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ON THE HANKEL TRANSFORM OF FUNCTIONS FROM NIKOL'SKII TYPE CLASSES

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By definition, a function $f(x)$ on $\mathbb{R}$ belongs to the Lipschitz class $Lip(\alpha, p; \mathbb{R})$, $0 < \alpha \leq 1$, $p \geq 1$, if $f \in L^p(\mathbb{R})$ and

$$\|f(x-t) - f(x)\|_{L^p(\mathbb{R})} = O(t^\alpha)$$

as $t \to 0$.
It is known the following classical theorem of E. Titchmarsh:

Theorem 1 ([T], Theorem 84)

Let $f(x) \in Lip(\alpha, p; \mathbb{R})$, $1 < p \leq 2$, $0 < \alpha \leq 1$, *and* $\widehat{f}(\lambda)$ *the Fourier transform of* f. *Then* $\widehat{f}$ *belongs to the Lebesgue classes* $L^r(\mathbb{R})$ *for*

$$\frac{p}{p + \alpha p - 1} < r \leq \frac{p}{p - 1},$$

and the bounds $\frac{p}{p+\alpha p-1}$ *and* $\frac{p}{p-1}$ *be sharp, in the sense that the range of values of* r *cannot be extended.*

[T] Titchmarsh E. C., *Introduction to the theory of Fourier integrals*, Oxford: Clarendon Press, 1937.

There are many analogues of Theorem 1: for the Fourier series on compact Vilenkin groups, that is, on compact metrizable zero-dimensional Abelian groups (see [Onn1], [QLY], [Yo]); for the Fourier series on compact homogeneous manifolds ([DDR]); for the Fourier transform on finite-dimensional metrizable locally compact Abelian groups ([Bl1], [Bl2]).

[Onn1] Onneweer C. W, Absolute convergence of Fourier series on certain groups, Duke Math. J. 39 (1972), 599–609.
[QLY] Quek T. S., Leonard Y. H. Yap, *Absolute convergence of Vilenkin–Fourier series*, J. Math. Anal. Appl. 74 (1980), 1–14.
[Yo] Younis M. S., *On the absolute convergence of Vilenkin-Fourier series*, J. of Math. Anal. and Appl. 163, 15-19 (1992).
[DDR] Daher R., Delgado J., Ruzhansky M., *Titchmarsh theorems for Fourier transforms of Hölder-Lipschitz functions on compact homogeneous manifolds*, Monatsh. Math. 189, No. 1, 23-49 (2019).
[Bl1] Bloom W. R., *Absolute convergence of Fourier series on finite-dimensional groups*, Colloq. Math. 46, 97-103 (1982).
[Bl2] Bloom W. R., *A characterisation of Lipschitz classes on finite dimensional groups*, Proc. Am. Math. Soc. 59, 297-304 (1976).

We prove an analogue of Theorem 1 for the the Hankel transform of functions from Nikol'skiĭ type function classes on the half-line $[0, +\infty)$. Let us present necessary information from Bessel harmonic analysis (see, for example, [Lev51], [Kip97], [Tri01], [Pla07]). In what follows, α is a real number greater than $-\frac{1}{2}$.

[Lev51] Levitan B.M. Expansion in Fourier series and integrals with Bessel functions. Uspekhi Mat. Nauk. 1951; 6(2): 102-143. (Russian)

[Kip97] Kipriyanov I.A. Singular elliptic boundary value problems. Moscow: Nauka; 1997. (Russian).

[Tri01] Triméche K. Generalized harmonic analysis and wavelet packets. Gordon and Breach, Amsterdam; 2001.

[Pla07] Platonov S.S. Bessel harmonic analysis and approximation of functions on the half-line. Izvestiya: Mathematics. 2007; 71(5): 1001-1048.

Let

$$\mathcal{B} = \mathcal{B}_x := \frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}$$

be the Bessel differential operator and denote by $j_\alpha(x)$ the normalized Bessel function of the first kind, that is

$$j_\alpha(x) = \begin{cases} 2^\alpha \Gamma(\alpha+1)\, x^{-\alpha}\, J_\alpha(x) & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

where $J_\alpha(x)$ is the Bessel function of the first kind and $\Gamma(x)$ is the gamma function. The function $y = j_\alpha(x)$ satisfies the differential equation $\mathcal{B}y + y = 0$ with the initial conditions $y(0) = 1$ and $y'(0) = 0$.

All function classes consist of complex-valued functions on $\mathbb{R}_+$, but it is convenient to assume that the functions are extended to the whole of $\mathbb{R}$ as even functions. Let $C(\mathbb{R}_+)$ be the set of even continuous functions on $\mathbb{R}$ and let $C^{(k)}(\mathbb{R})$ be the set of even k-times differentiable functions on $\mathbb{R}$. For any $p \in (0, +\infty)$ let $L_{p,\alpha}$ be the class of all measurable functions $f(x)$ on $\mathbb{R}_+$ for which

$$\int_{\mathbb{R}_+} |f(x)|^p \, x^{2\alpha+1} \, dx < \infty.$$

For $p \in [1, +\infty)$ the class $L_{p,\alpha}$ is a Banach space with the norm

$$\|f\|_{p,\alpha} := \left(\int_{\mathbb{R}_+} |f(x)| \, x^{2\alpha+1} \, dx \right)^{1/p}.$$

For $p = \infty$ we denote by $L_{\infty,\alpha}$ the Banach space of all measurable functions $f(x)$ that are essentially bounded on $\mathbb{R}_+$ with the norm

$$\|f\|_{\infty,\alpha} = \|f\|_{\infty} := \operatorname{ess\,sup}_{x \in \mathbb{R}_+} |f(x)|.$$

Following [Lev51], we define for every $f \in C^{(2)}(\mathbb{R}_+)$ the generalized Bessel translation operator $T^y f(x) = u(x,y)$, $x, y \in \mathbb{R}_+$, to be the solution of the Cauchy problem

$$\mathcal{B}_y u(x,y) = \mathcal{B}_x u(x,y), \tag{1}$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial s}(x,0) = 0, \tag{2}$$

where $\mathcal{B}_x$ and $\mathcal{B}_y$ are Bessel differential operators acting with respect to the variables x and y, respectively. The solution of the Cauchy problem can be written in explicit form,

$$T^y f(x) = u(x,y) = c_\alpha \int_0^\pi f\left(\sqrt{x^2+y^2-2xy\cos\varphi}\right) (\sin\varphi)^{2\alpha}\, d\varphi, \tag{3}$$

where

$$c_\alpha = \left(\int_0^\pi (\sin\varphi)^{2\alpha}\, d\varphi\right)^{-1} = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}.$$

By formula (3), the operator T^y can be extended to all functions $f \in L_{p,\alpha}$, $1 \le p \le \infty$, and to all locally integrable functions on $\mathbb{R}_+$ and the operator T^y is a linear continuous operator in the Banach space $L_{p,\alpha}$.
For every $f \in L_{p,\alpha}$ the differences $\Delta_h^k f(x)$ of order k $(k = 1, 2, 3, \dots)$ with step $h > 0$ and the modulus $\omega_k(f, \delta)_{p,\alpha}$ of smoothness of order k are defined by the formulae

$$\Delta_h^1 f(x) = \Delta_h f(x) := f(x) - T^h f(x),$$

$$\Delta_h^k f(x) := \Delta_h(\Delta_h^{k-1} f(x)), \quad k > 1; \tag{4}$$

$$\omega_k(f, \delta)_{p,\alpha} := \sup_{0<h\le\delta} \|\Delta_h^k f\|_{p,\alpha}, \quad \delta > 0. \tag{5}$$

respectively.

Let $r > 0$ be a real number, k be a natural number such that $2k > r$ (for instance, we can take $k = [r/2] + 1$, $[x]$ is the integer part of x).

Definition 1

A function $f(x)$ belongs to the space $H^r_{p,\alpha}$, $p \in [1, \infty]$, if $f \in L_{p,\alpha}$ and for some constant $A_f > 0$ we have

$$\omega_k(f;\delta)_{p,\alpha} \le A_f\,\delta^r, \qquad \delta > 0. \tag{6}$$

For $f \in H^r_{p,\alpha}$ we define the seminorm

$$h^r_{p,\alpha}(f) := \sup_{\delta>0} \frac{\omega_k(f,\delta)_{p,\alpha}}{\delta^r}. \tag{7}$$

$H^r_{p,\alpha}$ is a Banach space with norm

$$\|f\|_{H^r_{p,\alpha}} := \|f\|_{p,\alpha} + h^r_{p,\alpha}(f). \tag{8}$$

It can be proved that $H^r_{p,\alpha}$ do not depend on k (see, [Pla07]). Also, the paper [Pla07] contains other descriptions of the classes $H^r_{p,\alpha}$. The classes $H^r_{p,\alpha}$ are analogues of the classical Nikol'skiĭ function classes $H^r_p(\mathbb{R})$ on $\mathbb{R}$ (see [Nik75]).

[Nik75] Nikol'skii S.M. Approximation of functions of several variables and imbedding theorems. Grundlehren Math. Wiss., Vol. 205, Springer, New York - Heidelberg; 1975.

For every $f \in L_{1,\alpha}$ the Hankel transform is defined by

$$\mathcal{F} : f(x) \mapsto \widehat{f}(\lambda) = \int\limits_0^\infty f(x)\, j_\alpha(\lambda x)\, x^{2\alpha+1}\, dx, \qquad \lambda \in \mathbb{R}_+. \quad (9)$$

For every $g \in L_{1,\alpha}$ the inverse Hankel transform is given by

$$\mathcal{F}^{-1} : g(\lambda) \mapsto \widetilde{g}(x) = (2^\alpha \Gamma(\alpha+1))^{-2} \int\limits_0^\infty g(\lambda)\, j_\alpha(\lambda x)\, \lambda^{2\alpha+1}\, d\lambda, \quad (10)$$

that is $\mathcal{F}^{-1} = (2^\alpha \Gamma(\alpha+1))^{-2}\, \mathcal{F}$. Hankel transforms are sometimes also called Bessel transforms. In what follows, by $\widehat{f}(\lambda)$ we denote the Hankel transform of the function $f(x)$.

Suppose that $\mathcal{S}$ is the space of test functions, i.e., the set of all infinitely differentiable functions $\varphi(x)$ decreasing together with all derivatives faster than any power of $|x|^{-1}$. The space $\mathcal{S}$ is equipped with a topology and a locally convex space structure in the usual way. By $\mathcal{S}_+$ we denote the subspace in $\mathcal{S}$ consisting of even functions. The subspace $\mathcal{S}_+$ is equipped by the topology induced from the space $\mathcal{S}$. It is known that the direct and inverse Hankel transforms are mutually inverse automorphisms of the space $\mathcal{S}_+$.

For any $p \in [1, +\infty)$ let $p' := \frac{p}{p-1}$. If $1 < p \leq 2$ then for any function $f \in \mathcal{S}_+$ we have the Hausdorff – Young inequality

$$\|\mathcal{F}(f)\|_{p',\alpha} \leq A^{2/p'} \|f\|_{p,\alpha}, \tag{11}$$

where

$$A = 2^{\alpha}\,\Gamma(\alpha + 1). \tag{12}$$

It follows from (11) that Hankel transform $\mathcal{F} : \mathcal{S}_+ \to \mathcal{S}_+$ can be uniquely extended by continuity up to the linear continuous mapping of the Banach space $L_{p,\alpha}$, $1 < p \leq 2$, onto the Banach space $L_{p',\alpha}$. The extended mapping is also denoted $\mathcal{F} : f \mapsto \widehat{f}$ and it is called Hankel transform. The Hausdorff – Young inequality (11) remains true for any $f \in L_{p,\alpha}$, $1 \leq p \leq \infty$.

The following theorem is an analogue of Theorem 1 for the the Hankel transform.

Theorem 2

Let $f \in H^r_{p,\alpha}$, $1 \le p \le 2$, $r > 0$, and let $\widehat{f}$ be the Hankel transform of f. Then $\widehat{f}$ belongs to the Lebesgue classes $L_{q,\alpha}$ for

$$\frac{p}{rp/(2\alpha+2)+p-1} < q \le \frac{p}{p-1} \tag{13}$$

and the bounds $\frac{p}{rp/(2\alpha+2)+p-1}$ and $\frac{p}{p-1}$ be sharp, in the sense that the range of values of r cannot be extended.

Also note the limiting case of inequalities (13) as $\alpha \to -1/2$. For $\alpha = -1/2$, the Bessel operator takes the form $\mathcal{B} = d^2/dx^2$ and the generalized Bessel translation becomes

$$T^y f(x) = \frac{1}{2}(f(x+y) + f(x-y));$$

we also have $j_{-1/2}(x) = \cos x$ and the Hankel transform coincides with the Fourier cosine transform. For the case $\alpha = -1/2$ the inequalities (13) coincide with the Titchmarsh inequalities.

For any $a > 0$ and for any function f on $\mathbb{R}_+$ let

$$(\Gamma^a f)(x) = f(ax) \qquad (14)$$

be the dilatation operator. Then we have

$$\mathcal{F}(\Gamma^a f) = a^{-2\alpha-2}\Gamma^{1/a}(\mathcal{F}(f)), \qquad f \in L_{p,\alpha}, \quad 1 \leq p \leq 2,$$

that is,

$$\mathcal{F}(f(ax)) = a^{-2\alpha-2}\,\widehat{f}(\lambda/a), \qquad (15)$$

if $\mathcal{F}(f(x)) = \widehat{f}(\lambda)$.

The convolution of functions $f(x)$ and $g(x)$ on $\mathbb{R}_+$ is defined by the relation

$$(f * g)(y) := \int_0^\infty (T^y f(x))\, g(x)\, x^{2\alpha+1}\, dx. \qquad (16)$$

The convolution is well defined if the integral on the right-hand side of (16) is well defined (in particular, when $f, g \in L_{1,\alpha}$, which implies that $f * g$ belongs to $L_{1,\alpha}$).
The convolution has the following properties.
1) It is associative, that is,

$$(f * g) * h = f * (g * h).$$

2) It is commutative, that is,

$$f * g = g * f.$$

3) If $f \in L_{1,\alpha}$ and $g \in L_{p,\alpha}$, $1 \le p \le \infty$, then $f * g$ and $g * f$ are well defined and belong to $L_{p,\alpha}$. In this case

$$\|f * g\|_{p,\alpha} \le \|f\|_{1,\alpha} \|g\|_{p,\alpha}. \tag{17}$$

4) The Hankel transform of the convolution of functions is equal to the product of their Hankel transforms, that is,

$$\widehat{(f * g)}(\lambda) = \widehat{f}(\lambda)\,\widehat{g}(\lambda), \tag{18}$$

for any $f \in L_{1,\alpha}$, $g \in L_{p,\alpha}$, $1 \le p \le 2$.

Let $g(x)$ be any nonnegative function on $\mathbb{R}_+$, satisfying the conditions:

$$g(x) \in L_{1,\alpha}; \tag{19}$$

$$\int_0^{\infty} g(x)\, x^{2\alpha+1}\, dx = 1; \tag{20}$$

$$\int_0^{+\infty} g(x)\, x^{2k+2\alpha+1}\, dx < \infty. \tag{21}$$

For any $n \in \mathbb{N}$ we define the function

$$K_n(x) := \sum_{j=1}^{2k-1} d_j \left(\frac{n}{j}\right)^{2\alpha+2} \left(\Gamma^{n/j} g\right)(x). \tag{22}$$

Proposition 1

Let $f \in L_{p,\alpha}$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$. *The following inequality holds for any* $n = 1, 2, 3, \ldots$*:*

$$\|K_n * f - f\|_{p,\alpha} \leq c\,\omega_k\left(f; \frac{1}{n}\right)_{p,\alpha}, \tag{23}$$

where $c > 0$ *is a constant independent of* f *and* n.

In what follows, let

$$g(x) = \mu\,(j_{\alpha+1}(x))^{2N}\,, \tag{24}$$

where N is any positive integer such that $N > k + \alpha + 1$ and μ is a constant such that (20) holds. This functions satisfies the conditions (19) — (21).

Proposition 2

The Hankel transform $\widehat{K_n}(\lambda)$ is a continuous function with compact support on the segment $[0, 2Nn]$.

Proof of sufficiency of the conditions (13)

We shall use the notations

$$I_n := [-N2^{n+1}, N2^{n+1}], \quad S_n := I_{n+1} \setminus I_n, \quad n \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}. \tag{25}$$

For any measurable subset $A \subset \mathbb{R}$ let $\mu(A)$ be the Lebesgue measure of A. We note that

$$\mu(I_n) = \mu(S_n) = N2^{n+2}. \tag{26}$$

Let $f(x) \in H^r_{p,\alpha}$, $1 \le p \le 2$, $\alpha > 0$, and $r \in \mathbb{R}$ satisfies the conditions (13). The bounds in the inequalities (13) can be expressed by $p' := \frac{p}{p-1}$. Then $\frac{p}{rp/(2\alpha+2)+p-1} = \frac{(2\alpha+2)p'}{rp'+2\alpha+2}$ and the conditions (13) have the form

$$\frac{(2\alpha+2)p'}{rp'+2\alpha+2} < q \le p'. \tag{27}$$

Let us prove that $\widehat{f} \in L_{q,\alpha}$. The case $q = p'$ is obvious since $\widehat{f} \in L_{p',\alpha}$ for any function $f \in L_{p,\alpha}$. In what follows we assume that $q < p'$. We denote $c, c_1, c_2, \ldots$ positive constants that do not depend on n but can depend on p, k, r, q and some unimportant parameters.

It follows from Proposition 1 that for any $n \in \mathbb{Z}_+$ we have the inequality

$$\|K_{2^n} * f - f\|_{p,\alpha} \leq c\,\omega_k(f; 2^{-n})_{p,\alpha} \leq c_1 2^{-rn}, \qquad (28)$$

where c_1 is a positive constant. It follows from Proposition that

$$\mathrm{supp}\widehat{K_{2^n}} \subseteq I_n, \qquad (29)$$

hence

$$\widehat{K_{2^n}}(\lambda) = 0 \quad \forall \lambda \in S_n. \qquad (30)$$

It follows from (18) that

$$\mathcal{F}(K_{2^n} * f - f)(\lambda) = \left(\widehat{K_{2^n}}(\lambda) - 1\right) \widehat{f}(\lambda). \qquad (31)$$

Next, the proof that $\widehat{f} \in L_{q,\alpha}$ we consider separately for cases $p > 1$ and $p = 1$. Let $p > 1$. Using the Hausdorff – Young inequality (see (11)) and the inequality (28) we obtain that

$$\| \left(\widehat{K_{2^n}} - 1\right) \widehat{f}\|_{p',\alpha} \leq A^{2/p'} \|K_{2^n} * f - f\|_{p,\alpha} \leq c_2\, 2^{-rn}, \qquad (32)$$

where $c_2 = A^{2/p'} c_1$ is a constant. Using (30) we note that

$$\| \left(\widehat{K_{2^n}} - 1\right) \widehat{f}\|_{p',\alpha}^{p'} = \int\limits_0^\infty |\widehat{K_{2^n}}(\lambda) - 1|^{p'} |\widehat{f}(\lambda)|^{p'} \lambda^{2\alpha+1} d\lambda \geq \int\limits_{S_n} |\widehat{f}(\lambda)|^{p'} \lambda^{2\alpha+1}$$

whence, using (32), we obtain

$$\int\limits_{S_n} |\widehat{f}(\lambda)|^{p'} \lambda^{2\alpha+1} d\lambda \leq c_3\, 2^{-rnp'}, \quad n \in \mathbb{Z}_+, \qquad (33)$$

where $c_3 = (c_2)^{p'}$.

Using the Hölder inequality, we have that

$$\int\limits_{S_n} |\widehat{f}(\lambda)|^q \, \lambda^{2\alpha+1} d\lambda = \int\limits_{S_n} |\widehat{f}(\lambda)|^q \times 1 \, d\mu_\alpha(\lambda) \leq$$

$$\left(\int\limits_{S_n} \left(|\widehat{f}(\lambda)|^q\right)^{p'/q} d\mu_\alpha(\lambda) \right)^{q/p'} \left(\int\limits_{S_n} 1 \, d\mu_\alpha(\lambda) \right)^{(p'-q)/p'} \leq$$

$$c_4 \left(\int\limits_{S_n} |\widehat{f}(\lambda)|^{p'} \, d\mu_\alpha(\lambda) \right)^{q/p'} \left(2^{(2\alpha+2)n}\right)^{(p'-q)/p'}, \qquad (34)$$

where $c_4 = (D_\alpha)^{(p'-q)/p'}$. It follows from (33) and (34) that

$$\int\limits_{S_n} |\widehat{f}(\lambda)|^q \, \lambda^{2\alpha+1} d\lambda \leq c_5 \, \left(2^{-nrp'}\right)^{q/p'} 2^{(2\alpha+2)n(p'-q)/p'} = c_5 \, 2^{-\delta n}, \qquad (35)$$

where $c_5 = (c_3)^{q/p'} c_4$,

$$\delta = rq - \frac{(2\alpha + 2)(p' - q)}{p'} = \frac{rqp' - (2\alpha + 2)(p' - q)}{p'}. \qquad (36)$$

It follows from (27) that $\delta > 0$.

Using (35) we have

$$\sum_{n=0}^{\infty} \int_{S_n} |\widehat{f}(\lambda)|^q \, \lambda^{2\alpha+1} d\lambda \le c_5 \Big(\sum_{n=1}^{\infty} 2^{-n\delta} \Big) < \infty. \qquad (37)$$

Using the Hölder inequality we obtain that

$$\int_{I_0} |\widehat{f}(\lambda)|^q \, \lambda^{2\alpha+1} d\lambda = \int_{S_n} |\widehat{f}(\lambda)|^q \times 1 \, d\mu_\alpha(\lambda) \le$$

$$\Big(\int_{I_0} \left(|\widehat{f}(\lambda)|^q\right)^{p'/q} d\mu_\alpha(\lambda) \Big)^{q/p'} \Big(\int_{S_n} 1 \, d\mu_\alpha(\lambda) \Big)^{(p'-q)/p'} \le$$

$$\Big(\int_{I_0} |\widehat{f}(\lambda)|^{p'} \, d\lambda \Big)^{q/p'} (B_\alpha)^{(p'-q)/p'} \le c_6 \left(\|\widehat{f}\|_{p',\alpha} \right)^q < \infty, \qquad (38)$$

where $c_6 = (B_\alpha)^{(p'-q)/p'}$.

It follows from (37) and (38) that

$$\int_0^\infty |\widehat{f}(\lambda)|^q\,\lambda^{2\alpha+1}d\lambda = \int_{I_0} |\widehat{f}(\lambda)|^q\,\lambda^{2\alpha+1}d\lambda + \sum_{n=0}^{\infty}\int_{S_n} |\widehat{f}(\lambda)|^q\,\lambda^{2\alpha+1}d\lambda < \infty, \tag{39}$$

therefore $\widehat{f} \in L_{q,\alpha}$.

The main results of my report are published in the papers [Pl2] and [Pl3].

[Pl2] S. S. Platonov, *On the Hankel transform of functions from Nikol'ski type classes*, Integral Transforms and Special Functions (2020) 1-17, DOI: 10.1080/10652469.2020.1849184
[Pl3] S. S. Platonov, *On the Fourier transform of functions from the classes* $H_p^\alpha(\mathbb{R})$, Bollettino dell'Unione Matematica Italiana (2021) 1-16, DOI: 10.1007/s40574-021-00291-8