On the motivic t-structure on $\mathrm{DM}(k, \mathbb{Q})$

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An observation: there "should exist" a weight structure on $\mathrm{DM}(k, \mathbb{Q})$ with "nice properties".

These properties would yield some interesting statements on the (conjectural) motivic t-structures on $\mathrm{DM}^{gm}(k, \mathbb{Q}) \subset \mathrm{DM}(k, \mathbb{Q})$.

A problem: I currently do not know how to construct (a candidate for) this weight structure unconditionally. Yet recent abstract nonsense easily gives a candidate for the "big motivic t-structure".

Simple definitions and conventions

$\underline{C}$: a triangulated category; retracts = summands;

For $S \subset \operatorname{Obj} \underline{C}$, $\operatorname{Kar}_{\underline{C}}(S) = \{$retracts of $M \in S\}$;

$S^{\perp} = \{N \in \operatorname{Obj} \underline{C},\ \underline{C}(M, N) = \{0\}\ \forall M \in S\}$; ${}^{\perp}S = S^{\perp}_{\underline{C}^{op}}$;

S *Hom-generates* $\underline{C}$ if $(\cup_{i \in \mathbb{Z}} S[i])^{\perp} = \{0\}$;

S is (anti)connective if $S \perp S[i]\ \forall i > 0\ (< 0)$;

S is *(co)smashing* if $\underline{C}, S$ are $\coprod_{\underline{C}}$ ($\prod_{\underline{C}}$)-closed;

$\underline{C}$ is smashing $\implies$ $M \in \operatorname{Obj} \underline{C}$ is *compact* if $\underline{C}(M, -) : \underline{C} \to \underline{\mathrm{Ab}}$ respects coproducts;

$\underline{C}^c \subset \underline{C}$ = (the triangulated) subcategory of compact objects.

Weight and t-structures

Definition 1. [t-structures (homological convention).]

$(\underline{C}_{t \le 0}, \underline{C}_{t \ge 0}) \subset \operatorname{Obj} \underline{C}$: strict;

(i) $\underline{C}_{t \le 0} \subset \underline{C}_{t \le 0}[1]$ and $\underline{C}_{t \ge 0}[1] \subset \underline{C}_{t \ge 0}$.

(ii) $\underline{C}_{t \ge 0}[1] \perp \underline{C}_{t \le 0}$.

(iii) $\forall M \in \operatorname{Obj} \underline{C}$ $\exists$ distinguished triangle

$$L_t M \to M \to R_t M \to L_t M[1] : \qquad (1)$$

such that $L_t M \in \underline{C}_{t \ge 0}, R_t M \in \underline{C}_{t \le 0}[-1]$.

Definition 2. [$(\underline{C}, w)$ is a weight structure if]

(i) $\underline{C}_{w\le 0}, \underline{C}_{w\ge 0} \subset \operatorname{Obj}\underline{C}$ are retraction-closed.
(ii) $\underline{C}_{w\le 0} \subset \underline{C}_{w\le 0}[1]$; $\underline{C}_{w\ge 0}[1] \subset \underline{C}_{w\ge 0}$.
(iii) **Orthogonality**. $\underline{C}_{w\le 0} \perp \underline{C}_{w\ge 0}[1]$.
(iv) **Weight decompositions**. $\forall M \in \operatorname{Obj}\underline{C}$

$$\exists\ L_wM \to M \to R_wM \to L_wM[1]:$$

$L_wM \in \underline{C}_{w\le 0}$ and $R_wM \in \underline{C}_{w\ge 0}[1]$.

Definition 3. 1. $\underline{Ht} \subset \underline{C}$; $\operatorname{Obj}\underline{Ht} = \underline{C}_{t=0} = \underline{C}_{t\le 0} \cap \underline{C}_{t\ge 0}$; similarly for w.

2. $\forall i \in \mathbb{Z}$: $\underline{C}_{t,w\le,\ge i} := \underline{C}_{t,w\le,\ge 0}[i]$.
t is left (resp. right) non-degenerate if $\cap_{i\in\mathbb{Z}}\underline{C}_{t\ge i} = \{0\}$ (resp. $\cap_{i\in\mathbb{Z}}\underline{C}_{t\le i} = \{0\}$); similarly for w.

3. w if (co)smashing if $\underline{C}$, $\underline{C}_{w\le 0}$ and $\underline{C}_{w\ge 0}$ are; similarly for t.

Remark 4. $\underline{Hw}$ is connective; $\underline{Ht}$ is anti-connective.

Some abstract nonsense

Proposition 5 ([KeN13]). $\underline{B} \subset \underline{C}^c$ is small, anti-connective, abelian semi-simple, & Hom-generates $\underline{C}$ $\implies$ $(\underline{C}, w_{\underline{B}} = ((\cup_{i>0}\underline{B}[i])^{\perp}, (\cup_{i<0}\underline{B}[i])^{\perp}))$ is a smashing and cosmashing non-degenerate weight structure.

Theorem 6 ([Bon19]+[BoS19] + in progress). Assume $\underline{C}$ is compactly generated (or satisfies BRD); $(\underline{C}, w)$ is smashing and cosmashing.

1. $t = (\underline{C}_{w\le 0}, {}^{\perp}\underline{C}_{w\le -1} = {}^{\perp}(\cup_{i<0}\underline{C}_{w=i}))$ is a smashing t-structure. t restricts to t^c on $\underline{C}^c$.
2. t is right (resp. t^c is left) non-degenerate if w is so.
3. If w is left (resp. right) non-degenerate then $\underline{C}_{w\ge 0}$ ($\underline{C}_{w\le 0}$) = the smallest (co)smashing extension-closed class $\subset \operatorname{Obj}\underline{C}$ that contains $\underline{C}_{w=i}$ for $i \ge 0$ (≤ 0).
4. If $w = w_{\underline{B}}$ (see Proposition 5) then t is compactly generated and t^c is bounded.
5. $\underline{Hw}$ is $\underline{C}$-smashing and co-smashing. $\underline{Ht}$ ($\supset \underline{Ht}^c$) $\subset \operatorname{AddFun}(\underline{Hw}, \underline{\mathrm{Ab}}) =$ {functors that respect products (and coproducts)}.

Now take $\underline{C} = \mathrm{DM}(k, \mathbb{Q})$, $H^c : DM_{gm}(k, \mathbb{Q}) \to \underline{\mathrm{Ab}}$ be a "classical Weil cohomology theory" (de Rham or singular in characteristic 0;
$H^{et} : M \mapsto \mathcal{H}^{et}_{\mathbb{Q}_\ell}(M \otimes_{\mathrm{Spec}\ k} \mathrm{Spec}\ k^{alg})$).

Theorem 7. 1. H^c extends to $H \cong \underline{C}(-, R_H)$.

2. R_H is *pure injective*, that is, $\underline{C}(M, f) = 0$ $\forall M \in \mathrm{Obj}\, \underline{C}^c \implies \underline{C}(f, R_H) = 0$ ([Kra00]).

3. Any $\coprod_I R_H$ is a retract of $\prod_J R_H$ and vice versa.

4. $\exists t_H$ on $\underline{C}$, $\underline{C}_{t\geq 0} = {}^{\perp}\{R_H[i] : i < 0\}$ ([LaV20, Theorem 5.2]; since $\underline{C}$ is *algebraic*).

5. Standard conjectures $\implies$ one can take semi-simple mixed (= sums of pure) motives for $\underline{B}$ in Proposition 5 (see [Bon15]), $t_H = t$, R_H bigenerates w (cf. Theorem 6(3)), and $\underline{C}_{w=0} = \mathrm{Kar}_{\underline{C}}(\{\coprod_I R_H\})$.

Question 8. 1. Can one construct an unconditional candidate for w?

2. When does an object R "purely bigenerate" w ("strong connectivity"??)?

3. Is "algebraic" (and *compactly generated*?) necessary for [LaV20, Theorem 5.2]?

A weight splitting of R_H

Proposition 9. 1. $\exists t_{\mathrm{Chow}}$: $\underline{Ht_{\mathrm{Chow}}} \cong \mathrm{AddFun}(\mathrm{Chow}^{op}, \underline{\mathrm{Ab}})$.
2. $R_H \cong \bigoplus_{i\in\mathbb{Z}} t_{\mathrm{Chow}=i} R_H[i]$

Proof. 1. Apply the $\underline{C}$-connectivity of Chow$(=\mathrm{Chow}(k,\mathbb{Q}))$.
2. Weight spectral sequences for H^c degenerate at E_2. It remains to apply [Bon12, §3] and pure injectivity of H and all $t_{\mathrm{Chow}\geq i}H$.

□

Question 10. Can this isomorphism be chosen to be compatible with $m_H : R_H \otimes R_H \to R_H$ (cf. [Ayo14])?

Remark 11. $R_H \otimes R_H$: difficult! Take $\underline{C} \otimes \underline{C}$;
$\mathrm{RHom}(X \boxtimes Y, Z \boxtimes T) \cong \mathrm{RHom}(X, Z) \otimes \mathrm{RHom}(Y, T)$.
$\Longrightarrow$ $\mathrm{Chow}(k,\mathbb{Q}) \boxtimes \mathrm{Chow}(k,\mathbb{Q})$ gives $w_{\boxtimes}$ on $\underline{C} \otimes \underline{C}$,
$\{\otimes^*(t_{\mathrm{Chow}=i}R_H)\} \cup \underline{C}_{t_{\mathrm{Chow}=0}} \boxtimes \underline{C}_{t_{\mathrm{Chow}=0}} \subset t_{\boxtimes}\underline{C} \otimes \underline{C}$
($\otimes^*$ = right adjoint to $\otimes : \underline{C} \otimes \underline{C} \to \underline{C}$).

Modify $m'_H : R_H \boxtimes R_H \to \otimes^* R_H$ to make it "compatible with the unit" and prove "that there are no bad components" by the induction on "t-defect".

An "explicit" R_H

Assume k is algebraically closed, char $k \neq \ell$.

Take $\underline{D}$ = subcategory of $\mathrm{DM}(k, \mathbb{Z}_{(\ell)})$ generated by $\widehat{\mathrm{Chow}^{eff}(k, \mathbb{Q})}$; $\mathbf{1} = M^{\mathrm{gm}}(pt)$. $\mathbf{1}/\ell^n$ represents $\mathbb{Z}/\ell^n\mathbb{Z}$-étale cohomology ([SuV96]+duality) $\implies$ $R' := \hat{\mathbf{1}}_\ell = \varprojlim \mathbf{1}/\ell^n$ gives $\mathbb{Z}_\ell$-étale cohomologyon $\underline{D}^c$; thus $R_H \cong R' \otimes \mathbb{Q}$ in $\underline{C} = \underline{D}_{\mathbb{Q}}$ ("coherent with" Theorem 7(5)).

Next, $C := \mathrm{Cone}(\mathbf{1} \to R')$ is $\mathbb{Q}$-linear $\implies$ $\mathbb{Q} \otimes \mathbf{1} \to R \to C \to \mathbb{Q} \otimes \mathbf{1}[1]$ is distinguished.

Question 12. $\mathrm{RHom}(\mathrm{Cone}(\mathbf{1} \to R'), \mathbb{Q} \otimes \mathbf{1}) =?$

$\mathrm{RHom}(R', R') \cong \mathbb{Z}_\ell[0]$ (an adjunction).

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