The tame site

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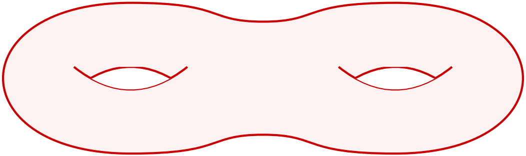

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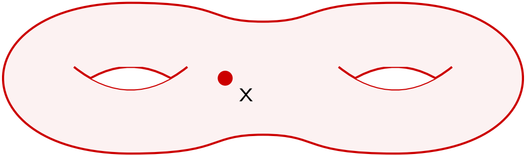

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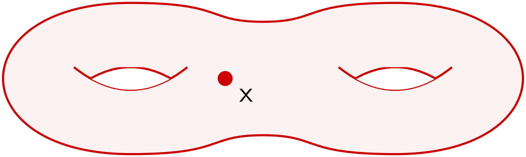

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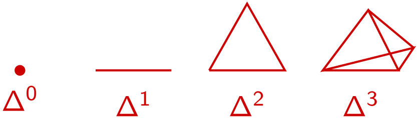

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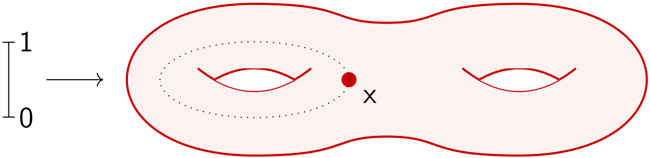

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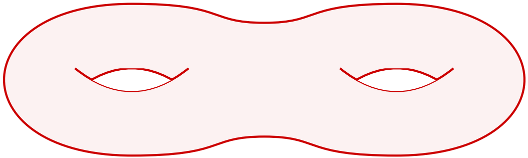

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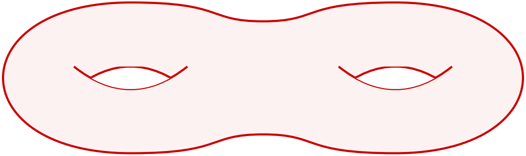

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$$\ldots \to \bigoplus_{\sigma \in \hom(\Delta^2, X)} R[\sigma] \to \bigoplus_{\sigma \in \hom(\Delta^1, X)} R[\sigma] \to \bigoplus_{\sigma \in \hom(\Delta^0, X)} R[\sigma] \to 0$$

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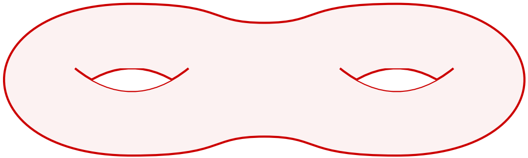

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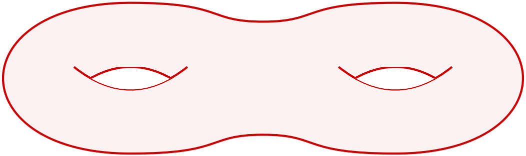

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$$H^n_{sing}(X,R) \cong H^n(X,R_X)$$

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- finiteness: if X is well behaved (e.g. a compact manifold) cohomology groups are finitely generated.

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$$H_n^S(X, M) = \mathrm{Tor}_n(\mathrm{Cor}(\Delta^\bullet, X), M),$$
$$H_S^n(X, M) = \mathrm{Ext}^n(\mathrm{Cor}(\Delta^\bullet, X), M),$$

where $\mathrm{Cor}(\Delta^i, X)$ is the group of correspondences: free abelian group over all integral subschemes of $\mathbb{A}^i \times_k X$ that are finite and surjective over $\mathbb{A}^i$.

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- but: if $\operatorname{char} k = p > 0$ and $p | \#M$, $H^n(X_{\mathrm{et}}, M)$ is not well behaved.

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$$\frac{\partial}{\partial y}(y^p - y - f(x)) = py^{p-1} - 1 = -1$$

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Problem: wild ramification at ∞ ($\mathbb{P}^1 \setminus \mathbb{A}^1 = \{\infty\}$)

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An extension $(K, K^+)|(k, k^+)$ of valued fields is

- unramified if $K^{sh} = k^{sh}$ (strict henselizations),
- tame(ly ramified) if $[K^{sh} : k^{sh}]$ is prime to the residue characteristic p.
- wild(ly ramified) if $p|[K^{sh} : k^{sh}]$.

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We want to study étale morphisms $Y \to X$ that are tamely ramified at the boundary $\bar{X} \setminus X$ in an appropriate sense.

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We want to study étale morphisms $Y \to X$ that are tamely ramified at the boundary $\bar{X} \setminus X$ in an appropriate sense.

Why should this work? The tame fundamental group $\pi_1^t(X/S, \bar{x})$ already exists and has good properties.

Expected properties

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- $H^n_t(X, M)$ is finite for finite M,
- homotopy invariance: $H^n_t(X \times_k \mathbb{A}^1, M) = H^n_t(X, M)$,
- base change theorems (proper and smooth)

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- Connection to Suslin cohomology:
$$H^n_t(X, M) \cong H^n_S(X, M).$$

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adic tame site
over adic space $\mathrm{Spa}(X, S)$: étale morphisms of adic spaces s.t. residue field extensions are tame

my own project

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connection

$\mathbb{A}^1$-homotopy theory

by Morel, Voevodsky ...

adic tame site
over adic space $\mathrm{Spa}(X, S)$: étale morphisms of adic spaces s.t. residue field extensions are tame

my own project

Construction of the tame site

For a tame site associated with $X \to S$ two complementary approaches seem promising:

algebraic tame site
étale morphisms $Y \to X$, tameness condition on coverings $(Y_i \to Y)_{i \in I}$

joint work with Alexander Schmidt

comparison theorem

adic tame site
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connection

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Table of Contents

Here: Only discretely ringed adic spaces, i.e., all rings are equipped with the discrete topology.

Valuations

Valuations of a ring A:

$$v : A \to \Gamma \cup \{0\}$$

- Γ: totally ordered group, multiplicative notation,
- v is multiplicative,
- $v(1) = 1$,
- v satisfies the strong triangle inequality:

$$v(a+b) \leq \max\{v(a), v(b)\}.$$

Connection to valuations of fields

For a ring A there is a bijection

$$\begin{aligned}
\{\text{valuations } v : A \to \Gamma \cup \{0\}\}/_\sim &\longleftrightarrow \left\{ (x, \mathcal{O}) \;\middle|\; \begin{array}{c} x \in \operatorname{Spec} A \\ \mathcal{O} \subseteq k(x) \text{ val. ring} \end{array} \right\} \\
v &\mapsto (\operatorname{supp} v, \mathcal{O}_v) \\
(A \to k(x) \xrightarrow{v_{\mathcal{O}}} \Gamma \cup \{0\}) &\leftarrow\!\shortmid (x, \mathcal{O}).
\end{aligned}$$

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- $\operatorname{supp} v = \{a \in A \mid v(a) = 0\}$ is a prime ideal of A
- $\mathcal{O}_v = \{a \in k(\operatorname{supp} v) \mid v(a) \leq 1\}$ is the corresponding valuation ring

Adic spectrum

Huber pair: (A, A^+), where

- A is a ring
- A^+ is a subring of A that is integrally closed in A.

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Example

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k algebraically closed, $(A, A^+) = (k(T), k[T])$. There are two types of points:

- The trivial valuation of $k(T)$ and
- for each $a \in k$ the valuation v_a: Every element of $k(T)$ can be written in the form $(T-a)^n g/h$ mit $n \in \mathbb{Z}$, $g, h \in k[T]$ mit $(T-a) \nmid g, h$. Then

$$v_a((T-a)^n g/h) = e^{-n}$$

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corresponding valuation ring: $k[T]_{(T-a)}$

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There are three types of points in $\mathrm{Spa}(A, A^+)$:

- The trivial valuation of $k(T)$,
- The valuations v_a of $k(T)$ from the previous example, and
- The trivial valuation on on $k[T]/(T - a) \cong k$ for each $a \in k$.

Visualization

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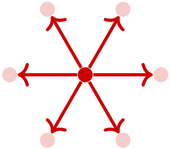

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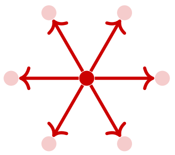

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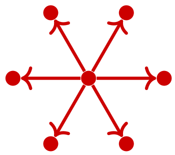

Generalization

S separated scheme, $X \to S$ morphism of schemes.

$$\mathrm{Spa}(X,S) = \left\{ (x,\mathcal{O}) \mid \begin{array}{c} x \in X \\ \mathcal{O} \subseteq k(x) \text{ valuation ring s.t.} \\ \exists \operatorname{Spec} \mathcal{O} \to S \text{ comp. with } \operatorname{Spec} k(x) \to X \end{array} \right\}$$

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compatible means that
$$\begin{tikzcd} \mathrm{Spec}\, k(x) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\,\mathcal{O} \arrow[r] & S \end{tikzcd}$$
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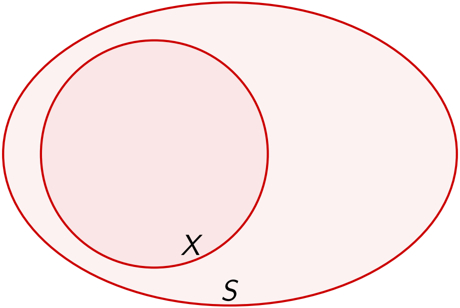
Assume $X \to S$ is an open immersion. Then we can visualize $\mathrm{Spa}(X, S)$ as:



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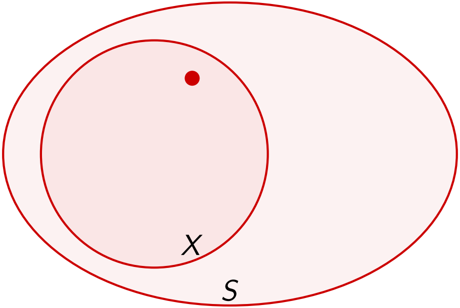

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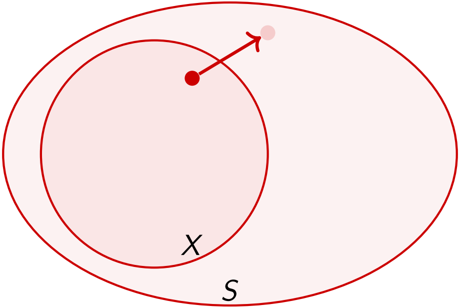

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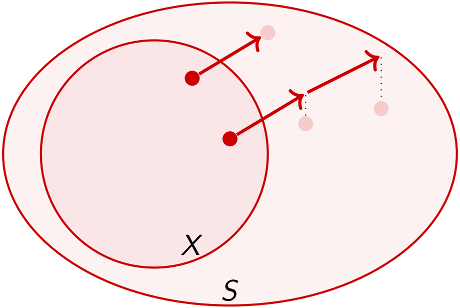

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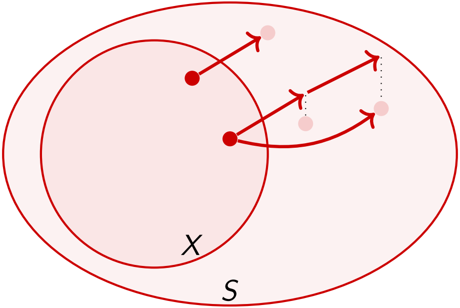

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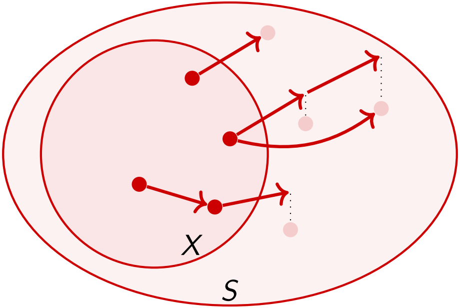

Morphisms

A commutative diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

induces a morphism of adic spaces

$$\begin{aligned} \mathrm{Spa}(f,g) : \mathrm{Spa}(X',S') &\longrightarrow \mathrm{Spa}(X,S), \\ (x',\mathcal{O}') &\mapsto (x = f(x'), \mathcal{O} = \mathcal{O}' \cap k(x')) \end{aligned}$$

Topology

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The topology of $\mathrm{Spa}(X, S)$ is generated by all $\mathrm{Spa}(U, T)$ coming from diagrams

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- $U \to X$ an open immersion,
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If $X \to S$ is an open immersion and $U \to T$ is dominant (hence an open immersion), then $T \to S$ is birational.

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Tame morphisms of adic spaces

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X' \arrow[r, "f"] \arrow[d, "dominant"'] & X \arrow[d, "dominant"] \\
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$\mathrm{Spa}(f, g) : \mathrm{Spa}(X', S') \to \mathrm{Spa}(X, S)$ is étale if and only if

- f is étale
- g is the composition of an integral with a f. t. morphism

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(f,g) is tame if it is étale and $\forall (x', \mathcal{O}') \in \mathrm{Spa}(X', S')$ mapping to $(x, \mathcal{O}) \in \mathrm{Spa}(X, S)$, $k(x')|k(x)$ is tamely ramified w.r.t. $\mathcal{O}'$.

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For a morphism of schemes $X \to S$: adic tame site $\mathrm{Spa}(X, S)_t$.

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Comparison theorem [H., Schmidt]:

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Suppose S is affine and of characteristic $p > 0$. Let $\mathcal{F}$ be a sheaf on $(X/S)_t$.

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Comparison theorem [H., Schmidt]:
Suppose S is affine and of characteristic $p > 0$. Let $\mathcal{F}$ be a sheaf on $(X/S)_t$.
Then there is a sheaf $\mathcal{F}'$ on $\mathrm{Spa}(X, S)_t$ such that

$$H^i((X/S)_t, \mathcal{F}) \cong H^i(\mathrm{Spa}(X, S)_t, \mathcal{F}').$$

Expected properties

k algebraically closed field of characteristic $p > 0$, X/k variety

- $H^n_t(X/k, M) = H^n_{\mathrm{et}}(X, M)$ if $p \nmid \#M$ or if X/k is proper,
- the fundamental group of the tame site is the existent $\pi_1^t(X/t, \bar{x})$,
- compatibility of tame cohomology with products $X \times_k Y$,
- $H^n_t(X/k, M)$ is finite for finite M,
- homotopy invariance: $H^n_t(X \times_k \mathbb{A}^1, M) = H^n_t(X, M)$,
- base change theorems (proper and smooth),

Expected properties

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- $H_t^n(X/k, M) = H_{\mathrm{et}}^n(X, M)$ if $p \nmid \#M$ or if X/k is proper, ✓
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Expected properties

k algebraically closed field of characteristic $p > 0$, X/k variety

- Poincaré duality: If X/k is smooth of dimension d and $pM = 0$, there is a perfect pairing

$$H^n_{t,c}(X, M) \times \mathrm{Ext}^{d-n}_X(M, \nu(d)) \to \mathbb{Z}/p\mathbb{Z},$$

- Cohomological purity: If X/k is smooth and $Z \hookrightarrow X$ is a smooth closed subvariety of codimension c,

$$H^n_{t,Z}(X, \nu(r)) \cong H^{n-c}(Z, \nu(r-c)),$$

- Connection with Suslin cohomology:

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$$H^n_{t,Z}(X, \nu(r)) \cong H^{n-c}(Z, \nu(r-c)), \qquad \text{for curves or if } r = 0$$

- Connection with Suslin cohomology:

$$H^n_t(X, M) \cong H^n_S(X, M). \qquad \text{work in progress (with A. Schmidt)}$$