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1. Projective Modules

Recall that if R is a ring, then a (finitely generated) *projective* R-module P is a summand of a free module $R^{\oplus n}$; in other words, there is some R-module Q and an isomorphism of R-modules

$$R^{\oplus n} \simeq P \oplus Q.$$

Serre's Problem (1955): Is every finite generated projective module over a polynomial ring $F[x_1, \ldots, x_d]$ over a field F free. Serre proved earlier that any such projective module P is *stably free*, that is $P \oplus R^{\oplus k} \simeq R^{\oplus m}$ for some k and m.

In (January of) 1976, Suslin and Quillen independently and essentially simultaneously proved the following theorem.

Theorem. *(Suslin, Quillen, 1976) Let*

$$R = F[x_1, x_2, \ldots, x_d]$$

be a polynomial algebra over a field F. Then every finitely generated projective R-module P is free.

Due to Serre's result it suffices to prove the *cancellation property*: for a finitely generated projective R-module: given an isomorphism $P \oplus R \simeq R^{\oplus m}$ for some $m > 0$, then P is isomorphic to the free R-module $R^{\oplus(m-1)}$.

Suslin observes that this is equivalent to proving that the group $\mathrm{GL}_n(R)$ acts transitively on the set of unimodular rows of length n, where a row is *unimodular* if its entries generate R as an R-module.

Let $P \oplus R \simeq R^n$ for a commutative ring R and an R-module P. Then the element $(0,1)$ in $P \oplus R$ corresponds to a unimodular row $(x_1, x_2, \ldots, x_n)$ in R^n. Then P is free, $P \simeq R^{n-1}$, (that is the cancellation holds) if and only if the unimodular row can be completed to an invertible $n \times n$ matrix over R.

This always holds when $n = 2$:
If $x_1 y_1 + x_2 y_2 = 1$, then the matrix

$$\begin{pmatrix} x_1 & x_2 \\ -y_2 & y_1 \end{pmatrix}$$

is invertible. The answer is negative in general if $n \geqslant 3$. But if $n = 3$, the row (x_1, x_2, x_3^2) can be completed.

Theorem. *(Suslin, 1982) Let* $m_1, m_2, \ldots,$ m_n *be positive integers. Given a unimodular row* $(x_1, x_2, \ldots, x_n)$ *over a commutative ring* R*, the unimodular row*

$$(x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n})$$

can always be completed to an invertible $n \times n$ *matrix over* R *if and only if the product* $m_1 m_2 \cdots m_n$ *is divisible by* $(n-1)!$.

2. K_1 AND STABILIZATION

Let R be a ring and $\mathrm{GL}_n(R)$ the general linear group. We have a sequence of subgroups

$$\mathrm{GL}_1(R) \subset \mathrm{GL}_2(R) \subset \cdots \subset \mathrm{GL}_n(R) \subset \cdots$$

and set $\mathrm{GL}(R) = \cup\, \mathrm{GL}_n(R)$.

Let $E_n(R) \subset \mathrm{GL}_n(R)$ be the subgroup generated by elementary matrices $1 + re_{ij}$, $r \in R$, $E(R) = \cup E_n(R)$.

Then $E(R)$ is normal in $\mathrm{GL}(R)$ and

$$K_1(R) := \mathrm{GL}(R)/E(R).$$

If $E_n(R)$ is normal in $\mathrm{GL}_n(R)$, we have a homomorphism

$$\mathrm{GL}_n(R)/E_n(R) \to \mathrm{GL}(R)/E(R) = K_1(R)$$

Theorem. *(Suslin, 1977) Let R be a commutative ring. Then $E_n(R)$ is normal in $\mathrm{GL}_n(R)$ for every $n \geqslant 3$.*

Theorem. *(Suslin, 1977) Let A be a Noetherian commutative ring,*

$$B = A[x_1, x_2, \ldots, x_d].$$

If $n \geqslant \max(3, \dim(A) + 2)$, then

the homomorphism

$$\mathrm{GL}_n(B)/E_n(B) \to K_1(B)$$

is an isomorphism.

3. K_2 AND THE BRAUER GROUP

Let F be a field. Consider central simple finite dimensional algebras over F.

Example. Suppose F contains a primitive degree n root ξ_n of 1. Let $a, b \in F^\times$. Write $(a, b)_n$ for the F *cyclic* algebra of dimension n^2 (degree n) with basis $u^i v^j$,

$i, j = 0, 1, \ldots, n-1$ and relations

$$u^n = a, \ v^n = b, \ uv = \xi_n vu.$$

If $n = 2$ cyclic algebras are called *(generalized) quaternion algebras.*

Let A and B be two central simple finite dimensional algebras over F. We say that A and B are *Brauer-equivalent* if $M_n(A) \simeq M_m(B)$. The set $\mathrm{Br}(F)$ of all equivalence classes is a group with respect to the tensor product, called the *Brauer group* of F.

If A is a cyclic algebra of degree n, then

$[A]^n = 1$, that is $[A] \in \mathrm{Br}(F)[n]$.

Example. 1. Every algebra of degree 2 is a quaternion algebra.

2. If $\deg(A) = 4$ and $[A]^2 = 1$, then A is the tensor product of two quaternion algebras: $A \simeq Q_1 \otimes Q_2$.

3. There is A of degree 8 such that $[A]^2 = 1$ but A is not tensor product of three quaternion algebras (Amitsur, Roven, Tignol).

On the other hand

$$M_2(A) \simeq Q_1 \otimes Q_2 \otimes Q_3 \otimes Q_4$$

is the tensor product of four quaternion algebras (Tignol), hence in $\mathrm{Br}(F)$ the class $[A]$ is a product of the classes of quaternion algebras.

Albert conjectured in 1932 that for any field F the group $\mathrm{Br}(F)[2]$ is generated by the classes of quaternion algebras.

Milnor defined the *Milnor K-theory ring*

$K_*^M(F)$ of a field Fas the tensor algebra

$$T^*(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times)^{\otimes 2} \oplus (F^\times)^{\otimes 3} \oplus \cdots$$

modulo the Steinberg relations $\{a \otimes b\}$ if $a + b = 1$. The class of the tensor $a_1 \otimes \cdots \otimes a_n$ in $K_n^M(F)$ is denoted by $\{a_1, \ldots, a_n\}$.

Let Γ_F be the absolute Galois group of a field F, i.e., $\Gamma_F = \mathrm{Gal}(F_{sep}/F)$, where F_{sep} is a separable closure of F. Let M be a discrete Γ_F-module over F we can consider *Galois cohomology groups*

$$H^n(F, M) := H^n(\Gamma_F, M).$$

For example, Γ_F acts on the group $F^\times_{sep}$ of nonzero elements of F_{sep}. We have

$$\begin{aligned} H^0(F, F^\times_{\text{sep}}) &= F^\times, \\ H^1(F, F^\times_{\text{sep}}) &= 1, \text{ (Hilbert's Thm 90)}, \\ H^2(F, F^\times_{\text{sep}}) &= \text{Br}(F). \end{aligned}$$

If m is an integer prime to $\text{ch}(F)$ and $M = \mu_m$ is the Γ_F-module of m-th roots of unity in F_{sep}. The Kummer exact sequence of Γ_F-modules

$$1 \to \mu_m \to F^\times_{\text{sep}} \xrightarrow{m} F^\times_{\text{sep}} \to 1$$

yields a connecting homomorphism

$$l : F^\times = H^0(F, F^\times_{\text{sep}}) \to H^1(F, \mu_m).$$

The cup-product in Galois cohomology yields a homomorphism

$$(F^\times)^{\otimes n} \to H^n(F, \mu_m^{\otimes n})$$

for every $n \geqslant 0$ sending the tensor $a_1 \otimes \cdots \otimes a_n$ to $l(a_1) \cup \cdots \cup l(a_n)$. This factors through the

norm residue homomorphism

$$\boxed{h_n : K_n^M(F)/mK_n^M(F) \to H^n(F, \mu_m^{\otimes n})}$$

It was conjectured by Bloch and Kato that h_n is an isomorphism for every field F, every m prime to $\mathrm{ch}(F)$ and every n. The case $n = 1$ follows from Hilbert Theorem 90. One can assume that $m = p$ is prime.

When $n = 2$ and $m = p$ is prime, we may assume that $\xi_p \in F$, then

$$H^2(F, \mu_p^{\otimes 2}) = \mathrm{Br}(F)[p] \otimes \mu_p \simeq \mathrm{Br}(F)[p].$$

The norm residue homomorphism

$$h_2 : K_2^M(F)/pK_2^M(F) \to \mathrm{Br}(F)[p]$$

takes $\{a, b\}$ to the class of the cyclic algebra $(a, b)_p$.

The Albert Problem is the surjectivity of h_2 in the case when $n = p = 2$.

Suslin suggested using Severi-Brauer varieties of central simple algebras. For a csa A of degree p over F there is the

Severi-Brauer variety $SB(A)$ that is a twisted form of the projective space $\mathbb{P}^{p-1}$ and $SB(A)$ has a point over a field extension K/F if and only if the algebra A is split over K.

If $A = (a, b)_p$ is a cyclic algebra, then using the machinery of algebraic K-theory one compares the properties of the norm residue homomorphisms h_2 over F and the function field of $SB(A)$. Over the function field the algebra $(a, b)_p$ is split and the 2-symbol $\{a, b\}$ is divisible by p.

Taking a "composite"of the function field of $SB(A)$ for all cyclic algebras of degree p we get a field extension F_∞/F such that the map h is an isomorphism of trivial groups over F_∞. Going down, we see that h_2 is an isomorphism over F.

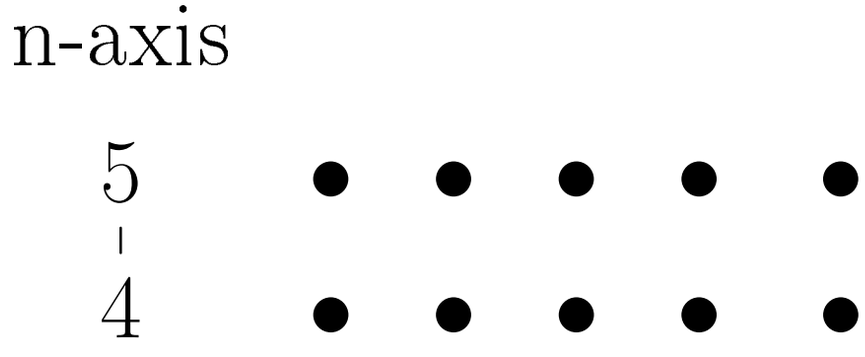
Theorem. *(M-Suslin, 1982) Let F be a field, m an integer prime to $\mathrm{ch}(F)$ and $\xi \in F$ a primitive m-th root of unity. Then the norm residue homomorphism*

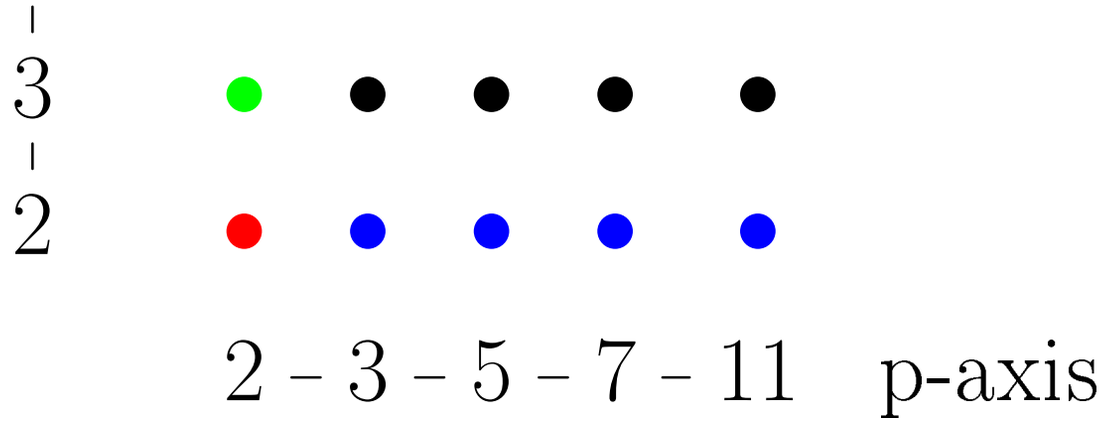
$$h_2 : K_2^M(F)/mK_2^M(F) \to \mathrm{Br}(F)[m]$$

is an isomorphism. This proves the Bloch-Kato Conjecture for $n = 2$ and arbitrary m.

In particular, the subgroup $\mathrm{Br}(F)[m]$ of the Brauer group is generated by the classes of cyclic algebras $(a,b)_m$ for $a,b \in F^\times$.

The case $n=3$ and $p=2$ of the Bloch-Kato Conjecture was completed by M-Suslin and independently by Rost.





In full generality, the Bloch-Kato Conjecture was proved by Voevodsky and Rost.

As corollary of this theorem about h_2 Suslin proved the following case of Serre's Conjecture II stating that all torsors of a simply connected semisimple algebraic group over a field F of cohomological dimension $\leqslant 2$ are trivial.

Let A be a central simple algebra of degree n over a field F and let

$$\mathrm{Nrd} : A^{\times} \to F^{\times}$$

be the *reduced norm* homomorphism. Write $\mathrm{SL}_1(A)$ for the kernel of Nrd viewed as a homomorphism of algebraic groups. The group $\mathrm{SL}_1(A)$ is simply connected of type A_{n-1}.

Theorem. *(Suslin, 1982) Serre's Conjecture II holds for* $\mathrm{SL}_1(A)$ *for any central simple algebra* A.

The set of $\mathrm{SL}_1(A)$-torsors over F is $F^\times / \operatorname{Nrd}(A^\times)$. Thus, Serre's Conjecture II for $\mathrm{SL}_1(A)$ is equivalent to the surjectivity of Nrd.

Suslin proved the surjectivity by first reducing to the case when n is a prime number and A is a central simple algebra of degree n and then showed that an element $a \in F^\times$ is in the image of the reduced norm map if and only if

$$l(a) \cup [A] = 0 \quad \text{in} \quad H^3(F, \mu_n^{\otimes 2}),$$

where $[A]$ is the class of the algebra A in the subgroup $H^2(F, \mu_n) = \mathrm{Br}(F)[n]$ of the Brauer group of F.

4. Milnor and Quillen K-theories

We define the *indecomposable group* $K_3(F)_{\mathrm{ind}}$ as the cokernel of

$$K_3^M(F) \to K_3^Q(F).$$

The following theorem of determines the torsion and co-torsion of $K_2(F)$ and $K_3(F)_{\mathrm{ind}}$.

Theorem. *(M-Suslin, 1990) Let m be an integer prime to* $\mathrm{ch}(F)$. *Then there is an exact sequence*

$$0 \to H^0(F, \mu_m^{\otimes 2}) \to K_3(F)_{\mathrm{ind}} \xrightarrow{m} K_3(F)_{\mathrm{ind}}$$

$$\to K_2(F) \xrightarrow{m} K_2(F) \to H^2(F, \mu_m^{\otimes 2}) \to 0.$$

If $p = \mathrm{ch}(F) > 0$, *then the group* $K_3(F)_{\mathrm{ind}}$ *is uniquely p-divisible.*

Theorem implies that the group $K_3(F)_{\mathrm{ind}}$ is never trivial! The motivic spectral (constructed considerably later) yields an isomorphism $K_3(F)_{\mathrm{ind}} \simeq H^1_M(F, \mathbb{Z}(2))$.

One application of the theorem is the computation that $K_3(\mathbb{Q})_{\text{ind}}$ is a cyclic group of order m, where m is the largest integer such that $i^2 - 1$ is divisible by m for all i prime to m. This integer m equals 24. One can deduce from this that

$$K_3^Q(\mathbb{Q}) = \mathbb{Z}/48\mathbb{Z},$$

a result of Lee and Szczarba.

The next theorem gives an information about relationship between the Milnor K-theory $K_n^M(F)$ and Quillen K-theory $K_n^Q(F)$ of a field F , showing it is closely related to the homological stability for GL_n.

Theorem. *(Suslin, 1982) If F is an infinite field, then*

$$H_n(\mathrm{GL}_n(F)) \xrightarrow{\sim} H_n(\mathrm{GL}(F)),$$

$$H_n(\mathrm{GL}_n(F))/\operatorname{Im}\{H_n(\mathrm{GL}_{n-1}(F))\} \simeq K_n^M(F).$$

Moreover, the natural composition

$$K_n^M(F) \to K_n^Q(F) \to H_n(\mathrm{GL}(F)) \simeq H_n(\mathrm{GL}_n(F)) \to K_n^M(F)$$

is multiplication by $(n-1)!$.

5. Suslin's Rigidity Theorem

Theorem. *(Suslin 1983) Let $F_0 \subseteq F$ be an inclusion of algebraically closed fields, and let n be a positive integer invertible in F_0. Then the induced map on mod-n Quillen K-theory is an isomorphism:*

$$K_*^Q(F_0;\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} K_*^Q(F;\mathbb{Z}/n\mathbb{Z}).$$

Equivalently, the natural maps

$$K_*^Q(F_0)[n] \to K_*^Q(F)[n] \quad \text{and}$$

$$K_*^Q(F_0)/n \to K_*^Q(F)/n$$

are isomorphisms.

The key point of the proof is the following rigidity statement: If X is a smooth, connected affine variety over F and x, x' are F-rational points of X, then the two induced maps $K_*(X,\mathbb{Z}/n) \rightrightarrows K_*(F,\mathbb{Z}/n)$ are equal.

Suslin deduces the following statement (a conjecture of Lichtenbaum):

Theorem. *(Suslin 1983-84) Let F be an algebraically closed field and let n be a positive integer invertible in F. Then*

$$K_{2i}^{Q}(F, \mathbb{Z}/n) = \mathbb{Z}/n, \; K_{2i+1}^{Q}(F, \mathbb{Z}/n) = 0.$$

6. Suslin's Complex

In 1987, Suslin introduced the complex $\mathrm{Sus}_*(X)$ associated with a variety X over a field F as follows:

For each n, consider the algebraic n-simplex

$$\Delta^n = \operatorname{Spec} F[t_0, \ldots, t_n]/(t_0 + \cdots + t_n - 1).$$

Let $\mathrm{Sus}_n(X)$ be the group of *finite correspondences* on $\Delta^n \times X$, that is the free abelian group generated by closed, integral subschemes

$$Z \subset \Delta^n \times X$$

that are finite and surjective over Δ^n.

These groups form a simplicial abelian group. Passing to the associated chain complex yields *Suslin's complex*.

Its homology groups

$$H_n^{\mathrm{Sus}}(X) := H_n(\mathrm{Sus}_*(X))$$

are called *Suslin homology* of X.

To give a finite irreducible correspondence $Z \subset \Delta^n \times X$ of degree d over Δ^n is the same as to give a morphism fromΔ^n to the d-th symmetric power $S^d(X)$ of X.

Motivation for the definition of Suslin's complex comes from the Dold-Thom theorem in algebraic topology which asserts that the homotopy groups of the simplicial abelian group $Sing(\coprod_{d=0}^{\infty} S^d(T))^+)$ can be naturally identified with the integral homology of a CW complex T.

Suslin and Voevodsky proved the following theorem.

Theorem. *(Suslin-Voevodsky, 1996) If X is a quasi-projective variety over $\mathbb{C}$, then the natural map*

$$H_n^{\mathrm{Sus}}(X, \mathbb{Z}/m\mathbb{Z}) \ \to \ H_n(X(\mathbb{C}), \mathbb{Z}/m\mathbb{Z})$$

is an isomorphism.

7. Motivic cohomology theories

There are *motivic complexes* of étale sheaves with transfers $\mathbb{Z}(q)$ for $q \geqslant 0$. In fact, $\mathbb{Z}(0)$ and $\mathbb{Z}(1)$ are quasi-isomorphic to the sheaves $\mathbb{Z}$ and $\mathbb{G}_m$ placed in degree 0 and 1, respectively.

One defines the *motivic* and *étale motivic* cohomology groups of a smooth variety X by

$$H^p_M(X, \mathbb{Z}(q)) := H^p_{Zar}(X, \mathbb{Z}(q)),$$
$$H^p_{et}(X, \mathbb{Z}(q)) := H^p_{et}(X, \mathbb{Z}(q)).$$

There is a natural homomorphism

$$H^n_M(X, \mathbb{Z}(q)) \ \to \ H^n_{et}(X, \mathbb{Z}(q)).$$

The integral *Beilinson-Lichtenbaum Conjecture* asserts that this is an isomorphism when $n \leqslant q$ and a monomorphism when $n = q + 1$.

When $q = n$ and $X = \mathrm{Spec}(F)$, the group $H^n_M(X, \mathbb{Z}(q))$ is canonically isomorphic to $K^M_n(F)$. In fact, the Bloch-Kato Conjecture is a special case of the Beilinson-Lichtenbaum Conjecture. It turned out that these two conjectures are equivalent.

Theorem. *(Suslin-Voevodsky, 2000) Let*

F be a field. Then the following two statements are equivalent:

1) The Bloch-Kato Conjecture holds for all fields.

2) The Beilinson-Lichtenbaum Conjecture holds for all smooth varieties over all fields.

The following theorem provides the analogue for algebraic K-theory of the Atiyah-Hirzeburch spectral sequence for topological K-theory.

Theorem. *(Friedlander-Suslin, 2002) Let*

X be a smooth quasi-projective variety over a field. Then there is a strongly convergent spectral sequence

$$E_2^{p,q} = H_M^{p-q}(X, \mathbb{Z}(-q)) \;\Rightarrow\; K_{-p-q}(X).$$