

Spectral Theory

of
Self-adjoint finitely cyclic operators

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Introduction to Matrix measure L^2 -spaces

{ relepis ^{knstnij} monografii / otuzoj pracy a la "dissertations" }

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I

Introduction

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Notation

{ There will be some introduction
in the near future... (and
the notation - too...) }

- I, 0 -

II. A detailed proof of The
"XMUE" Theorem
for selfadjoint cyclic operators

- II.0 -

II.1

II.1. The \times -multiplication Unitary Equivalence ("xMUE") Theorem. - The cyclic case.

Let \mathcal{H} be a complex Hilbert space.

Theorem (xMUE)

If A is a cyclic s.a. ^(operator) in \mathcal{H} , and φ is a cyclic vector for A , then A is unitary equivalent to T_{\times} in $L^2(\mu)$, where $\mu = E_{A, \varphi}$.

Moreover:

(i) $\text{Pol}_{\mu}(\mathbb{R})$ is a dense subspace of $L^2(\mu)$;

(ii) $\exists!$ $U \in \mathcal{B}(L^2(\mu), \mathcal{H})$ $\forall_{n \in \mathbb{N}_0}$ $U([x^n]) = A^n \varphi$;

(iii) the unique U from (ii) is a unitary operator from $L^2(\mu)$ onto \mathcal{H} and it is defined by

$$U([f]) = f(A)\varphi, \quad f \in L^2(\mu), \quad (1.1)$$

and

$$A = U T_{\times} U^{-1}. \quad (1.2)$$

Proof - Part 1 {the standard consequences of FC...}

Define linear $\tilde{U}: L^2(\mu) \rightarrow \mathcal{H}$ by formula

$$\tilde{U}f := f(A)\varphi, \quad f \in L^2(\mu)$$

Note that the RHS above is a correct definition of an element of \mathcal{H} by STh + FCTh (and \tilde{U} is linear), because

$$D(f(A)) = \{y \in \mathcal{H} : \int_{\mathbb{R}} |f|^2 dE_{A,y} < +\infty\} \quad (1.3)$$

and for $y \in D(f(A))$

$$\|f(A)y\| = \left(\int_{\mathbb{R}} |f|^2 dE_{A,y} \right)^{1/2} = \|f\|_{\mu}, \quad (1.4)$$

so by $f \in L^2(\mu)$ and $\mu = E_{A,\varphi}$ for $y = \varphi$ we get $\varphi \in D(f(A))$ and

$$\|\tilde{U}\varphi\|^2 = \|f(A)\varphi\|^2 = \|f\|_{\mu}^2 = \|[f]\|_{\mu}^2. \quad (1.5)$$

Moreover, by (1.5), $\{f \in L^2(\mu) : \|f\|_{\mu} = 0\} \subset \text{Ker } \tilde{U}$, so \tilde{U} can be factorized (to fact) $U: L^2(\mu) \rightarrow \mathcal{H}$ by formula

$$U([f]) := \tilde{U}(f), \quad f \in L^2(\mu), \quad (1.6)$$

and U is bounded operator given by (1.1) being also an isometry on its range (image).

One of the "standard" consequences (typical exercise) of STh + FCTH is:

$$\forall_{n \in \mathbb{N}_0} \mathbb{X}^n(A) = A^n \quad \left\{ \begin{array}{l} \leftarrow \text{the power of } A \text{ (recurs. def.)} \\ \leftarrow \text{the function } \mathbb{X}^n \text{ of } A \end{array} \right.$$

hence — we start to use the cyclicity here — knowing that φ is cyclic vector, we have in particular that

$$\varphi \in \mathcal{D}(A^n) \quad (\text{for any } n \in \mathbb{N}_0), \text{ so by (1.3)}$$

for $f = \mathbb{X}^n$ and $g = \varphi$ we get $[\mathbb{X}^n] \in L^2(\mu)$,

$$\text{and } \text{Pol}_\mu(\mathbb{R}) = \text{lin}\{[\mathbb{X}^n] : n \in \mathbb{N}_0\} \subset L^2(\mu).$$

We also have

$$\forall_{n \in \mathbb{N}_0} U([\mathbb{X}^n]) = \tilde{U} \mathbb{X}^n = \mathbb{X}^n(A) \varphi = A^n \varphi, \quad (1.6)$$

so U is one of the bounded operators as in (ii).

Moreover, by ~~the~~ cyclicity of φ for A $\text{lin}\{A^n \varphi : n \in \mathbb{N}_0\}$ is dense in \mathcal{H} , so the isometry U is onto \mathcal{H} , i.e.

U is a unitary transformation from $L^2(\mu)$ to \mathcal{H} ,

and $\text{Pol}_\mu(\mathbb{R})$ is also dense in $L^2(\mu)$.

Therefore we proved (i) and the " \supseteq " part of (ii), but the " \subseteq " (the uniqueness) part of (ii) is obvious

by the continuity requirement for the operator, and by the density from (i) (just proved). And thus, it remains only to prove (1.2) from (iii).

— We shall do this in the next part of the proof in full generality. But let us make here a remark, that this ^{remaining part} is quite obvious if A is e.g. a bounded operator. Indeed — observe, that for any $n \in \mathbb{N}_0$ we obviously have:

$$\begin{aligned} U^{-1} A (A^n \varphi) &= U^{-1} A^{n+1} \varphi = [X^{n+1}] = [X \cdot X^n] \\ &= T_X [X^n] = T_X \underbrace{U^{-1}(A^n \varphi)}_{\text{by } U[X^n] = A^n \varphi}, \end{aligned} \quad (1.7)$$

which means, that the equality

$$A y = U T_X U^{-1} y$$

is true for any y from a linear dense set $\{A^n \varphi : n \in \mathbb{N}_0\}$.

So (1.2) holds with the extra assumption: $A \in \mathcal{B}(\mathcal{H})$.

Note also, that the argumentation based on (1.7) works with a weaker assumption than the boundedness of A .

(Namely) with the assumption, that φ is an essential cyclic vector for A . In such a case, both A and $U T_X U^{-1}$ are s.a. extensions of $A|_{\text{lin}\{A^n \varphi : n \in \mathbb{N}_0\}}$, which is ess. s.a., so they are equal.

But we see, that it is not so obvious, that our assertion remains true, without the essential cyclicity assumption. The proof above does not work, because of "the domain details". But the result is true, however! There exist several proofs of (1.2). One of them just ^(directly) "patches the holes" in the proof above, but it is long and "technical". We shall use another one proof, based on a certain convenient idea (which we plan to use later - for the generalized result)

II.2 The "delicate" part of the proof of "xMUETH"

Proof of "xMUETH" - Part 2 {the final (general case) part}

The ^(main) idea of the proof of (1.2) is to omit the difficulties related to the domains of unbounded operators, and to prove "(1.2)" for "spectral projections of A instead of A ". - Namely, we shall prove that

$$\forall \omega \in \text{Borel}(\mathbb{R}) \quad E_{T_x}(\omega) = E_{U^{-1}AU}(\omega) \quad (2.1)$$

which would give (1.2) by STh + FCTh ("s.a. operator is the spectral integral of x with respect to its spectral measure").

So, fix $\omega \in \text{Borel}(\mathbb{R})$. The spectral resolution of T_x in $L^2(\mu)$

is well known:
$$E_{T_x}(\omega) = T_{x_\omega}$$

We also have $E_{U^{-1}AU}(\omega) = U^{-1}E_A(\omega)U$, hence

we should prove only:

$$UT_{x_\omega} = E_A(\omega)U.$$

But both sides above are bounded operators on $L^2(\mu)$, so by the linear density of $\{[x^n] : n \in \mathbb{N}_0\}$ (already proved) it suffices

to check, that for any $n \in \mathbb{N}_0$

$$\begin{aligned}
 U_{T_{\chi_\omega}}([X^n]) &= E_A(\omega) U([X^n]) \\
 &\quad \parallel \text{(det. of } T_f) \quad \parallel \text{(see (ii))} \\
 U([X_\omega X^n]) &\} E_A(\omega) (A^n \varphi). \quad \} \text{(2.2)} \\
 &\quad \underbrace{\qquad\qquad\qquad}_{??} \text{ - is it true?}
 \end{aligned}$$

But we have $U([X_\omega X^n]) = (X_\omega X^n)(A) \varphi$ by (1.6).

And also $E_A(\omega) (A^n \varphi) = (X_\omega(A) X^n(A)) \varphi = (X_\omega X^n)(A) \varphi$,
 (because " $f(A) \cdot g(A) \subset (f \cdot g)(A)$ ", by FCTh.), so
 (2.2) holds! □

The above idea and a similar proof of details will be used soon in the "more sophisticated" case - the generalization of \times MUETH to finitely cyclic s.a. operators.

III Matrix measure, L^2 -space for
matrix measure and multiplication by
function operators

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III . 1 →

III.1 Matrix measure and its trace measure

Consider a set Ω and \mathcal{M} - a σ -algebra of subsets of Ω . Let $d \in \mathbb{N}$ and consider a $d \times d$ complex matrix valued function M on \mathcal{M} , i.e.

$$M: \mathcal{M} \rightarrow M_d(\mathbb{C}).$$

Definition III.1 (matrix measure)

M is a matrix measure (on $\mathcal{M}, d \times d$) iff

- 1) M is countably additive [abbrev.: c.add.] in the norm sense in $M_d(\mathbb{C})$;
- 2) $M(\omega) \geq 0$ *) for any $\omega \in \mathcal{M}$.

In other words, matrix measure is just the " $M_d(\mathbb{C})$ -vector non-negative measure". Surely, norm sense of the c.add. above is equivalent to c.add. "on each matrix term".

A simplest example of matrix measure is the $d=1$ case,

*) Here $A \geq B$ is in the Hilbert space \mathbb{C}^d sense:

$$A \geq B \iff \forall x \in \mathbb{C}^d \quad \mathbb{R} \ni \langle Ax, x \rangle_{\mathbb{C}^d} \geq \langle Bx, x \rangle_{\mathbb{C}^d} \in \mathbb{R}.$$

In particular, both A, B should be self-adjoint, and

$$A \geq B \iff A - B \geq 0 \text{ (is non-negative)}. \text{ And } A \leq B \text{ means}$$

$$B \geq A \dots$$

because then, if we identify $M_1(\mathbb{C})$ and \mathbb{C} , matrix measure is just a finite measure ^(*).

Note, that we omit "non-negative" in the name "matrix measure" (although that would be ^{somewhat} more consistent with the terminology in ^(*) below (as e.g. "X measure")) by tradition.

For a dxd matrix measure M and $i, j = 1, \dots, d$ let us define $M_{ij} : \mathcal{M} \rightarrow \mathbb{C}$ by

$$M_{ij}(\omega) := (M(\omega))_{ij}, \quad \omega \in \mathcal{M}. \quad (1.1)$$

EEEx Proposition III.2

If M is a matrix measure, then:

- (a) $M(\emptyset) = 0$;
- (b) (finite add.) $\forall \omega, \omega' \in \mathcal{M} \quad (\omega \cap \omega' = \emptyset \Rightarrow M(\omega \cup \omega') = M(\omega) + M(\omega'))$;
- (c) (monotonicity) $\forall \omega, \omega' \in \mathcal{M} \quad (\omega' \subset \omega \Rightarrow M(\omega') \leq M(\omega))$;
- (d) $\forall_{i, j = 1, \dots, d} M_{ij}$ is a complex measure, $M_{ij} = \overline{M_{ji}}$, M_{ii} is a finite measure.

* Let us fix some "measure types" terminology here:

- measure (without extra adjective/name as "real", "vector", "complex" and so on) = any c. add. function into $[0; +\infty]$ (we need not to say "non-negative")
- complex/real/X-vector (or X)/vector measure = any c. add. function into $\mathbb{C}/\mathbb{R}/X$ / a normed space (here: X - a normed space)
- finite measure: such a measure μ , that $\mu(\mathbb{R}) \in [0; +\infty)$.

Note, that a measure is a complex measure iff it is a finite measure.

Note, that the point (c) gives the following important property of any matrix measure M :

$$\forall_{\substack{\omega, \omega' \in \mathcal{M} \\ \omega' \subset \omega}} M(\omega) = 0 \Rightarrow M(\omega') = 0. \quad (1.2)$$

(Ex) The above follows from: $0 \leq A \leq 0 \Rightarrow A = 0$, which is true (and easy to check) for the "matrix \leq ".

The property (1.2) shows how similar is any matrix measure to "usual" measures (observe that the analog of (1.2) for complex or vector measures is generally not true!)

A "stronger similarity" to measures follows from the next fact, stating that any matrix measure is absolutely continuous with respect to a certain measure (moreover - a finite measure). For M - a matrix measure, denote by tr_M the function from \mathcal{M} into \mathbb{C} given by

$$\text{tr}_M(\omega) := \text{tr}(M(\omega)), \quad \omega \in \mathcal{M}. \quad (1.3)$$

By d) of Proposition III.2 we know, that tr_M is a finite measure, as sum of d -finite measures.

Proposition III.3

If M is a matrix measure, then M and each M_{ij} for $i, j = 1, \dots, d$ are absolutely continuous with respect to tr_M , i.e., $\forall_{\omega \in \mathcal{M}} (\text{tr}_M(\omega) = 0 \Rightarrow (M(\omega) = 0 \text{ and } \forall_{i,j=1,\dots,d} M_{ij}(\omega) = 0))$.

Proof

The assertion for M follows directly from the following matrix property:

(EE_x) If $A \in M_d(\mathbb{C})$ and $A \geq 0$, then $A \leq \text{tr}(A) \cdot I$ (1.4)

(just diagonalize $A \dots$), and this obviously gives the assertion for any complex measure M_{ij} , too. III

We get even more by (1.4): for a matrix measure M on \mathcal{M}

$$\forall \omega \in \mathcal{M} \quad 0 \leq M(\omega) \leq \text{tr}_M(\omega) I. \quad (1.4')$$

Now, using Proposition III.3, ^{and Radon-Nikodym Theorem} let us choose for each $i, j = 1, \dots, d$ a density $d_{M, ij} : \Omega \rightarrow \mathbb{C}$ of M_{ij} with respect to tr_M - note, that we have a free choice of $d_{M, ij}$ up to the tr_M a.e. equality only. Then we define $D_M : \Omega \rightarrow M_d(\mathbb{C})$ by $(D_M(t))_{i, j} := d_{M, ij}(t)$, $t \in \Omega$, $i, j = 1, \dots, d$.

So, D_M is also a function defined up to the tr_M a.e. equality. - By \mathbb{D}_M we denote the class of all such ^(measurable) functions

Hence, finally, for any $D_M \in \mathbb{D}_M$

$$\forall \omega \in \Omega \quad M(\omega) = \int_{\omega} D_M d\text{tr}_M \quad \left(= \int_{\omega} D_M(t) d\text{tr}_M(t) \right), \quad (1.5)$$

where the integral above is just the standard Lebesgue integral of the $M_d(\mathbb{C}) \simeq \mathbb{C}^{(d^2)}$ -valued function with respect to the measure tr_M (the integral is defined coordinatewise, by the appropriate scalar valued function integrals).

Let us give some names, to the above denoted objects determined by matrix measure M .

Definition III.4 (trace measure, trace density, ...)

We call:

- tr_M - the trace measure
- \mathbb{D}_M - the trace density class
- each $D_M \in \mathbb{D}_M$ - the trace density of M .

The trace density cannot be just an arbitrary "matrix-valued" function (L^1 is just by Radon-Nikodym th.)!

Proposition III.5

Any trace density D_M of a matrix measure M satisfies:

$D_M(t)$ is selfadjoint and $0 \leq D_M(t) \leq I$ for tr_M a.e. $t \in \Omega$.

Proof

We shall use here the following general property of matrix (finite dimensional vector) valued function integrals:

If μ is a measure in (Ω, \mathcal{M}) , $\omega \in \Omega$, $\varphi: \mathbb{C}^k \rightarrow \mathbb{C}^l$ is linear and $f: \Omega \rightarrow \mathbb{C}^k$ is integrable with respect to μ , then $\varphi \circ f: \Omega \rightarrow \mathbb{C}^l$ is also integrable and

$$\varphi \left(\int_{\omega} f d\mu \right) = \int_{\omega} \varphi \circ f d\mu. \quad (1.6)$$

Now let $\omega \in \mathcal{M}$ and $x \in \mathbb{C}^d$. By (1.5) and by (1.6) used to $\varphi: M_d(\mathbb{C}) \rightarrow \mathbb{C}$, $\varphi(A) = \langle Ax, x \rangle_{\mathbb{C}^d}$ we have

$$0 \leq \langle M(\omega)x, x \rangle_{\mathbb{C}^d} = \left\langle \left(\int_{\omega} D_M d\text{tr}_M \right) x, x \right\rangle_{\mathbb{C}^d} = \int_{\omega} \left\langle D_M^{(t)} x, x \right\rangle_{\mathbb{C}^d} d\text{tr}_M(t).$$

Since ω was arbitrary, we get

$$\forall x \in \mathbb{C}^d \quad \left\langle D_M^{(t)} x, x \right\rangle_{\mathbb{C}^d} \geq 0 \quad \text{for } \text{tr}_M \text{ a.e. } t \in \Omega \quad (1.7)$$

Similarly, by (1.4')

$$0 \leq \langle (\operatorname{tr}_M(\omega)I - M(\omega))x, x \rangle_{\mathbb{C}^d} = \left\langle \left(\int_{\omega} I d\operatorname{tr}_M \right) x, x \right\rangle_{\mathbb{C}^d} - \left\langle \left(\int_{\omega} D_M d\operatorname{tr}_M \right) x, x \right\rangle_{\mathbb{C}^d} = \int_{\omega} \langle (I - D_M(t))x, x \rangle_{\mathbb{C}^d} d\operatorname{tr}_M(t)$$

and

$$\forall x \in \mathbb{C}^d \text{ for } \operatorname{tr}_M \text{ a.e. } t \in \Omega \quad \langle (I - D_M(t))x, x \rangle_{\mathbb{C}^d} \geq 0 \quad (1.8)$$

Let now C be a countable dense subset of \mathbb{C}^d . By (1.7), (1.8) and the countable additivity of tr_M we can choose a set

$$\omega_0 \in \mathcal{N} \text{ such that } \operatorname{tr}_M(\omega_0) = 0 \text{ and}$$

$$\forall t \in \Omega \setminus \omega_0 \quad \forall x \in C \quad \left(\langle D_M(t)x, x \rangle_{\mathbb{C}^d} \geq 0 \text{ and } \langle (I - D_M(t))x, x \rangle_{\mathbb{C}^d} \geq 0 \right)$$

Hence, by the continuity of $D_M(t) : \mathbb{C}^d \rightarrow \mathbb{C}^d$ and the joint continuity of the scalar product $\langle \cdot, \cdot \rangle_{\mathbb{C}^d}$ (used for "each fixed t "),

we get $\forall t \in \Omega \setminus \omega_0 \quad \forall x \in \underline{\mathbb{C}^d} \dots \dots$ (as above), so

$$\forall t \in \Omega \setminus \omega_0 \quad 0 \leq D_M(t) \leq I \quad (\text{in particular, for } t \in \Omega \setminus \omega_0$$

we get the selfadjointness, since we consider here complex space \mathbb{C}^d)

□

* Recall that in complex Hilbert space \mathcal{H} for $A \in \mathcal{B}(\mathcal{H})$
 $(\forall x \in \mathcal{H} \quad \langle Ax, x \rangle_{\mathcal{H}} \in \mathbb{R}) \implies A$ is s.a. (self-adjoint)

- III.1.7 -

Remark III.6

Using polarization formula for the sesquilinear, conjugate-symmetric form $\mathbb{C}^d \times \mathbb{C}^d \ni (x, y) \mapsto \langle Ax, y \rangle_{\mathbb{C}^d}$

where $A \in M_d(\mathbb{C})$ and $0 \leq A \leq I$ we easily get $\forall_{i,j=1,\dots,d} |\langle Ae_i, e_j \rangle| \leq 2$.

Hence, all the terms of trace density D_M of a matrix measure M are not only tr_M integrable functions, but they are also bounded by 2, i.e. they satisfy:

$$\forall_{i,j=1,\dots,d} |(D_M(t))_{ij}| \leq 2 \quad \text{for } \text{tr}_M \text{ a.e. } t \in \Omega. \quad (1.9)$$

The formula (1.5) says in particular that each matrix measure on \mathcal{M} has the form

$$M = F d\mu \quad (*) \quad (1.10)$$

for a measure μ on \mathcal{M} and a $d \times d$ -matrix valued, non-negative $L^1(\mu)$ function. Recall, that the above "d μ " notation is defined in such a general case just by:

$$\forall_{\omega \in \mathcal{M}} M(\omega) = \int_{\omega} F d\mu. \quad (1.10')$$

*) "d" for "d μ " and "d" for the \mathbb{C}^d dimension mean different objects... - III.1.8 - - sorry!...

So, we can say, that each matrix measure has quite a simple, well-understandable form.

Moreover, we have more information on "our" concrete F and μ , by our previous considerations: μ is a finite measure and $0 \leq F(t) \leq I$ for μ -a.e. $t \in \Omega$.

It is natural to ask, whether the opposite is true, i.e. : is each M of the form (1.10) a matrix measure? - The answer is simple.

Example III.7 (The general example of a matrix measure)

Let μ be a measure on \mathbb{M} and $F: \Omega \rightarrow M_d(\mathbb{C})$ let be a non-negative $L^1(\mu)$ -matrix valued function. Then $M: \mathbb{M} \rightarrow M_d(\mathbb{C})$ given by (1.10) (equiv. by (1.10')) is a matrix measure.

Moreover, for such a matrix measure M , we have

$$\text{tr}_M = \text{tr} F d\mu, \quad (1.11)$$

$$\text{tr}_M(\{t \in \Omega : \text{tr} F(t) = 0\}) = 0 \quad (1.12)$$

and for each $D_M \in \mathcal{D}_M$

$$D_M(t) = \frac{1}{\text{tr} F(t)} \cdot F(t) \text{ for } \text{tr}_M\text{-a.e. } t \in \{s \in \Omega : \text{tr} F(s) \neq 0\} \quad (1.13)$$

- III.1.9.

Proof

Non-negativity is a simple calculation based on (1.6)

(Etx) for φ as in Proof of Propos. III.5. And c. add. is a standard measure theory argument. Similarly (1.11) follows from (1.6) for tr-functional and hence, (1.12) and (1.13) are obvious. □

For a more concrete example see Example III.15 (p. III.2.10).

We end this part of Section III by the following observation concerning sets of zero M -matrix measure.

Remark III.8

If M is a matrix measure on \mathcal{W} and $\omega \in \mathcal{W}$, then TFCAE \Leftarrow the following conditions are equivalent

(i) $M(\omega) = 0$ (ii) $\text{tr}_M(\omega) = 0$ (iii) $D_M(t) = 0$
for tr_M a.e. $t \in \omega$ i.e. $\text{tr}_M(\{t \in \omega : D_M(t) \neq 0\}) = 0$.
Moreover, $\text{tr}_M(\{t \in \Omega : D_M(t) = 0\}) = 0$.

Proof

(ii) \Rightarrow (i) is already known by Prop. III.3 (the abs. cont. of M w.r. to tr_M).
(i) \Rightarrow (ii) - obviously: $\text{tr}_M(\omega) = \text{tr}(M(\omega)) = \text{tr} 0 = 0$, if (i) holds.
(iii) \Rightarrow (ii) - obvious, since $\{t \in \omega : \dots\} \subset \omega$. (iii) \Rightarrow (i)

$M(\omega) = \int_{\omega} D_M d\text{tr}_M = \int_{\omega} D_M d\text{tr}_M + \int_{\omega \setminus \omega} 0 d\text{tr}_M = 0 + 0 = 0$. Now for $\Omega' := \{t \in \Omega : D_M(t) = 0\}$ we get $\text{tr}_M(\Omega') = 0$ by (i) \Leftrightarrow (ii). □

The equivalence "(i) \Leftrightarrow (ii)" is very important. It says that we managed to get much more than we wanted to get, and we got in Proposition III.3 — the absolute continuity of M with respect to tr_M . Here we got in some sense also the opposite: "the absolute continuity of tr_M with respect to M ". It shows, that the choice of tr_M as a measure which "contains" possibly large information on our matrix measure M was very good and precise.

— III.1.11 —

III.2. \mathbb{C}^d -vector functions space $L^2(M)$,
 semi-norm $\|\cdot\|_\mu$ and "semi-scalar" product in $L^2(M)$,
 and the zero-layer $L_0^2(M)$

When μ is a measure on a σ -algebra \mathcal{M} of subsets of a set Ω , then the construction of the space $L^2(\mu)$ of the appropriate class of scalar functions starts from defining of the functions space $L^2(\mu)$ with the standard seminorm $\|\cdot\|_\mu$ and with the "semi-scalar product" $\langle \cdot, \cdot \rangle_\mu$, given by

$$\langle f, g \rangle_\mu = \int_{\Omega} f \bar{g} d\mu, \quad \|f\|_\mu = \langle f, f \rangle_\mu^{1/2}.$$

Then we define $L_0^2(\mu) := \{f \in L^2(\mu) : \|f\|_\mu = 0\}$,
 to use the construction of quotient space $L^2(\mu) / L_0^2(\mu) =: L(\mu)$
 which leads to a norm space with the norm $\|\cdot\|_\mu$
 induced by the scalar product $\langle \cdot, \cdot \rangle_\mu$, given by

$$\langle [f], [g] \rangle_{\mu_2} := \langle f, g \rangle_\mu, \quad f, g \in L^2(\mu).$$

Moreover, $L(\mu)$ is a Hilbert space.

Our main goal in Section III is to "perform" the analog construction of the space $L^2(M)$ for a matrix measure M instead of a measure μ . And here, in subsection III.2 we shall start with

the function space $L^2(M)$ consisting of some \mathbb{C}^d -functions $f: \Omega \rightarrow \mathbb{C}^d$ (here M is a $d \times d$ -matrix measure on \mathbb{R}).

So, let us first define $\llbracket f \rrbracket_M \in [0; +\infty]$ for each such a \mathbb{R} -measurable f by the formula:

$$\llbracket f \rrbracket_M := \int_{\Omega} \langle D_M(t) f(t), f(t) \rangle_{\mathbb{C}^d} d\text{tr}_M(t). \quad (2.1)$$

Note, that the integral above exists (but can be $+\infty$), since f is measurable and $D_M(t) \geq 0$ for $\text{tr}_M(t)$ -a.e. $t \in \Omega$, and the choice of $D_M \in \mathcal{D}_M$ does not matter here (and below...). Next we denote/define:

- $L^2(M) := \{f: \Omega \rightarrow \mathbb{C}^d : f \text{ is } \mathbb{R}\text{-measurable, } \llbracket f \rrbracket_M < +\infty\}$,

- $\| \cdot \|_M : L^2(M) \rightarrow [0; +\infty)$,

$$\|f\|_M := \llbracket f \rrbracket_M^{1/2}, \quad f \in L^2(M) \quad (2.2)$$

- $\langle \cdot, \cdot \rangle_M : L^2(M) \times L^2(M) \rightarrow \mathbb{C}$, $(f, g \in L^2(M))$,

$$\langle f, g \rangle_M := \int_{\Omega} \langle D_M(t) f(t), g(t) \rangle_{\mathbb{C}^d} d\text{tr}_M(t), \quad (2.3)$$

but to check, that the RHS of (2.3) is properly defined for $f, g \in L^2(M)$ and consider $\gamma_{f, g}: \Omega \rightarrow \mathbb{C}$ given by

$$\gamma_{f, g}(t) := \langle D_M(t) f(t), g(t) \rangle_{\mathbb{C}^d}, \quad t \in \Omega. \quad (2.4)$$

We need:

Lemma III.9

If $f, g \in L^2(M)$, then $\int_{fg} \in L^1(\text{tr}_M)$.

Proof

We can assume, that $D_M(t) \geq 0$ for any $t \in \Omega$.
So, for $t \in \Omega$, using the " $\sqrt{\cdot}$ " of non-negative matrix, we get

$$\begin{aligned} \left| \int_{fg}(t) \right| &= \left| \langle \sqrt{D_M(t)} f(t), \sqrt{D_M(t)} g(t) \rangle_{\mathbb{C}^d} \right| \leq \\ &\leq \left(\langle \sqrt{D_M(t)} f(t), \sqrt{D_M(t)} f(t) \rangle_{\mathbb{C}^d} \right)^{1/2} \cdot \left(\langle \sqrt{D_M(t)} g(t), \sqrt{D_M(t)} g(t) \rangle_{\mathbb{C}^d} \right)^{1/2} \\ &= \left(\langle D_M(t) f(t), f(t) \rangle_{\mathbb{C}^d} \right)^{1/2} \cdot \left(\langle D_M(t) g(t), g(t) \rangle_{\mathbb{C}^d} \right)^{1/2}. \end{aligned} \tag{2.5}$$

Now, using the Schwarz inequality for integrals, by (2.1) we obtain

$$\int_{\Omega} \left| \int_{fg}(t) \right| d\text{tr}_M(t) \leq \langle\langle f \rangle\rangle_M^{1/2} \cdot \langle\langle g \rangle\rangle_M^{1/2}. \quad \square$$

Having all our main objects well-defined, we study their expected properties.

Fact III.10

$L^2(M)$ is a \mathbb{C} -linear subspace of the space of all M -measurable \mathbb{C}^d functions on Ω (with the standard linear ^(pointwise) operations).

Moreover $\langle\langle \cdot, \cdot \rangle\rangle_M$ is a sesquilinear, nonnegative and conjugate-symmetric form on $L^2(M)$.

Proof

If $f \in L^2(M)$ and $z \in \mathbb{C}$, then $\langle\langle zf \rangle\rangle_M = |z|^2 \langle\langle f \rangle\rangle_M$, by (2.1), so $zf \in L^2(M)$. If $f, g \in L^2(M)$, then

$\langle\langle f+g \rangle\rangle_M = \int_{\Omega} \chi_{f+g, f+g} d\text{tr}_M$ by (2.4), and we have

$$|\chi_{f+g, f+g}(t)| \leq |\chi_{f,f}(t)| + |\chi_{g,g}(t)| + |\chi_{f,g}(t)| + |\chi_{g,f}(t)|,$$

so $\chi_{f+g, f+g} \in L^1(\text{tr}_M)$ by Lemma III.8, i.e.

$\langle\langle f+g \rangle\rangle_M < +\infty$, and $f+g \in L^2(M)$. Thus $L^2(M)$

is a \mathbb{C} -linear subspace of the measurable \mathbb{C}^d -functions space.

Non-negativity is obvious from $D_M(t) \geq 0$ for tr_M a.e. t .

Sesquilinearity and conjugate-symmetry is obvious by the same properties of $\langle \cdot, \cdot \rangle_{\mathbb{C}^d}$ and by the self-adjointness of $D_M(t)$ for tr_M a.e. $t \in \Omega$. □

Note here that (obviously ...), for any $f \in L^2(M)$

$$\langle\langle f \rangle\rangle_M = \langle\langle f, f \rangle\rangle_M \quad (2.5')$$

which explains that the ambiguity of the notation $\langle\langle \cdot \rangle\rangle_M$ is not very serious...

We can use now the following abstract result
(more or less "well-known" ...)

Proposition III.11 "On the quotient scalar product space"

Let \mathcal{L} be linear space over \mathbb{C} and let $\langle\langle \cdot, \cdot \rangle\rangle$ be a sesquilinear, non-negative conjugate-symmetric $*$) form $**$ on \mathcal{L} and let $\|\cdot\| : \mathcal{L} \rightarrow [0; +\infty)$ be given by

$$\|x\| := \langle\langle x, x \rangle\rangle^{1/2}, \quad x \in \mathcal{L}.$$

Then:

- (1) $\|\cdot\|$ is a seminorm in \mathcal{L} ;
- (2) the "semi-Schwarz" inequality holds:

$$|\langle\langle x, y \rangle\rangle| \leq \|x\| \cdot \|y\|, \quad x, y \in \mathcal{L};$$

(3) $\mathcal{L}_0 := \{x \in \mathcal{L} : \|x\| = 0\} \subset_{\text{lin}} \mathcal{L};$

(The quotient space)

(4) $\mathcal{L} := \mathcal{L} / \mathcal{L}_0$ is a normed scalar product space with the norm $\|\cdot\|$ and the sc. prod. $\langle \cdot, \cdot \rangle$ defined (and "well-defined") by

$$\|[x]\| := \|x\|, \quad \langle [x], [y] \rangle := \langle\langle x, y \rangle\rangle, \quad x, y \in \mathcal{L};$$

$*$) In fact, non-negativity follows from sesquilinearity + non-negativity (by polarisation formula).

$**$) such a form (ses., non-neg., con.-symm.) is also called

- III. 2.5 - "semi scalar product".

(5) the norm $\|\cdot\|$ in L is the norm induced by $\langle \cdot, \cdot \rangle$ in L , i.e. $\|[x]\|^2 = \langle [x], [x] \rangle$ for any $[x] \in L$.

(Ex) We skip the \pm standard proof.

This result is just what we need, to construct the space $L^2(M)$, which we shall describe in the next subsection in more details. But by (1) we see in particular, that $\|\cdot\|_M$ introduced in (2.2) is a seminorm in $L^2(M)$. Following the notation of (3) we define:

$$L_0^2(M) := \{f \in L^2(M) : \|f\|_M = 0\}. \quad (2.6)$$

Below we characterize this subspace in terms of the trace measure and ^{the} trace density of M .

Fact III.12 ("On the zero layer")

Let $f: \Omega \rightarrow \mathbb{C}^d$ be \mathbb{R} -measurable. TFCAE:

(a) $f \in L_0^2(M)$, (a') $\llbracket f \rrbracket_M = 0$,

(b) $f(t) \in \text{Ker } D_M(t)$ for tr_M -a.e. $t \in \Omega$.

We shall need an abstract result to prove this fact

Lemma III.13

Let A be a non-negative ^{bounded} operator in a complex Hilbert space \mathcal{H} . Then for $x \in \mathcal{H}$ TFCAE:

- (i) $x \in \text{Ker } A$
- (ii) $x \in \text{Ker} \sqrt{A}$
- (iii) $\langle Ax, x \rangle = 0$.

EEEx

Proof - Hint: $\langle Ax, x \rangle = \|\sqrt{A}x\|^2$.

Proof of Fact III.12

(a) \Leftrightarrow (a') - is obvious by the definition of $d^2(M)$ and by (2.2). Also **(b) \Rightarrow (a')** is simple since denoting $\Omega_0 = \{t \in \Omega : f(t) \in \text{Ker } D_M(t)\}$ we get

$$\begin{aligned} \langle f \rangle_M &= \int_{\Omega} \langle D_M(t)f(t), f(t) \rangle_{\mathbb{C}^d} d\text{tr}_M(t) = \int_{\Omega \setminus \Omega_0} \langle D_M(t)f(t), f(t) \rangle_{\mathbb{C}^d} d\text{tr}_M(t) \\ &= 0 \quad \text{if } \text{tr}_M(\Omega \setminus \Omega_0) = 0 \text{ (i.e. (b)) holds.} \end{aligned}$$

To get **(a') \Rightarrow (b)** assume (a'), we have

$$0 = \langle f \rangle_M = \int_{\Omega} \langle D_M(t)f(t), f(t) \rangle_{\mathbb{C}^d} d\text{tr}_M(t)$$

but the function $\Omega \ni t \mapsto \langle D_M(t) f(t), f(t) \rangle_{\mathbb{C}^d}$ is non-negative for tr_M a.e. $t \in \Omega$, so when the integral is 0, then also $\langle D_M(t) f(t), f(t) \rangle_{\mathbb{C}^d} = 0$ for tr_M a.e. t . Now, using Lemma III.13 for $A = D_M(t)$ we get $f(t) \in \text{Ker } D_M(t)$ for tr_M a.e. t . i.e. (b) holds. □

Let us discuss now "the problem of being in $L^2_0(M)$ ". The natural question is whether $\text{in } L^2(M)$ the characterization for $L^2_0(M)$, analogic to the "scalar" case, i.e., in $L^2(\nu)$ for $L^2_0(\nu) = \{f \in L^2(\nu) : \int |f|^2 d\nu = 0\}$, is true? Recall, that when ν is a measure in Ω (on \mathbb{R}), then for $f \in L^2(\nu)$ we have $f \in L^2_0(\nu)$ iff $\nu(\{t \in \Omega : f(t) \neq 0\}) = 0$. (2.7)

So, we could expect, that "the analogic condition" for the matrix measure $L^2(M)$ -space would be

$$M(\{t \in \Omega : f(t) \neq 0\}) = 0 \quad (2.7')$$

(here $f \in L^2(M)$).

"Unfortunately", the answer for such a question is "NOT"!

- III.2.8 -

Remarks III.14

Let M be a dxol matrix measure on M and let $f \in L^2(M)$. Denote:

$$\text{supp}_{L^2(M)}(f) := \{t \in \Omega : f(t) \notin \text{Ker } D_M(t)\},$$

$$\text{supp}_M(f) := \{t \in \Omega : f(t) \neq 0 \text{ and } D_M(t) \neq 0\}.$$

Then:

- (a) $\text{supp}_{L^2(M)}(f) \subset \text{supp}_M(f)$,
- (b) $f \in L^2_0(M) \iff \text{tr}_M(\text{supp}_{L^2(M)}(f)) = 0$,
- (c) f satisfies (2.7') $\iff \text{tr}_M(\text{supp}_M(f)) = 0$,
- (d) (2.7') $\implies f \in L^2_0(M)$
 but " \Leftarrow " generally doesn't have to hold, if $d > 1$.

Proof

(a) is clear (since " \supset " holds for the complements), (b) is just Fact III.12 and (c) follows from Remark III.8. Hence " \implies " of d) is true. To see that " \Leftarrow " can be untrue, let us see the following example (11)

Example III.15

Let $\Omega = \mathbb{R}$, $M = \text{Borel}(\mathbb{R})$, μ - the Lebesgue measure "restricted" to $[-1; 1]$, $d=2$, $F: \mathbb{R} \rightarrow M_{2 \times 2}(\mathbb{C})$

$$F(t) := \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \text{for } t \leq 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & \text{for } t > 0 \end{cases}, t \in \mathbb{R} \quad \text{and let } f: \mathbb{R} \rightarrow \mathbb{C}^2$$

$$f(t) := \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{for } t \leq 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for } t > 0 \end{cases}, t \in \mathbb{R}. \quad \text{Consider } M = F d\mu$$

- see Example III.7. We have $\text{tr} F \equiv 1$ so

$$\text{tr}_M = 1 d\mu = \mu \quad \text{and} \quad D_M = F. \quad \text{Moreover}$$

$$\langle\langle f \rangle\rangle_M = \frac{1}{2} \int_{[-1; 1]} \langle D_M(t) f(t), f(t) \rangle_{\mathbb{C}^2} dt = 0, \quad \text{so } f \in L^2(M),$$

and $f \in L^2_0(M)$ since $\text{supp}_M(f) = \emptyset$. And we have

$$\text{supp}_M(f) = \mathbb{R}, \quad \text{so} \quad M(\text{supp}_M(f)) = \mu(\text{supp}_M(f)) = \mu(\mathbb{R}) = 2.$$

$$\begin{aligned} \text{And } M(\{t \in \mathbb{R} : f(t) \neq 0\}) &= M(\mathbb{R}) = \int_{[-1; 0]} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} dt + \int_{(0; 1]} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} dt = \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = I. \end{aligned}$$

So, here $\|f\|_M = 0$, but $M(\{t \in \mathbb{R} : f(t) \neq 0\}) = I$.

I.e., some "trivial" elements of $L^2(M)$ can be "nontrivial" functions from the point of view

of the M matrix measure M (or the measure tr_M).

Note also, that our matrix measure is not only "some"

matrix measure. - It is a so-called probabilistic matrix measure, i.e. it satisfies an extra condition

$$M(\Omega) = I. \quad (2.8)$$

This is one of the reasons, which make the construction of the vectors of $L^2(M)$ from functions of $L^2(M)$ much more delicate/sophisticated, than the well-known construction of the vectors of $L^2(V)$ from functions of $L^2(V)$.

Mathematicians usually do not formally distinguish between $L^2(V)$ and $L^2(V)$, i.e. they say/write a "function f from $L^2(V)$ " instead of a "class" of functions $[f]$, and everybody is used to it, and understands what is "going on". But, because of the above delicacy (subtlety...) of the elements of $L_0(M)$, we shall not use the similar informality for $L^2(M)$ space!

- III. 2.11 -

III.3. The $L^2(M)$ space. Completeness.

Let us fix a dx-d matrix measure M on M - a σ -algebra for \mathcal{A}

In subsection III.2. the linear space $L^2(M)$ with the "semi-scalar product" $\ll \cdot, \cdot \gg_M$ and the related seminorm $\|\cdot\|_M$ were defined. We also defined $L_0^2(M) := \{f \in L^2(M) : \|f\|_M = 0\}$. Hence, according to Proposition III.11, defining the new space

$$L^2(M) := L^2(M) / L_0^2(M)$$

and $\langle \cdot, \cdot \rangle_M : L^2(M) \times L^2(M) \rightarrow \mathbb{C}$ by (3.1)

$$\langle [f], [g] \rangle_M := \ll f, g \gg_M \quad \text{for } [f], [g] \in L^2(M)$$

We obtain a scalar product space with the norm

$$\|[f]\|_M := \|f\|_M = \ll f, f \gg_M^{1/2}, \quad [f] \in L^2(M). \quad (3.2)$$

Recall, that by the definition of the quotient linear space, here, for any $f \in L^2(M)$

$$[f] := \{f+h : h \in L_0^2(M)\} \quad (*) \quad (3.3)$$

and the (linear) operations on the classes - the vectors - of $L^2(M)$ -

are as follows: $[f] + [g] := [f+g]$ and $z \cdot [f] := [zf]$ for $[f], [g] \in L^2(M)$.

* In particular, the on the choice of M , of $L^2(\mu)$ spaces, but we omit not lead to any misunderstandings.

operation "[]" depends similarly as in the case M (or μ) in the notation when it does

We follow the construction of scalar product space from a semi-scalar one

Our main goal now is to prove that in fact $L^2(M)$ is a Hilbert space.

Our proof is based on a convenient transformation of elements of $L^2(M)$, using the following simple, but crucial Observation: if $f: \Omega \rightarrow \mathbb{C}^d$ is measurable, then (see (2.1))

$$\begin{aligned}
 \langle\langle f \rangle\rangle_M &= \int_{\Omega} \langle D_M(t) f(t), f(t) \rangle_{\mathbb{C}^d} d\text{tr}_M(t) = \\
 &\left(\begin{array}{l} \text{if } f \in L^2(M) \\ \langle\langle f \rangle\rangle_M \end{array} \right) \int_{\Omega} \langle \sqrt{D_M(t)} f(t), \sqrt{D_M(t)} f(t) \rangle_{\mathbb{C}^d} d\text{tr}_M(t) = \\
 &\left. \begin{array}{l} \int_{\Omega} \sum_{j=1}^d |(\sqrt{D_M(t)} f(t))_j|^2 d\text{tr}_M(t) = \\ \sum_{j=1}^d \langle\langle (Ff)_j \rangle\rangle_{\text{tr}_M} \end{array} \right\} (3.3)
 \end{aligned}$$

where Ff denotes now a "new" function, $Ff: \Omega \rightarrow \mathbb{C}^d$, given by

$$(Ff)(t) := \sqrt{D_M(t)} f(t), \quad t \in \Omega \quad (3.4)$$

and for any measure μ on Ω and a measurable $g: \Omega \rightarrow \mathbb{C}$ we denote (see also page III.2.1)

$$\langle\langle g \rangle\rangle_{\mu} := \int_{\Omega} |g|^2 d\mu \in [0; +\infty]. \quad (3.5)$$

Recall that $L^2(\mu) = \{g: \Omega \rightarrow \mathbb{C} : g \text{-measurable and } \langle\langle f \rangle\rangle_{\mu} \in \mathbb{R}\}$ and $\|g\|_{\mu} = \langle\langle g \rangle\rangle_{\mu}^{1/2}$ for $g \in L^2(\mu)$.

Note, that - as usual - we choose $D_M \in \mathbb{D}_M$ such, that $\forall t \in \Omega$ $D_M(t) \geq 0$, so $Ff: \Omega \rightarrow \mathbb{C}^d$ is well (and "everywhere") defined. But this "simple calculation" (3.3) is somewhat delicate. We should be careful, because the last step - (3.3*) - in (3.3) is not obvious, and needs special explanation!

The question is whether we "have right" to claim, that $\langle\langle (Ff)_j \rangle\rangle_{t \in \Omega}$ have sense for any $j=1, \dots, d$.

And this is not a "quantitative" problem, but rather a "qualitative" one - the problem of measurability of all the functions $(Ff)_j$, $j=1, \dots, d$.

(i.e. - the measurability with respect to \mathcal{M} -in Ω and the Borel σ -algebra-in \mathbb{C}) or - equivalently - the measurability of Ff from Ω into \mathbb{C}^d (i.e. - with respect to \mathcal{M} -in Ω and the Borel σ -algebra-in \mathbb{C}^d). Note, that by the measurability of f ,

we can be sure that first 3 lines of (3.3) are ok, because the function given by $\Omega \ni t \mapsto \sum_{j=1}^d |(Ff(t))_j|^2$ is measurable,

which does not "automatically" give the measurability of each term $(Ff)_j$. But knowing that they all are measurable, we also get the crucial step (3.3*) ... (- being this more "quantitative" problem)

- III. 3.3. -

So, ^{to get the measurability of f} we shall use the following general result for a bounded self-adjoint operators in a Hilbert space \mathcal{H} (here we shall use it just for s.a. matrices from $M_{d \times d}(\mathbb{C})$, i.e. $\mathcal{H} = \mathbb{C}^d$).

Lemma III.16

Let $\phi \neq \emptyset \subset \mathbb{R}$ be an interval (= a convex subset) and $f: \mathcal{J} \rightarrow \mathbb{C}$ - a continuous function. Consider the transformation $\Phi_f: B_{\mathcal{J}}(\mathcal{H}) \rightarrow B(\mathcal{H})$, where $B_{\mathcal{J}}(\mathcal{H}) := \{A \in B(\mathcal{H}) : A \text{ is s.a. and } \sigma(A) \subset \mathcal{J}\}$, given by

$$\Phi_f(A) = f(A). \quad *)$$

Then Φ_f is continuous in operator-norm topology (in $B(\mathcal{H})$) and in the subspace topology sense in $B_{\mathcal{J}}(\mathcal{H}) \subset B(\mathcal{H})$. (see Appendix for the proof).

Now, we shall use the lemma to $\mathcal{J} := [0; +\infty)$ and $f: \mathcal{J} \rightarrow \mathbb{R}$, $f = \sqrt{\cdot}$. We know (see the construction of D_M in subsect. III.1) that $D_M: \Omega \rightarrow M_{d \times d}(\mathbb{C})$ is measurable (in \mathcal{M} -Borel($M_{d \times d}(\mathbb{C})$) sense) and $\forall t \in \Omega, D_M(t) \in B_{[0; +\infty)}(\mathbb{C}^d)$. By Lemma III.16

*) Note, that $\sigma(A)$ is compact, $\sigma(A) \subset \mathcal{J}$, so $f|_{\sigma(A)}$ is a bounded function, hence $f(A) \in B(\mathcal{H})$.

(hence also Borel)

$\sqrt{\cdot} : B_{[0,+\infty)}(\mathbb{C}^d) \rightarrow M_{d \times d}(\mathbb{C}) = B(\mathbb{C}^d)$ is continuous,

so the composition $\sqrt{D_M} : \Omega \rightarrow M_{d \times d}(\mathbb{C})$ is also measurable. Therefore, for fixed measurable f

also $Ff : \Omega \rightarrow \mathbb{C}^d$ given by (3.4) is measurable. Obviously, the measurability of Ff is equivalent to the measurability of each of $(Ff)_j, j=1, \dots, d$. So, we have proven the following result.

Fact III.17

Suppose that $D_M \in \mathcal{D}_M$ is chosen in such a way, that

$$\forall t \in \Omega \quad D_M(t) \geq 0. \tag{3.6}$$

Then:

- 1) the matrix valued function $\sqrt{D_M} : \Omega \rightarrow M_{d \times d}(\mathbb{C})$ is measurable
- 2) If $f : \Omega \rightarrow \mathbb{C}^d$ is measurable, then $Ff : \Omega \rightarrow \mathbb{C}^d$ is also measurable.
- 3) If $f : \Omega \rightarrow \mathbb{C}^d$ is measurable, then:
 - (a) $f \in L^2(M) \iff (Ff) \in (L^2(\text{tr}_M))^d$
 - (b) $f \in L^2_0(M)^* \iff (Ff) \in (L^2_0(\text{tr}_M))^d$
 - (c) $\langle\langle f \rangle\rangle_M = \sum_{j=1}^d \langle\langle f_j \rangle\rangle_{\text{tr}_M}$

* Recall that $L^2_0(\mu) := \{g \in L^2(\mu) : \langle\langle g \rangle\rangle_\mu = 0\}$, see (3.5).

- III.3.5 -

$$\|g\|_\mu^2$$

Note, that by (c), if $f \in L^2(M)$, i.e., $[f] \in L^2(M)$, then

$$\| [f] \|_M = \| f \|_M = \| [f] \|_{\text{tr}, d} = \| [f] \|_{\text{tr}, d}, \quad (3.7)$$

where we use the following notation, for any measure μ on M and for the functions $g \in (\mathbb{C}^d)^{\mathbb{R}}$ ^(and conventional) $*$ and their classes.

(i) We identify, as usual any $g: \mathbb{R} \rightarrow \mathbb{C}^d$ with the system $(g_1, \dots, g_d) \in (\mathbb{C}^{\mathbb{R}})^d$ to omit the use of any special notation for some "trivial" mappings

(ii) We denote by $[g]$ the class of functions determined by g in $(L^2(\mu))^d$ for $g \in (L^2(\mu))^d$ ^{the (one)} ~~bold~~ $[\cdot]$ instead of (usual) for $L^2(M)$

similarly as in (i) we identify the Hilbert space product $(L^2(\mu))^d$ of "d-times" $L^2(\mu)$ with the quotient space $(L^2(\mu))^d / (L_0^2(\mu))^d$ which is a "quotient scalar product space" obtained by the construction from Proposition III.11 (see p. Cii) below for the seminorm etc. notation). In particular, $[g]$ is identified with $([g_1]_{\mu}, \dots, [g_d]_{\mu})$.

Ex One can easily check that this identification is in fact an "obvious" unitary transformation of the Hilbert spaces...

* Recall the set theory notation: X^Y is the set of all the functions $f: Y \rightarrow X$, for sets X, Y .

(iii) Let us denote $\langle\langle g \rangle\rangle_{\mu, d} = \sum_{j=1}^d \langle\langle g_j \rangle\rangle_{\mu} \in [0; +\infty]$ for each measurable $g: \Omega \rightarrow \mathbb{C}^d$; for such g we have $g \in (L^2(\mu))^d$ iff $\langle\langle g \rangle\rangle_{\mu, d} \neq +\infty$.

For $g \in (L^2(\mu))^d$ and $[g] \in (L^2(\mu))^d$ we denote

$$\|g\|_{\mu, d} = \|[g]\|_{\mu, d} := \langle\langle g \rangle\rangle_{\mu, d}^{1/2} \quad (3.8)$$

which finally explains the notation used in (3.7), with $\mu = \text{tr}_M$. Let us keep here the notation $\mu := \text{tr}_M$ and $D := D_M$ for short.

Finally, after all those "formal-notation" explanations, thank to the linearity of $f \mapsto Ff$, we can define our key ^(linear) mapping:

$$\hat{F}: L^2(M) \rightarrow (L^2(\mu))^d$$

by the formula

$$\hat{F}[f] = [Ff], \quad f \in L^2(M) \quad ([f] \in (L^2(\mu))^d). \quad (3.9)$$

We can be sure, by Fact III.17, that it is a properly defined ^(linear) transformation, and, moreover, it is an isometry by (3.7)! So, since $(L^2(\text{tr}_M))^d$ is a Hilbert space (as a product of d Hilbert spaces), we could try to check whether \hat{F} is onto $(L^2(\text{tr}_M))^d$. - And, if it were true, the proof of the completeness of $L^2(M)$ would be complete... But, "unfortunately", it is NOT true! - the range $\text{Ran } \hat{F}$ is somewhat smaller... To understand this better ^{usually} it suffices to observe, that each Ff , for $f \in L^2(M)$

satisfies the extra condition:

(3.10)

$$Ff \in \mathcal{L}_D^2 := \left\{ g \in (L^2(\mu))^d : g(t) \in \text{Ran } D(t) \text{ for } \mu \text{ a.e. } t \in \Omega \right\} \quad (*)$$

(To check this, it suffices to recall (3.4) and the fact that for any matrix $A \geq 0$ we have $\text{Ker } \sqrt{A} = \text{Ker } A$ (see e.g. Lemma III.13 p. III.2.7), which, by s.a. of \sqrt{A} and A , gives

$$\text{Ran } \sqrt{A} = (\text{Ker } \sqrt{A})^\perp = (\text{Ker } A)^\perp = \text{Ran } A. \quad (**)$$

Hence, knowing that it "can happen for many t ", that $\text{Ran } D(t) \neq \mathbb{C}^d$, we define a "smaller" space:

$$\mathcal{L}_D^2 := \left\{ [g] \in (L^2(\mu))^d : g \in \mathcal{L}_D^2 \right\} \quad (3.11)$$

Fact III.18

\mathcal{L}_D^2 is a closed linear subspace of $L^2(M)$.

Proof.

Suppose that $g_n \in \mathcal{L}_D^2$ for each $n \in \mathbb{N}$, and $(g \in (L^2(\mu))^d)$, and that $[g_n] \rightarrow [g]$ in $(L^2(\mu))^d$. This means that for each $j = 1, \dots, d$

$$\| (g_n)_j - (g)_j \|_\mu \rightarrow 0 \quad (\text{see (3.8)}). \text{ Thus we can use now}$$

*) There is one "fragility" related to this " μ -a.e. $t \in \Omega$ " - see the end of the proof for explanations...

**) However this part of argumentation is not very important - with an equal effect we could put $\text{Ran } \sqrt{D(t)}$ as well, instead of $\text{Ran } D(t)$ in (3.10).

the well-known result on subsequences of L^p -convergent sequences, which are a.e. convergent. We can choose a joint subsequence "on each j ", i.e. let $\{k_n\}_{n \in \mathbb{N}}$ be a strongly increasing sequence from \mathbb{N} such that

$$g_{k_n}(t) \rightarrow g(t) \quad \text{for } \mu\text{-a.e. } t \in \Omega.$$

But $\text{Ran } D(t)$ is always a closed set (a linear subspace of \mathbb{C}^d), so $\left(\bigvee_{n \in \mathbb{N}} g_{k_n} \in L^2_D \right) \Rightarrow g \in L^2_D$, hence $[g] \in L^2_D$. The linearity is obvious by the linearity of $\text{Ran } D(t)$ for each t . □

Hence L^2_D is also a Hilbert space, and by (3.9), (3.10) and (3.11) we have

$$\hat{F} : L^2(M) \rightarrow L^2_D.$$

Now, to finish our proof of the completeness of $L^2(M)$ it suffices to prove the result below.

Fact III.19

\hat{F} is a unitary transformation between $L^2(M)$ and L^2_D .

Proof

Having the isometricity of \hat{F} just proved, we need only to prove that $\text{Ran } \hat{F} \supset L^2_D$.

Let $[g] \in L^2_D$, i.e. $g \in L^2_D$ and define

$f: \Omega \rightarrow \mathbb{C}^d$ by the formula

$$f(t) := G(D(t))g(t), \quad t \in \Omega, \quad (3.12)$$

where (as usual) $G(A)$ denotes the matrix (operator) being the function G of the (operator) $A \geq 0$, $A \in M_d(\mathbb{C})$, where $G: [0; +\infty) \rightarrow \mathbb{R}$ is given by

$$G(x) = \begin{cases} 0 & x=0 \\ 1/\sqrt{x} & x>0 \end{cases}. \quad (3.13)$$

Note, that, as usual we choose $D(t) \geq 0$ for any $t \in \Omega$ (to make sense for (3.12) for any t). To finish our proof we have to check:

$$f \in L^2(M), \quad (3.14)$$

$$[Ff] = [g], \quad (3.15)$$

because with those properties we obtain by (3.9)

$$[f] \in L^2(M) \text{ and } [g] = [Ff] = \hat{F}[f] \in \text{Ran } \hat{F}.$$

To get (3.14) let us first check, that f is measurable.

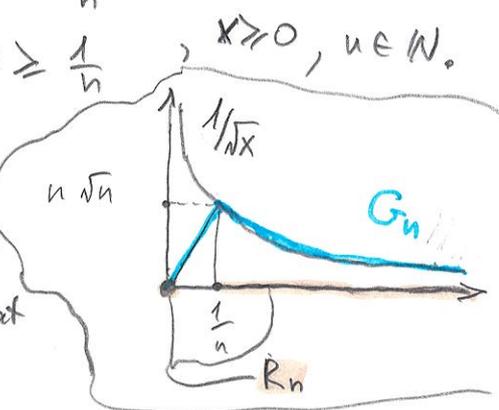
If G were continuous, then we would get it by the same argumentation, which proved that Ff was measurable for measurable f (i.e. by Lemma III.16 - see pp. III.3.4/5).

Our G is not continuous, but lets "approximate" it, in a way, by the following continuous functions $G_n : [0; +\infty) \rightarrow \mathbb{C}$:

$$G_n(x) := \begin{cases} n\sqrt{n}x & \text{for } x < \frac{1}{n} \\ \frac{1}{\sqrt{x}} & \text{for } x \geq \frac{1}{n} \end{cases}, \quad x \geq 0, n \in \mathbb{N}.$$

So, G_n and G are equal after the restriction to $[\frac{1}{n}, +\infty) \cup \{0\} =: R_n$.

Thus, for any matrix $A \geq 0$ in $M_d(\mathbb{C})$ we can fix some $N(A) \in \mathbb{N}$ such that



$$\forall_{n \geq N(A)} G(A) \subset R_n,$$

because $G(A)$ is a finite set contained in $[0; +\infty)$. Hence

$$\forall_{n \geq N(A)} G_n(A) = G(A). \quad (3.16)$$

Now defining $f_n : \Omega \rightarrow \mathbb{C}^d$ for any $n \in \mathbb{N}$ by

$$f_n(t) := G_n(D(t))g(t), \quad t \in \Omega \quad (3.17)$$

we see that f_n is measurable, by the continuity of G_n , and by (3.16), (3.17) and (3.12)

$\{f_n\}_{n \in \mathbb{N}}$ is pointwise convergent to f , because

$$\forall_{t \in \Omega} \forall_{n \geq N(D(t))} f_n(t) = f(t).$$

Now, having the measurability, we have by (3.12) also

$$\langle\langle f \rangle\rangle_{\mathcal{H}} = \int_{\Omega} \|\sqrt{D(t)} f(t)\|_{\mathbb{C}^d}^2 d\mu(t) = \int_{\Omega} \|P(t)g(t)\|_{\mathbb{C}^d}^2 d\mu(t), \text{ with}$$

$$P(t) = \sqrt{D(t)} \cdot G(D(t)). \quad (3.18)$$

By Sth+FCth for matrix $D(t)$, using the fact that for any

$x \in [0; +\infty)$ we have $\sqrt{x} G(x) = \begin{cases} 0 & x=0 \\ 1 & x>0 \end{cases}$

we see that $P(t) = E_{D(t)}^{((0; +\infty])} = I - E_{D(t)}^{(\{0\})} = I - P_0(t)$

where $P_0(t) = E_{D(t)}^{(\{0\})}$ is simply the orthogonal projection in \mathbb{C}^d onto $\text{Ker } D(t)$. Thus $P(t)$ is the orthogonal projection onto $\text{Ran } D(t) = (\text{Ker } D(t))^\perp$. In particular for

any $t \in \Omega$ $\|P(t)g(t)\|_{\mathbb{C}^d} \leq \|g(t)\|_{\mathbb{C}^d}$, so finally

$$\langle\langle f \rangle\rangle_M \leq \langle\langle g \rangle\rangle_{\mu, d} < +\infty,$$

and (3.14) holds. But by (3.4), (3.12) and (3.18)

$$\forall t \in \Omega \quad (Ff)(t) = \sqrt{D(t)} G(D(t)) g(t) = P(t) g(t)$$

and we assumed, that $g \in L^2_D$ which means ^{that} $g(t) \in \text{Ran } D(t) = \text{Ran } P(t)$ for μ -a.e. $t \in \Omega$. Now, taking

$$\omega := \{t \in \Omega : g(t) \in \text{Ran } D(t)\},$$

and changing the values of g in each $t \in \Omega \setminus \omega$ into 0 (which belongs to $\text{Ran } D(t)$) we get

$$Ff = g$$

without changing the the class $[g]$, so (3.15) also holds.

But ... - be careful! - There is still one "fragility" (see also *) p. III. 3.8.) ! We knew only, that $g(t) \in \text{Ran } D(t)$ for μ a.e. $t \in \Omega$ before this change on $\Omega \setminus \omega$ and this means ONLY, that $\Omega \setminus \omega$ is contained in a μ -zero measure set. But μ can be not a complete measure, so it does not automatically mean, that $\Omega \setminus \omega \in \mathcal{M}$ (equivalently $\omega \in \mathcal{M}$). And we need it, to know, that after our change we get really a measurable function g ...

- III. 3.12 -

So to end the proof in all the details, we still need the following abstract result

Lemma III.20

If $A: \Omega \rightarrow M_d(\mathbb{C})$ and $f: \Omega \rightarrow \mathbb{C}^d$ are measurable, then the set $\{t \in \Omega: f(t) \in \text{Ran } A(t)\}$ is measurable.

(Ex)

Proof (some key hints) Let $v \in \mathbb{C}^d$ and $B \in M_d(\mathbb{C})$

- 1) $v \in \text{Ran } B$ iff $v \in \text{lin}\{B_1, \dots, B_d\}$ iff $\text{dist}(v, \text{lin}\{B_1, \dots, B_d\}) = 0$, where B_i - the i -th column of B
- 2) there exists a subset of $\{B_1, \dots, B_d\}$ forming a base system for $\text{lin}\{B_1, \dots, B_d\}$ (including the case of the \emptyset subset for $\{0\}$ space...)
- 3) there exist an "analytic" formula for $\text{dist}(v, \text{lin}\{W_1, \dots, W_k\})$, when (W_1, \dots, W_k) is linearly independent (the formula uses Gram determinants for some systems of vectors formed from v, W_1, \dots, W_k).

III.20 (Lemma III.20)

III.19 (Fact III.19)

III. 4. The subspace $S(M)$ of simple "functions" and $L^2_\Sigma(M)$

We consider here two important subspaces of $L^2(M)$ for a matrix measure M on \mathcal{M} - a σ -algebra of subsets of a set Ω (as before - we fix here Ω , \mathcal{M} and M). The first - $S(M)$ is the subspace consisting of \mathbb{C} -simple "functions" (i.e., of their classes in $L^2(M)$ sense), and the second, larger one, is denoted by $L^2_\Sigma(M)$ (and does not have any special name...). It is worth to note, that $L^2_\Sigma(M)$ plays sometimes the similar role, to the role of the Schwarz functions class (in $L^2(M)$) in the standard $L^2(\mathbb{R})$ (\mathbb{R} with the Lebesgue measure) space. We shall prove here, that both subspaces are dense in $L^2(M)$.

Let us consider first so-called "vector characteristic functions", being just products $\chi_\omega c$ of scalar characteristic functions χ_ω for $\omega \in \Omega$ by vectors $c \in \mathbb{C}^d$, i.e.

$$\chi_\omega c: \Omega \rightarrow \mathbb{C}^d,$$

$$(\chi_\omega c)(t) = \begin{cases} c & \text{for } t \in \omega \\ 0 & \text{for } t \notin \omega \end{cases}, \quad t \in \Omega,$$

so $\chi_\omega c$ is measurable, when $\omega \in \mathcal{M}$. Denote:

$$\bullet \text{ VCF}(\mathcal{M}) := \{ \chi_\omega c : c \in \mathbb{C}^d, \omega \in \mathcal{M} \}$$

and

$$\bullet \mathcal{S}(\mathcal{M}) := \text{lin VCF}(\mathcal{M}).$$

For any $\omega \in \mathcal{M}$ and $c \in \mathbb{C}^d$, by Proposition III.5 (p. III.1.6),

$$\left. \begin{aligned} \langle \chi_\omega c \rangle_{\mathcal{M}} &:= \int_{\Omega} \langle D_{\mathcal{H}}(\chi_\omega c)(t), (\chi_\omega c)(t) \rangle_{\mathbb{C}^d} d\text{tr}_{\mathcal{M}}(t) = \\ &= \int_{\omega} \langle D_{\mathcal{H}}(c), c \rangle_{\mathbb{C}^d} d\text{tr}_{\mathcal{M}}(t) \leq \|c\|_{\mathbb{C}^d}^2 \cdot \text{tr}_{\mathcal{M}}(\omega). \end{aligned} \right\} (4.1)$$

Hence $\text{VCF}(\mathcal{M})$ and $\mathcal{S}(\mathcal{M})$ are included in $L^2(\mathcal{M})$,
and $\mathcal{S}(\mathcal{M}) \subset_{\text{lin}} L^2(\mathcal{M})$.

Let us make here several simple (nomen omen...) observations.

Fact III.21

- 1) $f \in \mathcal{S}(\mathbb{W}) \Leftrightarrow \forall_{j=1, \dots, d} f_j$ is a scalar (\mathbb{W}) -simple function $\Leftrightarrow f$ is a \mathbb{C}^d -vector \mathbb{W} -measurable function with finite image $f(\Omega)$.
- 2) If $\omega \in \mathbb{W}$, $c, c' \in \mathbb{C}^d$, then

$$\langle M(\omega)c, c' \rangle_{\mathbb{C}^d} = \int_{\omega} \langle D_M(t)c, c' \rangle_{\mathbb{C}^d} d\text{tr}_M(t), \quad (4.2)$$

$$\langle \langle X_{\omega}c \rangle \rangle_M = \|X_{\omega}c\|_M^2 = \langle M(\omega)c, c \rangle_{\mathbb{C}^d}, \quad (4.3)$$

$$M(\omega)c = \int_{\omega} D_M(t)c d\text{tr}_M(t). \quad (4.4)$$

EEEx

Proof

* scalar simple function (\mathbb{W} -simple) - i.e. a linear (complex) combination of characteristic functions X_{ω} with $\omega \in \mathbb{W}$ (or - equivalently - an \mathbb{W} -measurable function $g: \Omega \rightarrow \mathbb{C}$ with $g(\Omega)$ -finite).

- III.4.3 -

Now we define the subspace $S(M) \subseteq_{\text{lin}} L^2(M)$ in the natural way

$$S(M) := \{[f] \in L^2(M) : f \in \mathcal{S}(W_2)\}^* \quad (4.5)$$

Let us denote by π_M the quotient mapping

$$\pi_M : L^2(M) \longrightarrow L^2(M),$$

given by

$$\pi_M(f) = [f], \quad f \in L^2(M).$$

$L^2(M)$	\supseteq_{lin}	$\mathcal{S}(W_2) = \text{lin } VCF(W_2)$
$\downarrow \pi_M$		$\downarrow \pi_M$
$L^2(M)$	\supseteq_{lin}	$S(M) = \text{lin } [VCF]$

By the linearity of π_M we easily obtain (see the diagram above)

$$S(M) = \pi_M(\mathcal{S}(W_2)) = \text{lin } [VCF], \quad (4.6)$$

where $[VCF] := \{[f] \in L^2(M) : f \in VCF(W_2)\} = \pi_M(VCF(W_2)).$

*) Note here a difference in notation: we used W_2

for $\mathcal{S}(W_2)$, because it was determined only by W_2 , and not

on the choice of M on W_2 , but M in $S(M)$ is important, because $[f]$ is

the class in the " $L^2(M)$ " sense, although we did not

show this

the $[]$ notation.

— III. 4.4 —

dependence on M for

Let us define now the second subspace — $L^2_\Sigma(M)$. First observe that if $f, g: \Omega \rightarrow \mathbb{C}^d$ then for each $t \in \Omega$ and $\omega \in \mathbb{R}^2$

$$\left\langle M(\omega)f(t), g(t) \right\rangle_{\mathbb{C}^d} = \sum_{i=1}^d (M(\omega)f(t))_i \overline{g_i(t)} = \sum_{i,j=1,\dots,d} f_j(t) \overline{g_i(t)} M_{ij}(\omega). \quad (4.7)$$

Now, inspired by the above formula, we would like to "integrate somehow with respect to M , instead of using a fixed t and ω ", to get a properly defined semi-scalar product defined on a certain "large" set of functions from Ω into \mathbb{C}^d . Namely we would like to $(L^2_\Sigma(M))$ define $\langle \cdot, \cdot \rangle_\Sigma: L^2_\Sigma(M) \times L^2_\Sigma(M) \rightarrow \mathbb{C}$ by (compare to (4.7))

$$\langle f, g \rangle_\Sigma := \sum_{i,j=1,\dots,d} \int_{\Omega} f_j \overline{g_i} dM_{ij}, \quad f, g \in L^2_\Sigma(M), \quad (4.8)$$

but we should properly define the space $L^2_\Sigma(M)$ of functions. In particular, each integral from the RHS

has to be a well-defined complex number.
 Recall*) that $M_{ij}(\omega) = (M(\omega))_{ij}$, and
 each M_{ij} is a complex measure on Ω , so we
 need $f_j \bar{g}_i \in L^1(M_{ij})$ **) for any $i, j = 1, \dots, d$.

In particular we need $f_i \bar{f}_i = |f_i|^2 \in L^1(M_{ii})$
 that is $f_i \in L^2(M_{ii})$, equivalently, for any $i = 1, \dots, d$.
 And "we can be glad" that the above condition
 is sufficient for us. — Namely, the following result holds

Proposition III. 22.

If $h_1 \in L^2(M_{ii})$, $h_2 \in L^2(M_{jj})$ for some $i, j = 1, \dots, d$,
 then $h_1 h_2 \in L^1(M_{ij})$ and

$$\left| \int_{\Omega} h_1 h_2 dM_{ij} \right| \leq \int_{\Omega} |h_1 h_2| d|M_{ij}| \leq \left(\int_{\Omega} |h_1|^2 dM_{ii} \right)^{1/2} \cdot \left(\int_{\Omega} |h_2|^2 dM_{jj} \right)^{1/2}. \quad (4.9)$$

Proof

This is a direct consequence of the following two

*) See Proposition III. 2 p. III. 1. 2.

**) Recall, that $h \in L^p(\mu) \Leftrightarrow \int |h|^p d|\mu| < +\infty$
 for complex measure μ , where $|\mu|$ is the variation
 measure for μ and $p \in [1; +\infty)$ (see also ...).

lemmas.

Lemma III.23

If M is a complex measure on \mathcal{W} , then for any $i, j = 1, \dots, d$

$$\forall \omega \in \mathcal{W} \quad |M_{ij}(\omega)| \leq \left(M_{ii}(\omega) M_{jj}(\omega) \right)^{1/2}$$

Proof

Fix $i, j = 1, \dots, d$

Let e_s be the s -th standard base vector in \mathbb{C}^d for any $s = 1 \dots d$. For any $\omega \in \mathcal{W}$, by the Schwarz inequality:

$$\begin{aligned} |M_{ij}(\omega)| & \stackrel{*)}{=} \left| \langle M(\omega) e_j, e_i \rangle_{\mathbb{C}^d} \right| = \left| \langle \sqrt{M(\omega)} e_j, \sqrt{M(\omega)} e_i \rangle_{\mathbb{C}^d} \right| \\ & \leq \left(\langle \sqrt{M(\omega)} e_j, \sqrt{M(\omega)} e_j \rangle_{\mathbb{C}^d} \langle \sqrt{M(\omega)} e_i, \sqrt{M(\omega)} e_i \rangle_{\mathbb{C}^d} \right)^{1/2} \\ & = \left(\langle M(\omega) e_j, e_j \rangle_{\mathbb{C}^d} \langle M(\omega) e_i, e_i \rangle_{\mathbb{C}^d} \right)^{1/2} = \left(M_{ii}(\omega) M_{jj}(\omega) \right)^{1/2}. \end{aligned}$$

Now fix $\omega \in \mathcal{W}$ and consider a certain disjoint \mathcal{W} -decomposition $\omega_1, \dots, \omega_n$ of ω . Using the above

*) Note, that "usually" $|M_{ij}(\omega)| \neq |M_{ji}(\omega)| \dots$

estimate to each of ω_s ($s=1, \dots, n$) we get (from the Schwarz inequality, again)

$$\begin{aligned} \sum_{s=1}^n |M_{ij}(\omega_s)| &\leq \sum_{s=1}^n (M_{ii}(\omega_s))^{1/2} (M_{jj}(\omega_s))^{1/2} \leq \\ &\leq \left(\sum_{s=1}^n M_{ii}(\omega_s) \right)^{1/2} \left(\sum_{s=1}^n M_{jj}(\omega_s) \right)^{1/2} = (M_{ii}(\omega) M_{jj}(\omega))^{1/2} \end{aligned}$$

So, taking the sup over all the decompositions, we get the assertion for ω . □

Lemma III.24 ("Generalized Schwarz inequality")

Suppose, that μ, μ_1, μ_2 are measures on \mathcal{M} , satisfying

$$\forall \omega \in \mathcal{M} \quad \mu(\omega) \leq (\mu_1(\omega) \mu_2(\omega))^{1/2}$$

If $f_i \in L^2(\mu_i), i=1,2$, then $f_1 \cdot f_2 \in L^1(\mu)$ and

$$\int_{\Omega} |f_1 f_2| d\mu \leq \left(\int_{\Omega} |f_1|^2 d\mu_1 \cdot \int_{\Omega} |f_2|^2 d\mu_2 \right)^{1/2}$$

The proof is placed in Appendix ...

Note, that taking $\mu = \mu_1 = \mu_2$ we get simply "usual Schwarz" result.

Now, having already Proposition III.24, we can give the necessary definition of $L^2_\Sigma(M)$:

$$L^2_\Sigma(M) := \left\{ f: \Omega \rightarrow \mathbb{C}^d : f \text{ is } \mathbb{R}^d\text{-measurable and } \forall_{j=1, \dots, d} f_j \in L^2(M_{jj}) \right\}.$$

With such $L^2_\Sigma(M)$ the formula (4.8) for $\langle\langle f, g \rangle\rangle_\Sigma$ has sense for $f, g \in L^2_\Sigma(M)$, by

Proposition III.24.

Moreover, we have

Fact III.25

1) $L^2_\Sigma(M) \subset_{\text{lin}} L^2(M)$

2) if $f, g \in L^2_\Sigma(M)$ then

3) $\mathcal{S}(M) \subset_{\text{lin}} L^2_\Sigma(M)$, $\langle\langle f, g \rangle\rangle_\Sigma = \langle\langle f, g \rangle\rangle_M$,

Proof

Suppose that $f, g \in L^2_\Sigma(M)$. We have

$$\langle\langle f, g \rangle\rangle_\Sigma = \sum_{ij=1, \dots, d} \int_\Omega f_j \overline{g_i} dM_{ij} = \sum_{ij=1, \dots, d} \int_\Omega \left(\mathbb{1}_M(t) \right)_{ij} f_j(t) \overline{g_i(t)} dt_{ij}(t)$$

- III.4.9 -

In particular (by Proposition III.22) for any i, j
 $(D_M(\cdot))_{ij} f_j \overline{g_i} \in L^1(\text{tr}_M)$,

so the sum over all i, j too, and

$$\begin{aligned} \langle\langle f, g \rangle\rangle_{\Sigma} &= \int_{\Omega} \sum_{i, j=1, \dots, d} (D_M(t))_{ij} f_j(t) \overline{g_i(t)} d\text{tr}_M(t) \\ &= \int \langle D_M(t) f(t), g(t) \rangle_{\mathbb{C}^d} d\text{tr}_M(t). \end{aligned} \quad (4.10)$$

In particular, for $f = g$: $\mathbb{C} \ni \langle\langle f, f \rangle\rangle_{\Sigma} = \langle\langle f \rangle\rangle_M$,

so $\langle\langle f \rangle\rangle_M < +\infty$ for $f \in L^2_{\Sigma}(M)$, i.e. "C" from 1) is proved, so (2.3) and (4.10) we get 2).

Obviously $L^2_{\Sigma}(M)$ is a linear space, because each $L^2(M_{ii})$ is. Hence "C" in 1) holds.

Now, to prove 3) it suffices to check that

for any $\omega \in \mathcal{M}$ and $c \in \mathbb{C}^d$ $X_{\omega} c \in L^2_{\Sigma}(M)$.

But for $j=1, \dots, d$ $(X_{\omega} c)_j = c_j X_{\omega} \in L^2(M_{jj})$ since M_{jj} is a finite measure (see Proposition III.2 d). □

Remark III.26

We see in particular, that the idea to define a semi-scalar product by (4.8) leads just to the product $\langle\langle \cdot, \cdot \rangle\rangle_M$, which we knew before and to the space $L^2_{\Sigma}(M)$ being "only" some part of our

"main function space" $L^2(M)$.

But there is one advantage of the construction $L^2_\Sigma(M)$ made above: the definition of the semi-scalar product for the elements of $L^2_\Sigma(M)$ can be done without the use of the trace density D_M for M .

Now, analogously as for $S(M)$ we define

$$\begin{aligned} L^2_\Sigma(M) &:= \pi_M(L^2_\Sigma(M)) = \\ &= \{[f] \in L^2(M) : f \in L^2_\Sigma(M)\}. \end{aligned}$$

So, by Fact III.25

$$S(M) \subset_{\text{lin}} L^2_\Sigma(M) \subset_{\text{lin}} L^2(M). \quad (4.11)$$

We end this part by the density result announced at the beginning of this subsection.

Fact III.27

Both $S(M)$ and $L^2_\Sigma(M)$ are dense subspaces of $L^2(M)$.

Proof

By (4.11) we need only to prove that $\overline{SCM} = L^2(M)$, i.e., that

$$SCM^\perp = \{0\} \quad (4.12)$$

Let $[f] \in SCM^\perp$; $[VCF] \in SCM$, so

$$\left. \begin{aligned} \forall c \in \mathbb{C}^d \quad \forall \omega \in M \quad 0 &= \langle [f], [VCF] \rangle_M = \langle \langle f, \chi_\omega c \rangle \rangle_M \\ &= \int_\omega \langle D_M(t) f(t), c \rangle_{\mathbb{C}^d} dt \end{aligned} \right\} (4.13)$$

For $c \in \mathbb{C}^d$ consider $f_c: \Omega \rightarrow \mathbb{C}$ given by

$$f_c(t) := \langle D_M(t) f(t), c \rangle_{\mathbb{C}^d}, \quad t \in \Omega.$$

By Lemma III.9 (p. III.2.3) we know that $f_c \in L^1(\text{tr}_M)$ and by (4.13)

$$\forall \omega \in M \quad \int_\omega f_c d\text{tr}_M = 0, \quad (4.14)$$

and by the well-known fact from measure theory we get

$f_c(t) = 0$ for tr_M -a.e. $t \in \Omega$. Hence, defining $Z_c := \{t \in \Omega : f_c(t) \neq 0\}$ we get $\text{tr}_M(Z_c) = 0$ for any $c \in \mathbb{C}^d$.

Choose any $\{c_n\}_{n \geq 1}$ in \mathbb{C}^d such, that $\{c_n : n \in \mathbb{N}\}$ is dense in \mathbb{C}^d and let

$$Z = \bigcup_{n \geq 1} Z_{c_n}. \quad \text{If } t \in \Omega \setminus Z = \bigcap_{n \geq 1} (\Omega \setminus Z_{c_n}),$$

then

$$\forall_{n \in \mathbb{N}} \quad 0 = \delta_{c_n}(t) = \langle D_M(t)f(t), c_n \rangle_{\mathbb{C}^d},$$

so, by the density in \mathbb{C}^d , $D_M(t)f(t) = 0$. But

$\text{tr}_M(Z) = 0$ (Z is a countable sum of tr_M -zero sets), hence we proved, that

$f(t) \in \text{Ker } D_M(t)$ for tr_M -a.e. $t \in \Omega$,

This means, that $[f] = 0$ by Fact III.12, i.e. (4.12) holds. □

It is worth noting, that in some papers the definition of the space $L^2(M)$ is not rigorous enough.

For instance, in some of them the authors define ^{"only"} $L^2_\Sigma(M)$ (or even only $h^2_\Sigma(M)$) instead of $L^2(M)$.

Observe, that $L^2_\Sigma(M)$, itself, is not a Hilbert space often.

It can't be, as a dense subspace of a Hilbert space $L^2(M)$, as long as it is not the whole $L^2(M)$.

- For some M it is true that $L^2_\Sigma(M) = L^2(M)$ (e.g. always for $d=1 \dots$), but not for any M !

- III.4.13 -