Formulas for the $\mathbb{A}^1$-degree

jt w Thomas Brazelton

and Stephen McKean

k a field, char $k \neq 2$

Let $f = (f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$

have only isolated zeros

$\Leftrightarrow (f_1, \ldots, f_n)$ is a complete intersection

<u>Goal</u>: Find a simple algebraic formula for the $\mathbb{A}^1$-degree of f

Definition of $\deg^{\mathbb{A}^1} f$:

Need Morel's $\mathbb{A}^1$-degree

$\deg^{\mathbb{A}^1} : [\mathbb{P}^n_k / \mathbb{P}^{n-1}_k, \mathbb{P}^n_k / \mathbb{P}^{n-1}_k] \to GW(k)$

Grothendieck-Witt ring of k

generators of $GW(k)$:

$\langle a \rangle := \left(\begin{matrix} (x,y) \to axy \\ k \times k \to k \end{matrix} \right)$

$a \in k^\times$

Def (Kass-Wickelgren): Let x be an isolated zero of $f: \mathbb{A}^n_k \to \mathbb{A}^n_k$. Find a Zariski nbhd U of x st $f^{-1}(0) \cap U = \{x\}$

$$\deg^{\mathbb{A}^1}\left(\mathbb{P}^n_k / \mathbb{P}^{n-1}_k \to \mathbb{P}^n_k / \mathbb{P}^n_k - \{x\} \simeq U / U - \{x\} \xrightarrow{f} \mathbb{A}^n_k / \mathbb{A}^n_k - 0 \simeq \mathbb{P}^n_k / \mathbb{P}^{n-1}_k \right)$$

$$=: \deg^{\mathbb{A}^1}_x f \qquad \text{local } \mathbb{A}^1\text{-degree}$$

$$\deg^{\mathbb{A}^1} f := \sum_{x : f(x) = 0} \deg^{\mathbb{A}^1}_x f$$

Different formulas for the $\mathbb{A}^1$-degree

- Cazanave: global formula for the $\mathbb{A}^1$-degree of a map $\mathbb{P}^1 \to \mathbb{P}^1$ (Bézoutian)
- Kass-Wickelgren: formula for local $\mathbb{A}^1$-degree for k-points (EKL-form)
- Brazelton-Burklund-McKean-Montoro-Opie: formula for local $\mathbb{A}^1$-degree for x a zero with $k(x)$ separable over k
- Brazelton-McKean-P.: formula for $\deg^{\mathbb{A}^1} f$ without any restrictions on the residue fields of the zeros

Cazanave's formula for the $\mathbb{A}^1$-degree of $f = (f_0 : f_1) : \mathbb{P}^1 \to \mathbb{P}^1$

$$x = \frac{x_1}{x_0}, \quad y = \frac{y_1}{y_0}$$

$$\text{Béz} := \frac{f_1(x) f_0(y) - f_1(y) f_0(x)}{x - y}$$

$$= \sum B_{ij} x^i y^j \in k[x,y]$$

(B_{ij}) is the Gram matrix of a non-degenerate symmetric bilinear form

<u>Thm</u> (Cazanave)

$$\deg^{\mathbb{A}^1} f = [(B_{ij})] \in GW(k)$$

<u>Ex</u>: $f_0 = 2x \quad f_1 = x^2 - y^2$

$$\text{Béz} = \frac{(x^2-1) \cdot 2y - (y^2-1) \cdot 2x}{x - y} = 2xy + 2$$

$$B_{ij} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$\rightsquigarrow \langle 2 \rangle + \langle 2 \rangle = \langle 1 \rangle + \langle 1 \rangle \in GW(k)$

Scheja - Storch duality for complete intersections $(f_1, \dots, f_n) \subseteq k[x_1, \dots, x_n]$

- 2 Endofunctors

$$F : \mathrm{Alg}_k^{f.g} \to \mathrm{Alg}_k^{f.g}$$
$$A \mapsto A \otimes_k A$$
$$G : \mathrm{Alg}_k^{f.g} \to \mathrm{Alg}^{f.g.}$$
$$A \mapsto \mathrm{Hom}_k(\mathrm{Hom}_k(A, k), A)$$

$$\chi : F \Rightarrow G$$
$$\chi_A : A \otimes_k A \to \mathrm{Hom}_k(\mathrm{Hom}_k(A,k), A)$$
$$a \otimes b \mapsto (\varphi \mapsto \varphi(a) \cdot b)$$

- $\ker\left(k[x_1, \dots, x_n] \otimes_k k[x_1, \dots, x_n] \xrightarrow{\mu} k[x_1, \dots, x_n]\right)$

$$\begin{gathered} \| \\ k[X_1, \dots, X_n, Y_1, \dots, Y_n] \end{gathered}$$

is generated by $X_j - Y_j$

Write $f_i(X_1, \dots, X_n) - f_i(Y_1, \dots, Y_n)$

$$= \sum_{j=1}^n \Delta_{ij} (X_j - Y_j)$$

$\Delta_{ij} \in k[X_1, \dots, X_n, Y_1, \dots, Y_n]$

$$Q := \frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_n)}$$ and

let $\rho : k[x_1, \dots, x_n] \to Q$ be the quotient map.

$$\Delta := \rho \otimes \rho (\det \Delta_{ij}) \in Q \otimes_k Q$$

Theorem (Scheja-Storch)

1) Δ is independent of the choice of Δ_{ij}. $\quad a \cdot \varphi = \varphi(a \cdot -)$

2) $\chi_Q(\Delta) : \mathrm{Hom}_k(Q, k) \to Q$ defines an isomorphism of Q-modules

3) $\Delta = \tau(\Delta)$

$$\tau : Q \otimes Q \to Q \otimes Q$$
$$a \otimes b \mapsto b \otimes a$$

Cor: $\Phi : Q \times Q \to k$

$$(a, b) \mapsto (\chi_Q(\Delta))^{-1}(a)(b)$$

is a non-degenerate symmetric bilinear form.

Claim: The class $[\Phi]$ of Φ in $GW(k)$ is the $\mathbb{A}^1$-degree

$$f = (f_1, \dots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$$

Pf: Let $m_1, \dots, m_s$ be the maximal ideals corresponding to the zeros of f.

$$Q = \bigoplus_{i=1}^{s} Q_{m_i}$$

and $\Phi = \bigoplus \Phi_i$ where Φ_i is the non-deg symm bilinear form on Q_{m_i} constructed in the exact same way.

Now the claim follows from

Theorem (Bachmann – Wickelgren)

$$\deg^{\mathbb{A}^1}_{m_i} f = [\Phi_i] \quad \text{in } GW(k)$$

Formula for Φ:

There is a multivariate version of the Bézoutian.

Let

$$D_{ij} := \frac{f_i(Y_1,\dots,Y_{j-1},X_j,\dots,X_n) - f_i(Y_1,\dots,Y_j,X_{j+1},\dots,X_n)}{X_j - Y_j}$$

$$\in k[X_1,\dots,X_n,Y_1,\dots,Y_n]$$

Let Béz be the image of $\det D_{ij}$ in $Q \otimes_k Q$ $\left(Q = \frac{k[x_1,\dots,x_n]}{(f_1,\dots,f_n)} \right)$

Choose a k-VS basis $a_1,\dots,a_m$ for Q.

$x^i \quad y^i$

$$\text{Béz} = \sum_{i,j=1}^{m} B_{ij}\, a_i \otimes a_j$$

Theorem (Brazelton - McKean - P.)

Assume $f = (f_1,\dots,f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ has only isolated zeros.

Then (B_{ij}) is the Gram matrix wrt basis $a_1, \dots, a_m$ for a non-deg symm bilinear form representing $\deg^{\mathbb{A}^1} f$ in $GW(k)$.

$$\frac{k[x]}{(f)} \qquad \begin{array}{ccc} 1, & x, \dots, & x^{m-1} \\ \parallel & & \parallel \\ a_1 & & a_m \end{array}$$

<u>Pf</u>: Need $\Delta_{ij} \in k[X_1 \dots X_n, Y_1, \dots, Y_n]$

$$f_i(X_1, \dots, X_n) - f_i(Y_1, \dots, Y_n) = \sum_{j=1}^{n} \Delta_{ij}(X_j - Y_j)$$

Observation: Can choose $D_{ij} = \Delta_{ij}$ (telescoping sum)

$$\rho \otimes \rho(\det \Delta_{ij}) = \Delta = \text{Béz}$$

Let $a_i^* \in \mathrm{Hom}_k(Q, k)$

st $a_i^*(a_j) = \delta_{ij}$

$$\Phi : Q \times Q \longrightarrow K$$
$$(c, d) \longmapsto (\chi_Q(\Delta))^{-1}(c)(d)$$

$$\underbrace{\chi_Q(\Delta)}_{\substack{\mathrm{Hom}_k(Q,k) \longrightarrow Q \\ \varphi \longmapsto \sum \varphi(a_i) b_i}} (a_i^*) = \sum_{j=1}^{m} a_i^*(a_j)\, b_j = b_i$$

$$\Delta = \sum_{i=1}^{m} a_i \otimes \underbrace{b_i}_{\sum_{j=1}^{m} B_{ij} a_j}$$

$a_1, \dots, a_m$ basis for Q

So $(\chi_Q(\Delta))^{-1}(b_i) = a_i^*$

so $\Phi(b_i, a_j) = (\chi_Q(\Delta))^{-1}(b_i)(a_j)$

$$= a_i^*(a_j)$$

$$= \delta_{ij}$$

$$\Phi(b_i, a_j) = \sum_{s=1}^{m} B_{is}\, \Phi(a_s, a_j)$$

So $\quad (B_{ij})_{ij}^{-1} = (\Phi(a_i, a_j))_{ij}$

$\uparrow$

Gram matrix of Φ wrt basis $a_1, \ldots, a_m$

In $GW(k)$ $\quad [(B_{ij})] = [\Phi]$

$$\langle a \rangle = \langle \tfrac{1}{a} \rangle \quad \text{in } GW(k)$$

$\square$

Example: p an odd prime

$k = \mathbb{F}_p(t)$

$f = (f_1, f_2) : \mathbb{A}^2_k \to \mathbb{A}^2_k$

$$\overset{\shortparallel}{(x_1^p - t, x_1 x_2)}$$

$$\Delta_{11} = \frac{(x_1^p - t) - (y_1^p - t)}{x_1 - y_1} \qquad \Delta_{12} = \frac{y_1^p - t - (y_1^p - t)}{x_2 - y_2} = 0$$

$$\Delta_{ij} \; \frac{f_i(y_1, \dots, y_{j-1}, x_j, \dots, x_n) - f_i(y_1, \dots, y_j, x_{j+1}, \dots, x_n)}{x_j - y_j}$$

$$\Delta_{21} = \frac{x_1 x_2 - y_1 x_2}{x_1 - y_1} = x_2 \qquad \Delta_{22} = \frac{y_1 x_2 - y_1 y_2}{x_2 - y_2} = y_1$$

$$\text{Béz} = \det \begin{vmatrix} \frac{x_1^p - y_1^p}{x_1 - y_1} & 0 \\ x_2 & y_1 \end{vmatrix}$$

$$= y_1 x_1^{p-1} + y_1^2 x_1^{p-2} + \dots + y_1^{p-1} \cdot x_1 + y_1^p$$

$\shortparallel$ t

$$B_{ij} = \begin{array}{c|cccc} & 1 & x_1 & \cdots & x_1^{p-1} \\ \hline 1 & t & & 0 & 1 \\ & & 0 & \iddots & \\ & 0 & \iddots & & 0 \\ y_1^{p-1} & 0 & 1 & & \end{array}$$

$$\rightsquigarrow \langle t \rangle + \frac{p-1}{2} \left(\langle 1 \rangle + \langle -1 \rangle \right) \in GW(k)$$

Let's use this in the setting of $\mathbb{A}^1$-enumerative geometry:

Kass-Wickelgren:

X = smooth + proper k-variety of dim n

$p : V \to X$ rk n VB that is **relatively oriented**

by $\rho : (\det TX)^{-1} \otimes \det V \xrightarrow{\sim} \mathcal{L}^{\otimes 2}$

where $\mathcal{L} \to X$ a line bundle

Let $\sigma : X \to V$ be a section with only isolated zeros.

<u>Def</u> (Kass-Wickelgren) $\mathbb{A}^1$-Euler number

local $\mathbb{A}^1$-degree with coordinates around x and trivialization of "compatible" with ρ

$$n^{\mathbb{A}^1}(V, \rho) := \sum_{x : \sigma(x) = 0} \deg_x^{\mathbb{A}^1}(\sigma, \rho)$$

$$\in GW(k)$$

By Bachmann - Wickelgren this is

independent of $\mathcal{G}$.

Ex (Kass-Wickelgren):
arithmetic count of lines on a cubic surface

$$n^{\mathbb{A}^1}(\mathrm{Sym}^3 S^\vee \to Gr(2,4))$$
$$= 15\langle 1\rangle + 12\langle -1\rangle \in GW(k)$$

$\mathbb{A}^1$- Euler characteric:

X smooth proper k-variety

$$\chi^{\mathbb{A}^1}(X) := n^{\mathbb{A}^1}(TX) \in GW(k)$$

Let $X = \mathbb{P}^n_k$.

Let ℓ be a ~~line~~ 1-dim subspace in k^{n+1} ~~through~~

~~0~~ $\rightsquigarrow [\ell] \in \mathbb{P}^n_k$

$$TX_{[\ell]} = \mathrm{Hom}(\ell, k^{n+1}/\ell)$$

So $\sigma_0, \dots, \sigma_n \in k[x_0, \dots, x_n]_1$
define a section σ of TX

$$\sigma([\ell]) = \left(\ell \xrightarrow{(\sigma_0|_\ell, \dots, \sigma_n|_\ell)} k^{n+1} \longrightarrow k^{n+1}/\ell\right)$$

Choose: $\sigma_0 = -x_n$

$\sigma_1 = -x_0$

$\vdots$

$\sigma_n = -x_{n-1}$

$\lambda \in k^\times$

$$\text{mod } \ell = \begin{pmatrix} \lambda \\ \lambda x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$\mathbb{A}^n_k = U_0 \subseteq \mathbb{P}^n_k$

$\sigma|_{U_0}$ triviales to

$$\sigma|_{U_0}(x_1, \dots, x_n) = \big(\sigma_1(1, x_1, \dots, x_n) - x_1\sigma_0(1, x_1, \dots, x_n),$$

$$\vdots$$

$$\sigma_n(1, x_1, \dots, x_n) - x_n\sigma_0(-)\big)$$

$$= (-1 + x_1x_n, -x_1 + x_2x_n, \dots, -x_{n-1} + x_n^2)$$

and $\chi^{\mathbb{A}^1}(\mathbb{P}^n_k) = \deg^{\mathbb{A}^1}(\vec{G}|_{u_0})$

$$\text{Béz} = \det \Delta_{ij} = \det \begin{pmatrix} x_n & & & & y_1 \\ -1 & \ddots & & 0 & \vdots \\ & \ddots & \ddots & & \vdots \\ 0 & & & x_n & y_{n-1} \\ & & & -1 & x_n + y_n \end{pmatrix}$$

$$= y_1 + x_n \cdot y_2 + x_n^2 y_3 + \dots + x_n^{n-1} y_n + x_n^n$$

$$= y_1 + x_n y_2 + x_{n-1} y_3 + x_{n-2} y_4 + \dots + x_2 y_n + x_1$$

So
$$B_{ij} = \underbrace{\begin{pmatrix} & & & 1 \\ & 0 & ⋰ & \\ & ⋰ & 0 & \\ 1 & & & \end{pmatrix}}_{n+1}$$

$$\Rightarrow \chi^{\mathbb{A}^1}(\mathbb{P}^n_k) = \begin{cases} \frac{n+1}{2}(\langle 1\rangle + \langle -1\rangle) & n \text{ odd} \\ \frac{n}{2}(\langle 1\rangle + \langle -1\rangle) + \langle 1\rangle & n \text{ even} \end{cases}$$

For a hypersurface X you would need "Nisnevich coordinates" around your zeros.

$x \in \sigma^{-1}(0)$

Need Zariski open nbhd $U \subseteq X$ of x + étale map $\varphi : U \to \mathbb{A}^n_k$ st φ induces an isomorphism on $k(x)$