On factorization of nuclear operators through $S_{s,p}$-operators

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On factorization of operators

We are interesting in some factorization theorems. The importance of theorems of such a kind is illustrated by two examples.

Example 1

A. Grothendieck noticed that every nuclear operator T in a Banach space X can be factored through a Hilbert space in such a way that this factorization

$$T : X \overset{A}{\to} H \overset{B}{\to} X$$

has the property that AB is a Hilbert-Schmidt operator. Therefore, the sequence of eigenvalues of T is in l_2.

A. Grothendieck, Produits tensoriels topologiques et espases nucléaires, Mem. Amer. Math. Soc., Volume 16, 1955, 196 + 140.

Example 2

G. Pisier has shown that if a convolution operator

$$f\star:\ M(G)\to C(G),$$

where G is a compact Abelian group and $f\in C(G)$, can be factored through a Hilbert space, then f has the absolutely summable set of Fourier coefficients.

G. Pisier applied this result to a characterization of Sidon sets.

G. Pisier, Factorization of Linear Operators and Geometry of Banach Spaces, Amer. Math. Soc., Providence, Rhode Island, CBMS Vol. 60, 1985.

A question of B. Mityagin

A natural question (B. Mityagin,
Aleksander Pelczynski Memorial Conference 2014, Poland):

- Is it true that a product of two nuclear operators in Banach spaces can be factored through an nuclear (that is, trace-class) operator in a Hilbert space?
- By usinng the Carlemann's example from

Carleman T., Über die Fourierkoeffizienten einer stetigen Funktion, A. M., 41 (1918), 377-384,

we showed that:

- The answer is negative.

O.I. Reinov, On product of nuclear operators, Function. Anal. and Appl., **51**:4 (2017), 90-91.

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On the talk

The talk will consists of two parts;

Part I

On factorization of operators through Lorentz, R. Schatten and J. von Neumann operators in Hilbert spaces. Applications to eigenvalues problems.

Part II

Generalizations of the result of G. Pisier and vector-valued cases.

On the talk

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Part II

Generalizations of the result of G. Pisier and vector-valued cases.

- $$U \in S_{p,q}(H):\ ||(\mu_n(U))||_{pq} := \left(\sum_{n\in\mathbb{N}} \mu_n(U)^q n^{q/p-1} \right)^{1/q} < +\infty,$$

  i.e., $(\mu_n(U)) \in l_{p,q}$ (singular numbers).
  We consider the case $0 < p, q < \infty$. Quasi-norm on $S_{p,q}(H)$: $\sigma_{p,q}(U) = ||(\mu_n(U))||_{pq}$. If $p = q$, then we get S_p, σ_p.
  $S_{\infty,\infty}(H)$ — all (compact) operators with the usual operator norm.

- $$S_{p,q} \circ S_{r,s} \subset S_{t,u},\ 1/t = 1/p + 1/r,\ 1/u = 1/q + 1/s;$$

  $$N_{p,q}(H) = S_{p,q}(H),\ 0 < p, q \le 1.$$

  The ideal $N_{p,q}$ $(0 < p, q \le 1)$ of $l_{p,q}$-nuclear operators can be defined a follows:

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  The ideal $N_{p,q}$ $(0 < p, q \leq 1)$ of $l_{p,q}$-nuclear operators can be defined a follows:

$l_{p,q}$-nuclear operators

- $T : X \to Y$ is $l_{p,q}$-nuclear $(0 < p, q \leq 1)$ if

  $$Tx = \sum_{k=1}^{\infty} a_k \langle x'_k, x \rangle y_k$$

  for all $x \in X$, where $(x'_k) \subset X^*, (y_k) \subset Y$, $||x'_k|| \, ||y_k|| \leq 1$, $(a_k) \in l_{p,q}$. Notation: $T \in N_{p,q}(X, Y)$.
  Quasi-norm $\nu_{p,q}(T) := \inf ||(a_n)||_{p,q}$. If $p = q$, then we get p-nuclear operators, $N_p(X, Y)$.

- A. Hinrichs, A. Pietsch, *p*-nuclear operators in the sense of Grothendieck, Math. Nachr., Volume 283, No. 2 (2010), 232–261.

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$S_{p,q}$-factorization

Definition

An operator $T : X \to Y$ can be factored through an operator from $S_{p,q}(H)$ (through $S_{p,q}$-operator), if there exist the operators $A \in L(X, H)$, $U \in S_{p,q}(H)$ and $B \in L(H, Y)$ such that $T = BUA$. We put

$$\gamma_{S_{p,q}}(T) = \inf ||A|| \, \sigma_{p,q}(U) \, ||B||.$$

Factorization theorem for $p = q$

Let us begin with the simplest situation.

Theorem

If $X_1, \dots, X_{n+1}$ are Banach spaces, $s_k \in (0,1]$ and $T_k \in N_{s_k}(X_k, X_{k+1})$ for $k = 1, 2, \dots, n$, then the product $T := T_n T_{n-1} \cdots T_1$ can be factored through an operator from S_r, where $1/r = 1/s_1 + 1/s_2 + \cdots + 1/s_n - (n+1)/2$.

Theorem is sharp.

For example, we have:

1) If an operator T in a Banach space is nuclear and $m > 1$, then T^m can be factored through an operator from S_r, where $r = 2/(m-1)$.

2) There exists a nuclear operator T in the space $C[0,1]$ (or in the space $L_1[0,1]$) such that for any $m > 1$ and $r < 2/(m-1)$ the operator T^m can not be factored through an operator from S_r.

Factorization theorem (general case)

Theorem

Let $m \in \mathbb{N}$. If $X_1, X_2, \ldots, X_{m+1}$ are Banach spaces, $0 < r_k, s_k < 1$ and $T_k \in N_{s_k,r_k}(X_k, X_{k+1})$ for $k = 1, 2, \ldots, m$, then the product

$$T := T_m T_{m-1} \cdots T_1$$

can be factored through an operator from $S_{s,r}$, where

$$1/s = 1/s_1 + 1/s_2 + \cdots + 1/s_m - (m+1)/2$$

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$$1/r = 1/r_1 + 1/r_2 + \cdots + 1/r_m - (m+1)/2.$$

Moreover,

$$\gamma_{S_{s,r}}(T) \leq c_0 \prod_{k=1}^{m} \nu_{s_k,r_k}(T_k).$$

If $s = r$, then $c_0 = 1$.

Space $\Sigma_{p,q}$

Let $0 < p < q < 1$. Consider the space $\Sigma_{p,q}$ of all unordered complex sequences α such that every usual sequence (α_k) that defines α lies in $l_{p,q}$. Define a metric $\rho_{p,q}$ on $\Sigma_{p,q}$ by setting

$$\rho_{p,q}(\alpha, \beta) := \inf(||\alpha_k - \beta_k||_{p,q}^{\min})^q$$

where $||\cdot||_{p,q}^{\min}$ is an equivalent q-norm on the space $l_{p,q}$; see, e.g., p. 238 in

A. Hinrichs, A. Pietsch, p -nuclear operators in the sense of Grothendieck, Math. Nachr. 283, No. 2, 232–261 (2010).

Here, the infimum is taken over all possible sequences (α_k) (respectively, (β_k)) from $l_{p,q}$, which define the unordered sequence α (respectively, β).

Eigenvalues problem for $\prod N_{s_k,r_k}$

Corollary

Let $m \in \mathbb{N}$, $X_1, X_2, \ldots, X_{m+1}$ are Banach spaces, $X_1 = X_{m+1}$, $0 < r_k, s_k < 1$ and $T_k \in N_{s_k,r_k}(X_k, X_{k+1})$ for $k = 1, 2, \ldots, m$. If

$$1/\tilde{s} = 1/s_1 + 1/s_2 + \cdots + 1/s_m - m/2,$$

$$1/\tilde{r} = 1/r_1 + 1/r_2 + \cdots + 1/r_m - m/2.$$

and $\tilde{s} \leq \tilde{r}$, then
1. the sequence of eigenvalues of the product $T := T_m T_{m-1} \cdots T_1$ belongs to $l_{\tilde{s},\tilde{r}}$ and
2. the natural mapping $T \to$ eigenvalues of T

$$\prod_{j=1}^{m} N_{s_{m-j+1},r_{m-j+1}}(X_{s_{m-j+1}}, X_{r_{m-j+1}}) \to \Sigma_{\tilde{s},\tilde{r}}$$

is continuous.

Part 1 of the previous theorem may be obtained in more general and stronger form:

Theorem

Let $m \in \mathbb{N}$. If $X_1, X_2, \ldots, X_{m+1}$ are Banach spaces, $X_1 = X_{m+1}$,
$0 < r_k, s_k < 1$,
$T_k \in N_{s_k,r_k}(X_k, X_{k+1})$ for $k = 1, 2, \ldots, m$,
then the sequence of eigenvalues of the product
$T := T_m T_{m-1} \cdots T_1$ belongs to $l_{p,q}$, where

$$1/p = 1/s_1 + 1/s_2 + \cdots + 1/s_m - m/2$$

$$1/q = \sum_{k=1}^{m} 1/r_k.$$

On sharpness: finite dimensional case

We will show a way to get a sharpness of the above facts for the case where $\tilde{s} = \tilde{r}$ (and, for every k, $s_k = r_k$).

Theorem

There exists a constant $G > 0$ such that for every $n \in \mathbb{N}$ one can find an operator $A_n : l_1^n \to l_1^n$ with the property:
If $m \in \mathbb{N}$, $s_k \in (0, 1]$ for $k = 1, 2, \ldots, m$,
$1/q = 1/s_1 + 1/s_2 + \cdots + 1/s_m - m/2$, $u \in (0, q]$ *and*
$1/s = 1/s_1 + 1/s_2 + \cdots + 1/s_m - (m+1)/2$, $t \in (0, s]$, *then*

$$||(\lambda_j(A_n^m))|_u = n^{1/u-1/q} \prod_{k=1}^{m} \nu_{s_k}(A_n),$$

$$\gamma_{S_t}(A_n^m) \geq G n^{1/t-1/s} \prod_{k=1}^{m} \nu_{s_k}(A_n).$$

On sharpness (continuation)

Now, if the last theorem is proved, taking a direct sum of infinitely many operators, we get:

Theorem

Let $m \in \mathbb{N}$, $s_k \in (0, 1]$ *for* $k = 1, 2, \dots, m$ *and*

$$1/s = 1/s_1 + 1/s_2 + \cdots + 1/s_m - m/2.$$

There exists a sequence of operators $T_k \in N_{s_k}(X_k, X_{k+1})$ *in Banach spaces such that the sequence of eigenvalues of the operator*

$$T := T_m T_{m-1} \cdots T_1$$

lies in $l_s \setminus \cup_{t<s} l_t$.
T *can be factored through an operator* $S_r(H), 1/r = 1/s - 1/2$, *but cannot be factorized through a* S_t*-operator, if* $t \in (0, r)$.

SECOND PART

Let's go to the second part of the talk.
Firstly, recall that we have:

Proposition

If $T \in N_q(X, Y)$ $(0 < q \leq 1)$, then T can be factored through an operator from $S_p(H)$, where $1/p = 1/q - 1$.

We are going to show that this result is sharp.
For this we need the following first generalization of the Pisier result, mentioned in the very beginning.

1st theorem

Theorem

Let $f \in C(G)$, $0 < q \leq 1$ *and* $1/p = 1/q - 1$. *Consider a convolution operator* $\star f : M(G) \to C(G)$.
The set $\hat{f}$ *of Fourier coefficients of* f *belongs to* l_q

if and only if

the operator $\star f$ *can be factored through a Schatten-von Neumann* S_p*-operator in a Hilbert space:*

$$\star f : M(G) \to H \underset{S_p(H)}{\longrightarrow} H \to C(G).$$

Moreover, if $\hat{f} \in l_q$, *then* $\gamma_{S_p}(\star f) = (\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^q)^{1/q}$.
On the other hand, $|| \star f|| = ||f||_{C(G)}$.

Main part of the proof

Let there exists $U \in S_p(H)$ such that

$$\star f = AUB: \ M(G) \xrightarrow{B} H \xrightarrow{U} H \xrightarrow{A} C(G).$$

If $j: C(G) \hookrightarrow M(G)$ is a natural injection, then Fourier coefficients of f are the eigenvalues of the operator $AUBj: C(G) \to M(G) \to C(G)$. Consider a diagram

$$C(G) \overset{j}{\hookrightarrow} M(G) \xrightarrow{B} H \xrightarrow{U} H \xrightarrow{A} C(G) \overset{j}{\hookrightarrow} M(G) \xrightarrow{B} H.$$

The operators $AUBj$ and $BjAU$ have the same sequences of eigenvalues. Since $B \in \Pi_2$, $j: C(G) \hookrightarrow L_2(G) \hookrightarrow M(G) \in \Pi_2$ and $U \in S_p$, we get that

$$(*) \quad BjAU \in S_p \circ S_1 \subset S_q,$$

where $1/q = 1 + 1/p$. Therefore, the eigenvalues of $AUBj$ lies in l_q. So $\{\hat{f}(n)\} \in l_q$.

Sharpness of factorizations for N_q-operators

Corollary

Let $f \in C(G)$, $0 < q \leq 1$ and $1/p = 1/q - 1$. TFAE:
1). The operator $\star f : M(G) \to C(G)$ is q-nuclear;
2). the operator $\star f$ can be factored through a Schatten-von Neumann S_p-operator in a Hilbert space.

Thus, *the fact, mentioned in Proposition above* (that is, 1) $\Longrightarrow$ 2)), *is sharp.*

Generalization for S_{p_1,p_2}-factorization

Theorem

In the previous theorem, change q by $0 < q_1, q_2 < 1$, p by $1/p_1 = 1/q_1 - 1$, $1/p_2 = 1/q_2 - 1$. Instead of l_q and S_p, consider l_{q_1,q_2} and S_{p_1,p_2}. The conclusion remains true.

Later we will formulate a more general result (Corollary 3 in the very end of the talk).

Vector-valued case

Let $f \in C(G)$ and $T \in L(X, Y)$ (in Banach spaces).
Denote by $M(G, X)$ the Banach space of all regular Borel X-valued measures of bounded variation, C(G,X) the Banach space of all continuous X-valued functions defined on G equipped with the supremum norm.
Note that

$$M(G) \otimes X \subset M(G) \widehat{\otimes} X \subset M(G, X),$$

where $\widehat{\otimes}$ is the porjective tensor product.
Define a map $T_f : M(G, X) \to C(G, X)$ by

$$T_f(\overline{\mu})(s) = \int_G f(s-t)\, dT\overline{\mu}(t), \ \overline{\mu} \in M(G, X).$$

Vector-valued case

Theorem

Let $f \in C(G)$, $0 < q_1, q_2 < 1$ *and* $1/p_1 = 1/q_1 - 1$, $1/p_2 = 1/q_2 - 1$.
Consider a convolution operator $\star f : M(G) \to C(G)$ *and an operator* $T : X \to Y$.
If the operator

$$T_f : M(G, X) \to C(G, X)$$

can be factored through an S_{p_1,p_2}*-operator,*
then the operators $f\star$ *and* T *possess the same property.*
In particular, the set $\hat{f}$ *of Fourier coefficients of* f *belongs to* l_{q_1,q_2}.
The same is true for the case where $p_1 = p_2 = \infty$ *(or* $q_1 = q_2 = 1$*).*

Vector-valued case

Corollary 1

Take $p := p_1 = p_2$ in the above conditions to get a result for factorization through the S_p-operators.

Corollary 2

Take $X = Y$, $T = \mathrm{id}_X$ and $p := p_1 = p_2 = \infty$ above to get a result of E. Saab: Under the corresponding conditions, $\hat{f} \in l_1$ and $X \cong H$.

Paulette Saab, Convolution Operators that Factor Through a Hilbert Space, Quaestiones Mathematicae, March 2008, 31(1): 79-87.

Vector-valued case

Partial converse of the theorem is true for the projective tensor product:

Theorem

Let $f \in C(G)$, $0 < q_1, q_2 < 1$ and $1/p_1 = 1/q_1 - 1$, $1/p_2 = 1/q_2 - 1$. Consider a convolution operator $\star f : M(G) \to C(G)$ and an operator $T : X \to Y$. Suppose that the operators $f\star$ and T can be factored through S_{p_1,p_2}-operators. If $p_2 \leq p_1$ (or $q_2 \leq q_1$), then the operator

$$T_f : M(G)\widehat{\otimes}X \to C(G,X)$$

has the same property.
In particular, it is true if $\hat{f} \in l_{q_1,q_2}$ and T is l_{q_1,q_2}-nuclear ($q_2 \leq q_1$).

Vector-valued case

Corollary 3

Let $f \in C(G)$, $0 < q_1, q_2 < 1$ or $q_1 = q_2 = 1$ and let $1/p_1 = 1/q_1 - 1$, $1/p_2 = 1/q_2 - 1$. Consider a convolution operator $\star f : M(G) \to C(G)$ and an operator $T : X \to Y$.
If $p_2 \leq p_1$ (or $q_2 \leq q_1$) and the space X has the Radon-Nikodym property (in particular, reflexive or isomorphic to a separable dual) then TFAE:
1) Operator T_f can be factored through an S_{p_1,p_2}-operator;
2) The operators $f\star$ and T can be factored through an S_{p_1,p_2}-operator;
3) $\hat{f} \in l_{q_1,q_2}$ and T can be factored through S_{p_1,p_2}-operator.

Remark

The case where $q_1 = q_2 = 1$, $T = \mathrm{id}_X$ is Theorem 3.1 in the paper of P. Saab.

Vector-valued case

Remark

If $q_1 = q_2 =: q$, then 1)–3) follow from the condition
4) The operators $f\star$ and T are q-nuclear.

Remark

Corollary 3 seems to be not valid in general case, that is, in the case where $q_2 > q_1$.
[A counterexample is not written down.]

Thank you for your attention!