A Gersten complex on real schemes

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joint work with H. Xie

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- Coherent duality is a prototype for several other duality theories

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- The duality theorems can be expressed in terms of the *Grothendieck six functors formalism*
- Major differences between Coherent duality and other duality theories

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- Dualizing objects are preserved by $f^!$

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  Projection formula: f proper,

$$Rf_* R\underline{Hom}(\mathcal{F}, f^!\mathcal{G}) \simeq R\underline{Hom}(Rf_*\mathcal{F}, \mathcal{G})$$

- In coherent duality, no suitable subcategory of *constructible objects* preserved by 6 functors

Dualizing complexes

A (coherent) *dualizing complex* over a scheme X is a complex $K \in D^b_{coh}(Qcoh(X))$ quasi-isomorphic to a bounded complex of injective $\mathcal{O}_X$-modules, such that for any $\mathcal{F} \in D^b_{coh}(Qcoh(X))$,

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- $f : Y \to X$ quasi-projective, K dualizing complex over X
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- If K and K' are two dualizing complexes over X, then
  $K' = K \otimes \mathcal{L}[n]$, where $n \in \mathbb{Z}$, $\mathcal{L}$ invertible $\mathcal{O}_X$-module

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- In particular, a scheme with a dualizing complex is universally catenary and has a codimension function

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- For $j \geqslant \dim(X) + 1$, $\underline{I}^j \simeq \underline{I}^{j+1}$.

Real schemes

- The *real spectrum* of a ring A is a topological space $\mathrm{Sper}(A)$

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- X_r depends functorially on X: $f : X \to Y \Rightarrow f_r : X_r \to Y_r$

Real schemes and real algebraic geometry

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- The *constructible subsets* of $\mathrm{Sper}(A)$ are boolean combinations of the $D(a)$'s (finite unions, finite intersections, complements)

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- Example: If X is separated of finite type over the real numbers $\mathbb{R}$, then the constructible subsets of X_r are precisely the semi-algebraic subsets of $X(\mathbb{R})$

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Theorem (Coste-Roy, Scheiderer)

There is a canonical equivalence of sites $X_r \simeq X_{ret}$.

Jacobson's theorem

Global signature map

$$\begin{aligned}\mathrm{Sign} : W(X) &\to C(X_r, \mathbb{Z}) \\ [\phi] &\mapsto ((x, P) \mapsto \mathrm{Sign}_P([i_x^* \phi]))\end{aligned}$$

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If $2 \in \mathcal{O}(X)^\times$, this induces an isomorphism of ret-sheaves

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- Jacobson's theorem can be extended to the twisted setting

Twisting by invertible sheaves

- $(X, \mathcal{O}_X)$ ringed space, $\mathcal{L}$ invertible $\mathcal{O}_X$-module, $\mathcal{F}$ sheaf on X with an action of $\mathcal{O}_X^\times$, define $\mathcal{F}(\mathcal{L})$ as the sheafification of

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Theorem (J.)

$A \in D_c(X_r) \Leftrightarrow \exists$ *finite stratification of* X_r *into constructible subsets* X_i *such that* $A_{|X_i}$ *is constant with perfect stalks*

The Gersten-Witt complex

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- Pass to I^∞ and sheafify $\Rightarrow$ complex $\underline{C}(X, \underline{I}^\infty, K)$

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  - Absolute purity (Scheiderer, J.-Xie): Z, X regular, $i : Z \to X$ closed immersion of codimension c, N normal bundle of i, $\mathcal{F}$ locally constant constructible (lcc) sheaf on X_r

$$i_r^! \mathcal{F} \simeq i_r^* \mathcal{F} \otimes \mathbb{Z}(\det(N)^{-1})[-c]$$

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where $\omega_{x/X}$ = determinant of the normal bundle of $x \to \mathrm{Spec}(\mathcal{O}_{X,x})$

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- By sheafifying we get a canonical resolution of $\mathcal{F}$ by acyclic sheaves (Scheiderer, J.-Xie)

$$\bigoplus_{x \in X^{(0)}} (i_{x_r})_* \mathcal{F}(\omega_{x/X}) \to \bigoplus_{x \in X^{(1)}} (i_{x_r})_* \mathcal{F}(\omega_{x/X}) \to \cdots$$

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- *Twisted transfer*: L/F finite field extension, $H =$ one-dimensional F-vector space

$$C(L_r, \mathbb{Z}(\mathrm{Hom}_F(L, H))) \to C(F_r, \mathbb{Z}(H))$$

  is defined using the trace form $L \to \mathrm{Hom}_F(L, F)$

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  Here $i_{x_r} : x_r \to X_r$ is the canonical map, and the terms $(i_{x_r})_*\mathbb{Z}(K_{k(x)})$ look like skyscraper sheaves

Main theorem

Theorem (J.-Xie)

(A) X excellent regular, $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then $\underline{C}(X_r, \mathcal{L}) \simeq \mathbb{Z}(\mathcal{L})$.

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Consequently, $K \mapsto \underline{C}(X_r, K)$ gives rise to a map
$\{\text{dualizing complexes over } X\} \to \{\text{dualizing objects in } D(X_r)\}$

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The first part is a counterpart of Jacobson's theorem for complexes, and the second part is because $\underline{C}(X_r, K)$ is a complex of acyclic sheaves

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- If we replace $\mathbb{R}$ by a real closed field, we obtain Delfs' semi-algebraic Borel-Moore homology

Thank you!