \leq 1$ and $6 \in k^\times$.

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If R is a smooth affine algebra of dimension 4 over an algebraically closed field k with $char(k) \neq 2, 3$ and Q is a projective R-module of rank 3 with $c_1(Q) = c_2(Q) = c_3(Q) = 0$, then Q is cancellative.

Topologically contractible varieties

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Example

The affine spaces $\mathbb{A}_k^n$ are the primordial examples of topologically contractible smooth affine k-schemes.

The generalized Serre question

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Generalized Serre question

If $X = Spec(R)$ is a topologically contractible smooth affine complex variety, then are all algebraic vector bundles on X trivial?

Result in dimension 1

Theorem (Serre, 1957)

Let R be a commutative Noetherian ring of dimension d. Then any finitely generated projective R-module P of constant rank $r > d$ is of the form $P \cong P' \oplus R^{r-d}$ for some projective R-module P' of constant rank d.

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Corollary (Gurjar, 1980)

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In fact, the map $(c_1, c_2) : \mathcal{V}_2(R) \xrightarrow{\cong} \mathrm{CH}^1(X) \times \mathrm{CH}^2(X)$ is a bijection.

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If $X = Spec(R)$ is a topologically contractible smooth affine complex variety of dimension 3, then all algebraic vector bundles on X are trivial if and only if $\mathrm{CH}^2(X)$ and $\mathrm{CH}^3(X)$ are trivial.

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and Kumar-Murthy proved in 1982 that

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Result in dimension 4

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If $X = Spec(R)$ is a smooth affine variety of dimension 4 over an algebraically closed field k with $6 \in k^{\times}$, all algebraic vector bundles on X are trivial if $\mathrm{CH}^i(X) = 0$ for $1 \leq i \leq 4$ and $\mathrm{H}^2(X, \mathbf{I}^3) = 0$.

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Sketch of proof.

$\mathrm{CH}^i(X) = 0$ for $i = 1, 2, 3, 4 \Rightarrow \tilde{K}_0(X) = 0$ *and* $\mathcal{V}_1(R)$ is trivial.

Result in dimension 4

Theorem (S., 2019)

If $X = Spec(R)$ is a smooth affine variety of dimension 4 over an algebraically closed field k with $6 \in k^{\times}$, all algebraic vector bundles on X are trivial if $\mathrm{CH}^i(X) = 0$ for $1 \leq i \leq 4$ and $\mathrm{H}^2(X, \mathbf{I}^3) = 0$.

Sketch of proof.

$\mathrm{CH}^i(X) = 0$ for $i = 1, 2, 3, 4 \Rightarrow \tilde{K}_0(X) = 0$ *and* $\mathcal{V}_1(R)$ is trivial.

$0 \to \tilde{K}_0(X) \to K_0(X) \xrightarrow{rk} \mathbb{Z} \to 0$ exact $\Rightarrow$ all v.b. stably trivial.

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$\mathrm{CH}^3(X) = \mathrm{CH}^4(X) = \mathrm{H}^2(X, \mathbf{I}^3) = 0 \Rightarrow W_{SL}(R) = 0$ (use the Gersten-Grothendieck-Witt spectral sequence) $\Rightarrow \mathcal{V}_2(R)$ trivial. □

Result in dimension 4

Corollary (S., 2019)

If $X = Spec(R)$ is a topologically contractible smooth affine complex variety of dimension 4, then all algebraic vector bundles on X are trivial if $\mathrm{CH}^i(X) = 0$ for $i = 2, 3, 4$ and $\mathrm{H}^2(X, \mathbf{I}^3) = 0$.

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Conjecture (Asok-Østvær, 2019)

If $X = Spec(R)$ is a topologically contractible smooth affine complex variety and $x \in X(\mathbb{C})$ a chosen base-point, then there exists an integer $n \geq 0$ such that $\Sigma^n_{\mathbb{P}^1}(X, x)$ is $\mathbb{A}^1$-contractible.

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Remark

A proof of the conjecture above would automatically yield an affirmative answer to the generalized Serre question on algebraic vector bundles in dimensions 3 and 4.