

III.5 Multiplication operators in $L^2(M)$

Multiplication operators in $L^2(\mu)$, for measures μ are basic operators for Spectral Theory. It is natural to try to find an analog of such kind of operators for matrix measure L^2 -spaces. So, assume, as usual, that M is a $d \times d$ matrix measure on \mathcal{S} - a σ -algebra of subsets of S , and let us think first about some transformations of $L^2(M)$, which could be possibly treated as "natural generalisations" of multiplications by scalar function F in $L^2(\mu)$. The first - "most general" guess is a "multiplication" by a $\text{matrix}^{\text{measurable}}$ function $A: S \rightarrow M_d(\mathbb{C})$ given by the formula

$$L^2(M) \xrightarrow{\text{lin}} D \ni f \mapsto Af \in L^2(M), \text{ where } (Af)(t) = A(t)f(t).$$

But here an important problem arises! - When we want to define the appropriate quotient factorization of such a transformation, we have to be shure, that for $f \in L_0(M)$ also $Af \in L_0(M)$. That is, assuming that $D_M(f)f(f) = 0$ for tr_M -a.e. $t \in S$, we would like to obtain $D_M(f)A(f)f(f) = 0$. For general case

of matrix measures M , i.e., quite a general class of different trace densities $M_\mu : \mathcal{S} \rightarrow M_d(\mathbb{C})$, it would be difficult to get this. — Something +/- like commutativity of $D_M(t)$ and $A(t)$ seems to be necessary. So — the form of $A(t)$ should be, in general, "very simple". Like $A(t) = F(t) \cdot I$, with a scalar function F ...

Let $F : \mathcal{S} \rightarrow \mathbb{C}$ be a \mathcal{M} -measurable function. We define the multiplication by F operator T_F in $L^2(M)$, as follows (note, that we extend the meaning of the symbol " T_F " onto all the L^2 matrix spaces) *

- $D(T_F) := \{[f] \in L^2(M) : Ff \in L^2(M)\} \quad (5.1)$
- $T_F[f] := [Ff] \quad \text{for } [f] \in D(T_F).$

We should check, that this is a proper definition of a linear operator $T_F : D(T_F) \hookrightarrow L^2(M)$. So observe first that the related operator J_F in $L^2(M)$ with the domain $\mathcal{D} := \{f \in L^2(M) : Ff \in L^2(M)\}$, given by $J_F f = Ff$ for $f \in \mathcal{D}$, is linear, and if

$f \in L_0^2(M)$, then $D_M(t) f(t) = 0$ for t -a.e. $t \in \mathcal{S}$,
be careful: D_M is not \mathcal{D} or D ...
so, the same is true for $F(t) \cdot D_M(t) f(t) = D_M(t)(F(t)f(t))$, which means that $Ff \in L_0^2(M)$ **)
(see Fact III.12). Hence $[Ff]$ does not depend on the choice of f for $[f]$, and T_F is linear.

*) for $f : \mathcal{S} \rightarrow \mathbb{C}^d$ $Ff : \mathcal{S} \rightarrow \mathbb{C}^d$ and $(Ff)(t) = F(t)f(t)$ for $t \in \mathcal{S}$.

**) Hence, in particular: $L_0^2(M) \subset \mathcal{D}$, and $J_F(L_0^2(M)) \subset L_0^2(M)$.

Let us recall the notion of the essential value set $VE_\mu(F)$ with respect to a measure μ on \mathbb{M} . We define first its complement:

$$VNE_\mu(F) := \bigcup_{\substack{\text{U open} \\ \text{"non 0"}}} \{U \subset \mathbb{C} : U - \text{open}, \mu(F^{-1}(U)) = 0\}$$

so $VE_\mu(F) := \mathbb{C} \setminus VNE_\mu(F)$.

We repeat the analogic definition for the matrix measure M , just by replacing μ by M above, and we get $VE_M(F)$, being a kind of ansatz of the set $F(\Omega)$ of all ^{the} values of F , containing just the "topologically important values of F from the point of view of the matrix measure $M"$ *) (note that $VE_M(F)$ is always a closed set in \mathbb{C} , as a complement of an open $VNE_M(F)$).

But note also, that we do not need to make some special studies of essential value set with respect to matrix measures — we can use just the well-known properties of this classical notion for measure!

Indeed: we know that $M(w) = 0$ for $w \in \mathbb{M}$ iff $\text{tr}_M(w) = 0$ (see Remark III.8 p. III.1.10). Hence

$$VE_M(F) = VE_{\text{tr}_M}(F). \quad (5.2)$$

(and analogically for $VNE_M \dots$).

* Note that because $- \underline{\text{III.5.3}} -$, in particular, $\overline{F(\Omega)} \supset VE_M(F)$, $\mathbb{C} \setminus F(\Omega) \subset VNE_M(F)$.

In particular, by the Lindelöf property of the (metric + separable) space $VNE_M(F)$
 $\text{tr}_M(F^*(VNE_M(F))) = 0$ and $M(F^*(VNE_M(F))) = 0$, (5.2')

We are ready now to formulate the theorem describing some basic operator and spectral properties of multiplication operators in $L^2(M)$ space.

Theorem III.28

Assume that $F, G : \Sigma \rightarrow \mathbb{C}$ are $M\mathbb{C}$ -measurable. Then:

1) T_F is densely defined and closed;

2) $\forall_{\lambda \in \mathbb{C} \setminus \{0\}} T_{\lambda F} = \lambda T_F$ (and $T_0 = 0^*$);

3) a) $T_F = 0$ iff $VE_M(F) \subset \{0\}$, b) $T_F = T_G$ iff $F(t) = G(t)$ for tr_M -a.e. $t \in \Sigma$ ***

4) $T_F \in B(L^2(M)) \Leftrightarrow VE_M(F)$ is bounded. Moreover for $T_F \in B(L^2(M))$

$$\|T_F\| = \underbrace{\sup\{|\lambda| : \lambda \in VE_M(F)\}}_{\text{notation: } \sup|VE_M(F)|}, \quad (5.3)$$

provided $L^2(M) \neq \{0\}$.

notation: $\sup|VE_M(F)|$

*) 0 denotes also the zero operator in $L^2(M)$ (with $D(0) = L^2(M)$).

**) F can happen only when $M(\Sigma) = 0$, which means that $VE_M(F) = \emptyset$.

***) Note the important difference between the condition for " $[f] = [g]$ " and for " $T_F = T_G$ "!

$$5) \quad T_F^* = T_F \quad \text{and} \quad D(T_F^*) = D(T_F).$$

$$6) \quad T_F T_G \subset T_{FG} \quad \text{and} \quad D(T_F T_G) = D(T_{FG}) \cap D(T_G). \quad \text{In particular } T_F T_G = T_{FG} \Leftrightarrow D(T_{FG}) \subset D(T_G).$$

If we assume additionally that $L^2(M) \neq \{0\}$, then: (5.4)

$$7) \quad S(T_F) = VE_M(F) = VE_{tr_M}(F)$$

$$8) \quad \sigma_p(T_F) = \{\lambda \in \mathbb{C} : \operatorname{tr}_M(F^{-1}(\{\lambda\})) \neq 0\};$$

9) If $\lambda_0 \in \sigma(T_F)$, then

$$(T_F - \lambda_0 I)^{-1} = T_H, \quad \text{where} \quad \} \quad (5.5)$$

$H: \mathcal{S} \rightarrow \mathbb{C}$ is any measurable function satisfying $H(t)(F(t) - \lambda_0) = 1$ for tr_M -a.e. $t \in \mathcal{S}$

10) If $F(\mathcal{S}) \subset \mathbb{R}$ then the operator function on the Borel subsets class $\operatorname{Borel}(\mathbb{R})$ on \mathbb{R} given by

$$E_{F,M}: \operatorname{Borel}(\mathbb{R}) \rightarrow \mathcal{B}(L^2(M)),$$

$$E_{F,M}(\omega) := T_{\chi_{F^{-1}(\omega)}} \quad \text{for } \omega \in \operatorname{Borel}(\mathbb{R})$$

is the (projection valued) spectral measure (i.e. the resolution of identity) for T_F .

Observe, that this long formulation is a matrix measure analog of the appropriate classical result for multiplication by function operators in $L^2(\mu)$ with a measure μ . Also the proof is similar.

Proof

Let us recall the notation $\gamma_{f,g}$ from section III.2. For any $f, g : \Omega \rightarrow \mathbb{C}^d$ (previously we assumed that $f, g \in L^2(M)$, but it is not necessary for the definition only)

$\gamma_{f,g} : \Omega \rightarrow \mathbb{C}$ and

$$\gamma_{f,g}(t) := \langle D_M(t)f(t), g(t) \rangle_{\mathbb{C}^d}, \quad t \in \Omega. \quad \} \quad (5.6)$$

and $\gamma_f := \gamma_{f,f}$.

Observe that:

$\gamma_{f,g}$	$= \overline{\gamma_{g,f}}$
$\gamma_{F,f}$	$= F ^2 \gamma_f$
$\gamma_{FF,f,g} = \gamma_{f,\overline{F}g} = F \gamma_{f,g}$	

} (5.7)

in particular, for any $\omega \in \Omega$

$$|\chi_\omega f| = \chi_\omega |f|. \quad (5.8)$$

Now, we shall say something about the proofs of each point of the theorem.

Ad. 1) Take $f \in L^2(M)$ and for any $n \in \mathbb{N}$ consider

$$\omega_n := \{t \in \Omega : |F(f)| \leq n\}.$$

Recall that for any measurable $g : \Omega \rightarrow \mathbb{C}^d$

$$\langle g \rangle_M = \int_{\Omega} g \, d\mu_M. \quad (5.9)$$

Now, using also (5.6 - 5.8), we can easily check that $f_n := \chi_{\omega_n} f$ for any $n \in \mathbb{N}$ belongs to $D(T_M)$, and by the Lebesgue dominated convergence theorem

(in particular) $\|[f] - [f_n]\|_M^2 \rightarrow 0$

so $D(T_M)$ is dense. And the closures we get later, as a consequence of 5) (the adjoint operator is always closed).

Ad 5)

Take any $f, g \in L^2(M)$ such that $[f] \in D(T_F)$, $[g] \in D(T_{\bar{F}})$ (obviously — by (5.9) and (5.7) — these two domains are equal). For any $h_1, h_2 \in L^2(M)$ we have

$$\langle [h_1], [h_2] \rangle_M = \int_M Y_{h_1, h_2} d\text{tr}_M , \quad (5.10)$$

so, by (5.7)

$$\begin{aligned} \langle T_F[g], [f] \rangle_M &= \int_M Y_{Fg, f} d\text{tr}_M = \int_M Y_{g, \bar{F}f} d\text{tr}_M = \\ &= \langle [g], T_{\bar{F}}[f] \rangle_M , \end{aligned}$$

and hence $T_{\bar{F}} \subset (T_F)^*$.

Assume now, that $[f] \in D((T_F)^*)$, so, let's fix $h \in L^2(M)$ satisfying: $[h] = (T_F)^*[f]$, i.e.:

$$\forall_{[g] \in D(T_F)} \quad \langle T_F([g]), [f] \rangle_M = \langle [g], [h] \rangle_M$$

In particular, we can use this for g_n of the form $g_n = \chi_{w_n} g$ for the previously defined $w_n - s$, $n \in \mathbb{N}$, and any $g \in L^2(\Omega)$.

$$\langle T_F[g_n], [f] \rangle_M = \langle [g_n], [h] \rangle_M , n \in \mathbb{N}, g \in L^2(\Omega) . \quad (5.11)$$

But using again (5.7) and (5.10)
we get

$$\text{and } \langle T_F [g_n], [f] \rangle_M = \langle [g], [x_{\omega_n}(\bar{F}f)] \rangle_M,$$

$$\langle [g_n], [h] \rangle_M = \langle [g], [x_{\omega_n}h] \rangle_M,$$

so, by (5.1) and the arbitrariness of $[g] \in L^2(M)$ we obtain:

$$\forall_{n \in \mathbb{N}} \quad [x_{\omega_n}(h - \bar{F}f)] = 0. \quad (5.12)$$

We shall prove that $(h - \bar{F}f) \in L^2(M)$. By:

Fact III.12, we have to check, that $\text{tr}_M(\tilde{\mathcal{R}}) = 0$, where

$$\tilde{\mathcal{R}} := \{t \in \mathbb{R} : (h - \tilde{F}f) \notin \text{Ker } D_M(t)\}.$$

But (5.12) means that $\text{tr}_M(\tilde{\mathcal{R}} \cap \omega_n) = 0$, for any $n \in \mathbb{N}$, so

$$0 = \text{tr}_M\left(\bigcup_{n \in \mathbb{N}} (\tilde{\mathcal{R}} \cap \omega_n)\right) = \text{tr}_M(\tilde{\mathcal{R}}). \quad \text{Finally } h - \bar{F}f, h \in L^2(M)$$

hence also $\bar{F}f \in L^2(M)$ and $[\bar{F}f] = [h] = (T_F)^*[f]$
i.e. also $(T_F)^* \subset T_{\bar{F}}$.

So, the proofs of 1) and 5) are ready. Part 2) is trivial, however it is good to remember "the jump" in the domain at $t=0$.

We shall use below the following notation for a function F , $\varepsilon > 0$, $\lambda \in \mathbb{C}$ and $c \in \mathbb{C}^d$:

$$\beta_{F, \lambda, \varepsilon, c} := X_{F^{-1}(B(\lambda, \varepsilon))} \cdot c \quad *) \quad (5.13)$$

(see p. III.4.2). By Fact III.21 we obtain:

Fact III.29

If $F: \Omega \rightarrow \mathbb{C}$ is M -measurable then:

(1) for any $\varepsilon > 0$, $\lambda \in \mathbb{C}$, $c \in \mathbb{C}^d$ $\beta_{F, \lambda, \varepsilon, c} \in L^2(M)$ and

$$\|[\beta_{F, \lambda, \varepsilon, c}]^*\|^2 = \langle M(F^{-1}(B(\lambda, \varepsilon)))c, c \rangle_{\mathbb{C}^d} \quad (5.14)$$

(2) if ε, λ, c are such, that $c \notin \text{Ker } M(F^{-1}(B(\lambda, \varepsilon)))$ then

$$\|[\beta_{F, \lambda, \varepsilon, c}]^*\| > 0 \text{ and } r \in \mathbb{R} \text{ can be chosen with}$$

$$\|[\beta_{F, \lambda, \varepsilon, \tilde{c}}]\| = 1 \text{ for } \tilde{c} = rc;$$

(3) $\omega \in M$, and for any $t \in \omega$ $s \leq |F(t)| \leq S$ for some $s, S \in [0; +\infty)$, then $[X_\omega \cdot c] \in D(T_F)$ for each $c \in \mathbb{C}^d$, and

$$s \| [X_\omega \cdot c] \|_M \leq \| T_F [X_\omega \cdot c] \|_M \leq S \| [X_\omega \cdot c] \|_M, \| [X_\omega \cdot c] \|_M = K_M(c, c) \quad **) \quad (5.15)$$

(4) for any $\varepsilon > 0$, $\lambda \in \mathbb{C}$ and $c \in \mathbb{C}^d$ $\beta_{F, \lambda, \varepsilon, c} \in D(T_F)$.

*) $B(\lambda, \varepsilon) := \{z \in \mathbb{C} : |\lambda - z| < \varepsilon\}$

**) see (III.4.3) ...

This fact will be a convenient tool in some next proofs.

Ad. 4

Consider first, $[f] \in D(T_F)$, $\|[f]\|_M \leq 1$. We have by (5.2')

$$\|T_F[f]\|_M^2 = \|[Ff]\|_M^2 = \int_{\mathbb{R}} |F(t)|^2 \langle D_M(t)f(t), f(t) \rangle_{\mathbb{C}^d} dt r_M(t) =$$

$$\int_{VNE_M(F)} \dots + \int_{VE_M(F)} \dots = \int_{VE_M(F)} \dots \leq (\text{sup}|VE_M(F)|)^2 \cdot \|f\|_M^2,$$

which proves " \leq " " \Leftarrow " and " \leq " of (5.3) from 4).

We shall prove now " \geq " of (5.3) also in the case when $VE_M(F)$ is unbounded (with the $\|T_F\|$ being $\cdot + \infty$ for the unbounded operator T_F case). So, let $s \in [0; +\infty)$

be such, that $s < \text{sup}|VE_M(F)|$. Then, using the def. of sup let us choose $\lambda_0 \in VE_M(F)$
such that $s < |\lambda_0|$ and

$$\text{let } \varepsilon := \frac{|\lambda_0| - s}{2}. \quad \text{Then for } w \in F^{-1}(B(\lambda_0, \varepsilon))$$

$$\forall_{t \in w} s \leq |F(t)| \leq |\lambda_0| + \varepsilon.$$

So, by Fact III.29, by (5.2') and by the definition of $VE_M(F)$ we see that $r_M(w) > 0$ and so we can choose $c \in \mathbb{C}^d$ such that for $f := \beta_{F, \lambda_0, \varepsilon, c}$ we have $\|[f]\|_M = 1$, $[f] \in D(T_F)$ and

$$\|T_F[f]\| \geq s.$$

Hence $\|T_F\| \geq s$, and by the arbitrariness of the choice of $s \in [0, \text{sup}|VE_M(F)|]$, we finally get (5.3) (in the expanded - "unbounded" case also), and (ii) is proved.

- III.5.11 -

Ad. 3.

In the trivial case $M(\mathcal{R}) = \emptyset$ also 3) is trivial. Suppose that $M(\mathcal{R}) \neq \emptyset$. Then $L^2(M) \neq \{0\}$, so by 4) we get a) (note, that $VE_M(F) \neq \emptyset$ then, because $VNE_E(F) \neq \emptyset$).

(Ex) Also " \Leftarrow " of b) obviously holds, so let us check " \Rightarrow ".

If $T_F = T_G$, then we have:

$$T_{F-G} \supset T_F - T_G = O_{D(F)}$$

where O_Y denotes the operator with the domain Y (for linear subspaces Y) which is constantly 0 on Y .

But $D(F)$ is dense, and T_{F-G} — closed $\} \text{by 1),}$ so

$T_{F-G} \supset \overline{O_{D(F)}} = O$, i.e. $T_{F-G} = O$. Hence we get our assertion by a). \star

* Note, that for each measurable function $H: \mathcal{R} \rightarrow \mathbb{C}$:
 $VE_M(H) \subset \{0\} \Rightarrow \mathbb{C} \setminus \{0\} \subset VNE_H(H) \Rightarrow$
 $\Rightarrow \text{tr}_M(\{t \in \mathcal{R}: H(t) \neq 0\}) = 0$.

[Ad. 6)]

Let $g \in L^2(M)$. Then, by the definition of the multiplication by function operator (; + the footnote **) p. III. 5. 2) and the definition of the product of unbounded operators, we obtain:

for $g \in L^2(M)$

$$(1) [g] \in D(T_F T_G) \Leftrightarrow \begin{cases} Gg \in L^2(M) & (5.16) \\ FGg \in L^2(M) & (5.17) \end{cases}$$

$$(2) [g] \in D(T_{FG}) \Leftrightarrow (5.17) \text{ holds}$$

$$(3) [g] \in D(T_G) \Leftrightarrow (5.16) \text{ holds}$$

Therefore $T_F T_G \subset T_{FG}$, and $D(T_F T_G) = D(T_{FG}) \cap D(T_G)$. Hence $T_F T_G = T_{FG} \Leftrightarrow D(T_{FG}) \cap D(T_G) = D(T_{FG}) \Leftrightarrow D(T_{FG}) \subset D(T_G)$.

Ad. 7)

We shall use the following

Lemma III.30

Let $G: \mathcal{R} \rightarrow \mathbb{C}$ be M -measurable, $\omega_0 := G^{-1}(\{0\})$.

(A) For any $f \in L^2(M)$

$$[f] \in \text{Ker } T_G \iff Gf \in L_0^2(M).$$

(B) If $c \in \mathbb{C}^d$, then $[X_{\omega_0} c] \in \text{Ker } T_G$.

(C) $\text{Ker } T_G = \{0\} \iff M(\omega_0) = 0$.

(D) Suppose, that $M(\omega_0) = 0$, and

let $H: \mathcal{R} \rightarrow \mathbb{C}$ be a measurable function, such that

$$H(t) = 1/G(t) \text{ for trn a.e. } t \in \mathcal{R} \setminus \omega_0. \text{ Then}$$

$$T_H T_G = I|D(T_G), \quad T_G T_H = I|D(T_H), \quad (5.18)$$

and $T_G: D(T_G) \rightarrow D(T_H)$ is a bijection onto $D(T_H)$ and

$T_H = (T_G)^{-1}$; moreover $(T_G)^{-1} \in B(L^2(M))$ iff $0 \in VNE_M(G)$.

Proof (of Lemma)

(E)

$$\beta(T_G) = VNE_M(G).$$

If $f \in L^2(M)$, then by the definition of T_G we have

$Gf \in L_0^2$, when $[f] \in \text{Ker } T_G$; but when $Gf \in L_0^2(M)$, then

in particular $Gf \in L^2(M)$, so $[f] \in D(T_G)$ and $T_G[f] = 0$.

This proves (A).

For $c \in \mathbb{C}^d$ we get $[X_{\omega_0} c] \in D(T_G)$ (by Fact III.29)

(P. (3)) (with $s = S = 0$) and $[X_{\omega_0} c] \in \text{Ker}(T_G)$. Hence (B) holds.

To prove (C) assume first that $\text{Ker } T_G = \{0\}$.
 Thus, by (B) and (5.15)

$$\forall_{c \in \mathbb{C}^d} 0 = \|[\chi_{\omega_0} c]\|_n^2 = \langle M(\omega_0)c, c \rangle,$$

therefore $M(\omega_0) = 0$.

Now, assume $M(\omega_0) = 0$ and let $[f] \in \text{Ker } T_G$.

Hence $Gf \in L^2(M)$. Recall now, that - by Remarks III.14 (b) - for any $g \in L^2(M)$ the condition $g \in L^2_0(M)$ is equivalent to $M(\text{supp}_{L^2(M)}(g)) = 0$, where

$$\text{supp}_{L^2(M)}(g) = \{t \in \mathbb{R} : D_M(t)g(t) \neq 0\}$$

(it is defined "up to" a M -measure zero set, due to the freedom of the choice of D_M , but let us fix some D_M here). So, we have:

$$\begin{aligned} 0 &\leq M(\text{supp}_{L^2(M)}(f)) = M(\omega_0 \cap \text{supp}_{L^2(M)}(f)) + M((\mathbb{R} \setminus \omega_0) \cap \text{supp}_{L^2(M)}(f)) \\ &= M(\{t \in \mathbb{R} : G(t) \neq 0 \text{ and } D_M(t)f(t) \neq 0\}) = \\ &= M(\{t \in \mathbb{R} : G(t) \neq 0 \text{ and } D_M(t)(G(t)f(t)) \neq 0\}) \leq M(\text{supp}_{L^2(M)}(Gf)) \end{aligned}$$

hence, also $M(\text{supp}_{L^2(M)}(f)) = 0$ and $[f] = 0$, \square
 i.e., $\text{Ker } T_G = \{0\}$.

Now, we consider H from (D). Since $M(\omega_0) = 0$, we have $H \cdot G = G \cdot H = 1$ tr_M -almost everywhere on S^2 (here 1 is the constant 1 function), hence, by parts 3)b) and 6) of Theorem III.2.8 (both parts - already proved!) we get $T_H T_G, T_G T_H \subset T_{\tilde{1}} = I$ and $D(T_{\tilde{1}}) = L^2(M)$, so (5.18) holds, T_G is "1-1" and $T_H = T_G$. Moreover, by parts 1), 3), 4) of the theorem:

$$T_G^{-1} \in B(L^2(M)) \Leftrightarrow T_{\tilde{H}} \in B(L^2(M)) \Leftrightarrow \exists_{r \in (1, \infty)} |VNE_M(\tilde{H})| \leq r, \text{ where } \tilde{H}(t) := \begin{cases} 1/G(t) & \text{for } t \in S^2 \\ 1 & \text{for } t \in \omega_0, \end{cases}$$

because $T_H = T_{\tilde{H}}$ (thanks to $M(\omega_0) = 0$).

$$\text{But } \exists_{r > 1} |VNE_M(\tilde{H})| \leq r \Leftrightarrow VNE_M(\tilde{H}) \supset \Pi_r := \{z \in \mathbb{C} : |z| > r\} \Leftrightarrow M(\tilde{H}^{-1}(\Pi_r)) = 0,$$

because $M(\tilde{H}^{-1}(VNE_M(\tilde{H}))) = 0$ by (5.2').

$$\text{But for } r > 1 \quad \tilde{H}^{-1}(\Pi_r) = G^{-1}(K(0, \frac{1}{r})). \text{ So, finally,}$$

$$T_G^{-1} \in B(L^2(M)) \Leftrightarrow 0 \in VNE_M(G) \quad \square$$

To obtain (E) it suffices to use (D) and the fact that if $M(\omega_0) \neq 0$, then $0 \notin g(T_G)$ since $\text{Ker } T_G \neq \{0\}$ by (C) and also $0 \notin VNE_M(G)$ because $VNE_M(G)$ is open subset of \mathbb{C} and if $0 \in VNE_M(G)$, then $0 = M(G^{-1}(VNE_M(G)))$ (see 5.2') and $M(G^{-1}(VNE_M(G))) \geq M(G^{-1}(\{0\}))$

$M(\omega_0) \geq 0$. □

- III. 5.16 -

Now, to obtain 7), it suffices to use (E) of the lemma to: $G = F - \lambda_0 I$ for each $\lambda_0 \in \mathbb{C}$ and the obvious observation, that $T_G = T_F - \lambda_0 I$ for such G .

- This gives $f(T_F) = VNE_M(F)$, so $\delta(T_F) = VNE_M(F)$. Similarly, we get 9) using (D) and (E) of the lemma for this G , and we get 8) using (C). It remains to prove 10).

Assume $F(\mathbb{R}) \subset \mathbb{R}$. To get 10) we should check that:

(i) $E := E_{F,M}$ is a spectral measure on $\text{Borel}(\mathbb{R})$

$$(ii) \int_{\mathbb{R}} x dE = T_F.$$

Property (i) means that

- E is additive
- $E(\mathbb{R}) = I$
- $E(\omega)^* = E(\omega)$ for any $\omega \in \text{Borel}(\mathbb{R})$
- $E(\omega \cap \omega') = E(\omega)E(\omega')$ for any $\omega, \omega' \in \text{Borel}(\mathbb{R})$

and

• $\forall [f] \in L^2(M) \quad E_{[f]} \text{ is a measure on } \text{Borel}(\mathbb{R})$ *)

To check it, observe that for $[f], [g] \in L^2(\mathbb{R})$

$$\forall_{\omega \in \text{Borel}(\mathbb{R})} \langle E_{[f]}, [g] \rangle_M(\omega) = \langle E(\omega)[f], [g] \rangle_M = \langle X_{F^{-1}(\omega)} f, g \rangle_M = \int_{F^{-1}(\omega)} f_{t,g} dt \nu_M(t) \quad (5.1c)$$

*) Recall that for a spectral measure E for a Hilbert space \mathcal{H} on σ -algebra M and for $x, y \in \mathcal{H}$ $E_{x,y}: M \rightarrow \mathbb{C}$ is given for $\omega \in M$ by and $E_x := E_{x,x}$.

and recall, that by Lemma III.9 we have

$$f_{f,g} \in L^1(\mu_M). \quad (5.20)$$

So, let us denote by $\mu_{f,g}$ the complex measure:

$$\mu_{f,g} := f_{f,g} d\mu_M. \quad (5.21)$$

By standard properties of complex measures we know that the variation (measure) of $\mu_{f,g}$ is given by

$$|\mu_{f,g}| = |f_{f,g}| d\mu_M. \quad (5.22)$$

In particular, for $\mu_f := \mu_{f,f}$ we have

$$\mu_f := f d\mu_M$$

and μ_f is a finite measure, $f \geq 0$. $(5.22')$

Recall that the complex measure $\mu_{f,g}$ can be "moved" from Ω onto $\text{Borel}(\mathbb{R})$ by the measurable $F: \Omega \rightarrow \mathbb{R}$ and the resulting complex measure $(\mu_{f,g})_F$ ($(\mu_f)_F$ for $g=f$, and then $(\mu_f)_F$ is a finite measure) is given by.

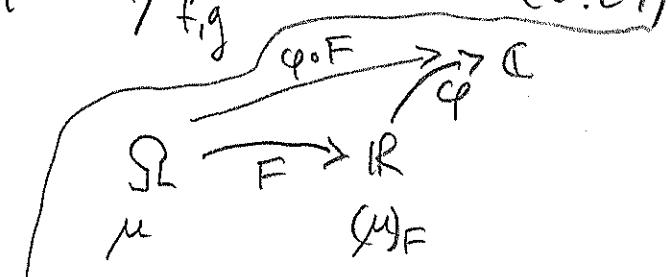
$$(\mu_{f,g})_F(\omega) = \mu_{f,g}(F^{-1}(\omega)), \quad \omega \in \text{Borel}(\mathbb{R}). \quad (5.23)$$

The "change of variable" rule for complex measures says, that for any Borel function $\varphi: \mathbb{R} \rightarrow \mathbb{C}$

$$\varphi \in L^1((\mu_{f,g})_F) \iff \varphi \circ F \in L^1(\mu_{f,g}) \quad (5.24)$$

and if $\varphi \in L^1((\mu_{f,g})_F)$, then

$$\int_{\Omega} \varphi d(\mu_{f,g})_F = \int_{\mathbb{R}} \varphi \circ F d\mu_{f,g} \quad (5.24')$$



Hence, finally

$$E_{[f], [g]} = (\mu_{f,g})_F$$

(5.25)

Now, we can come back to the last part of
(ii) and we see, that $E_{[f]} = (\mu_f)_F$, so
it is a finite measure, in particular.

To obtain (ii) let us notice first, that
both sides of (ii): T_F and $\int_X dE$ are self-adjoint
operators. So, it suffices to prove the inclusion

$$T_F \subset \int_X dE$$

to get (ii). Let $[f] \in D(T_F^R)$, so $Ff \in L^2(M)$ and

$$|F|^2 f_F = Ff \in L^1(M) \quad (5.26)$$

(see (5.7) and use (5.20) for Ff, Ff), which can be written as

$$|F|^2 \in \mathcal{L}^1(\mu_f) \quad (5.26')$$

Moreover we have by (5.10) and (5.7)

$$\begin{aligned} \forall [g] \in L^2(M) \quad & \langle T_F[f], [g] \rangle_M = \langle [Ff], [g] \rangle_M = \int_F f_{Ff, g} d\mu_M = \\ & = \int_F f \cdot \chi_{f, g} d\mu_M = \int_F f d\mu_{f, g}, \end{aligned} \quad \left. \right\} (5.27)$$

$$\text{and } F \in \mathcal{L}^1(\mu_{f,g})$$

by Lemma III.9. used for Ff, g .

We have to check, that $[f] \in D\left(\int_{\mathbb{R}} \chi dE\right)$ and

that

$$\forall [g] \in L^2(M) \quad \left\langle \left(\int_{\mathbb{R}} \chi dE \right)[f], [g] \right\rangle_M = \left\langle F[f], [g] \right\rangle_M. \quad (5.28)$$

But by (5.25), and by the definition of the "weak"

integration with respect to the spectral measure E ,

$$\text{hence: } [h] \in D\left(\int_{\mathbb{R}} \chi dE\right) \iff \chi \in L^2(E_{[h]}) = L^2((\mu_h)_F) \quad \text{iff } |\chi|^2 \in L^1((\mu_h)_F) \quad (5.29)$$

and for $[h] \in D\left(\int_{\mathbb{R}} \chi dE\right)$ and $[g] \in L^2(M)$ we have

$$\chi \in L^1((\mu_{h,g})_F), \quad \left\langle \left(\int_{\mathbb{R}} \chi dE \right)[h], [g] \right\rangle_M = \int_{\mathbb{R}} \chi d(\mu_{h,g})_F. \quad (5.30)$$

But using (5.24) to $\varphi = |\chi|^2$ and $g = f$, by (5.26)

we get $|\chi|^2 \circ F \in L^1(\mu_f)$, hence

$$|\chi|^2 \in L^1((\mu_f)_F).$$

Now, by (5.29) $[f] \in D\left(\int_{\mathbb{R}} \chi dE\right)$, and we can take $h = f$ in (5.30), i.e.

LHS of (5.28) equals $\int_{\mathbb{R}} \chi d(\mu_{f,g})_F$.

But, by (5.27), RHS of (5.28) equals $\int_{\mathbb{R}} F d\mu_{f,g}$, and

$F \in L^1(\mu_{f,g})$, so taking $\varphi = \chi$ in (5.25) and (5.25') we

see that $\int_{\mathbb{R}} F d\mu_{f,g} = \int_{\mathbb{R}} \chi d(\mu_{f,g})_F$, i.e. (5.28) holds.



IV

"X MUE" Theorem -

- The finitely cyclic case

-IV.0-

IV.1 →

IV.1. Finitely cyclic s.a. operators

Let A be a linear operator in a normed space X . Recall that A is cyclic iff there exists a cyclic vector $\varphi \in X$ for A , which means that:

$$\varphi \in D(A^\infty) \quad *) \quad \text{and} \quad \text{Orb}_A(\varphi) := \{A^n\varphi : n \in \mathbb{N}_0\} \quad \{ \quad (1.1)$$

is linearly dense in X .

We define here some notions, which generalize cyclicity. First observe that by the linearity of A we can consider the space $\text{lin}\{\varphi\}$ instead of an individual cyclic vector φ .

Definition IV.1

A subset $Y \subset X$ is a cyclic set for A iff
(or shortly - is cyclic for A)

- *) Recall that $D(A)$ denotes the domain of a linear operator A and $D(A^\infty) := \bigcup_{n \in \mathbb{N}} D(A^n)$.

$Y \subset D(A^\infty)$ and $\text{Orb}_A(Y) := \{A^n y : n \in \mathbb{N}_0, y \in Y\}$

(1.2)

is linearly dense* in X .

When $Y \subset X$, then we say also "a cyclic space" instead of "cyclic set".

Similarly, when $\vec{\varphi} = (\varphi_1, \varphi_2, \dots, \varphi_d)$ is not "only" a finite set $\{\varphi_1, \dots, \varphi_d\}$, but an (ordered) system of vectors (that is $\vec{\varphi} \in X^d$, and not $\vec{\varphi} \subset X$), then

we denote

$$\text{Orb}_A(\vec{\varphi}) := \text{Orb}_A(\{\varphi_1, \dots, \varphi_d\}) = \left\{ A^n \varphi_j : n \in \mathbb{N}_0, j = 1, \dots, d \right\}$$

and we say $\vec{\varphi}$ is a cyclic system (or is cyclic, for short) for A iff $\{\varphi_1, \dots, \varphi_d\}$ is a cyclic set for A .

We study here "finite cyclicity" defined as follows.

Definition IV.2

A is finitely cyclic iff there exists a finite dimensional cyclic space for A . Each such subspace of X is called a space of cyclicity for A .

(note, that this is more than to be a cyclic space, because of the extra finite dimension assumption).

* Note here, that assuming more - the density instead of linear density - we get the hypercyclicity notion.

Observe, that for any $\phi \neq \psi \in D(A^\infty)$

$$\text{lin } \text{Orb}_A(\psi) = \text{lin}(\text{Orb}_A(\text{lin}(\psi))), \quad (1.3)$$

because, by the linearity of A , $\text{lin}(\psi) \subset D(A^\infty)$ and

$$\text{Orb}_A(\text{lin}(\psi)) \subset \text{lin}(\text{Orb}_A(\psi)), \quad (1.3')$$

if only $\psi \in D(A^\infty)$.

Therefore we get:

Fact IV.3

Suppose that $\varphi_1, \dots, \varphi_d \in X$. TECAE:

- (i) A is finitely cyclic with $\text{lin}(\{\varphi_1, \dots, \varphi_d\})$ is a cyclic space for A
- (ii) $\{\varphi_1, \dots, \varphi_d\}$ is a cyclic set for A
- (iii) $\vec{\varphi} = (\varphi_1, \dots, \varphi_d)$ is a cyclic system for A .

Obviously, the cyclicity property of A is the finite cyclicity "with $d=1$ " and a cyclic set $\{\varphi_1, \dots, \varphi_d\}$ or a cyclic system $\vec{\varphi}$ is a "good" analog of a cyclic vector. But note, that d can be larger in general, than the dimension of the space $\text{lin}(\{\varphi_1, \dots, \varphi_d\})$.

IV.2 The spectral matrix measure

for a s.a. operator and a system of vectors

In "our" "xMUE" Theorem for ^{the} cyclic case (Section II) the crucial role played the spectral measure $\mu = E_{A,\varphi}$ for A and a fixed cyclic vector φ for A . As one can guess now, for finitely cyclic version of xMUE Th. we shall consider the multiplication by function $\vec{\varphi}$ operator $\vec{\varphi}$ in the space $L^2(M)$ for some matrix measure M on $\text{Borel}(\mathbb{R})$ ($\vec{\varphi}$ the ^{class of} Borel subsets of \mathbb{R}). This ^{matrix} measure will be called the spectral matrix measure for A and the (cyclic) system $\vec{\varphi}$, however we define it, and we denote it by $E_{A,\vec{\varphi}}$, also when we do not assume the cyclicity.

So, assume that: A is a self-adjoint operator in a Hilbert space \mathcal{H} , $\vec{\varphi} \in \mathcal{H}^d$, $\vec{\varphi} = (\varphi_1, \dots, \varphi_d)$, and that $E_A: \text{Borel}(\mathbb{R}) \rightarrow \mathcal{B}(\mathcal{H})$ is the projection-valued spectral measure for A (other name: the resolution of identity I for A) - see ...

Now consider

$$E_{A,\vec{\varphi}} : \text{Borel}(\mathbb{R}) \rightarrow M_d(\mathbb{C}), \quad \left. \right\} (2.1)$$

$$E_{A,\vec{\varphi}}(\omega) := \left(\langle E_A(\omega)\varphi_j, \varphi_i \rangle \right)_{i,j=1,\dots,d}, \quad \omega \in \text{Borel}(\mathbb{R})$$

(here $\langle \cdot, \cdot \rangle$ is for \mathcal{H}).

Lemma IV.4

If B is s.a. in a Hilbert space \mathcal{H} ,

$\varphi_1, \dots, \varphi_d \in D(B)$ and $C \in M_d(\mathbb{C})$ is given

by $C := (\langle B\varphi_j, \varphi_i \rangle)_{i,j=1,\dots,d}$

then C is s.a. If moreover $B \geq 0$, then also $C \geq 0$.

EE_x Proof. ...



Using this lemma, the fact that for any borel set ω $E_A(\omega)$ is an orthogonal projection (so, in particular, $E_A(\omega) \geq 0$) and the "weak complex measure" properties of E_A , we see that $E_{A,\vec{\varphi}}$ defined above is a matrix measure on $\text{Borel}(\mathbb{R})$.

Definition IV.5

$E_{A,\vec{\varphi}}$ (given by (2.1)) is called
the spectral matrix measure for A and $\vec{\varphi}$.

We shall prove now a convenient result, being our first connection joining spectral calculus for A and some vectors + classes of functions - from $L^2(E_{A,\vec{\varphi}})$.

Those "some vectors" would be just vectors from the $L^2_\Sigma(E_{A,\vec{\varphi}})$ space (see III.4).

And it is here, and also later, when you can see that the role \mathbb{M}_Σ of $L^2_\Sigma(M)$ in $L^2(M)$ (mentioned before) is analog now to the role of the Schwarz class functions $S(\mathbb{R})$ in $L^2(\mathbb{R})$ in some popular (e.g. Rudin...) proofs of the unitarity of the Fourier transform.

Theorem IV.6

Suppose that A is s.a. in \mathcal{H} and $\vec{\varphi} \in \mathcal{H}^d$ ($d \in \mathbb{N}$). Let M be ^{the} spectral measure for A and $\vec{\varphi}$ (i.e. $M = E_{A, \vec{\varphi}}$). Then:

(i) for any $f \in L^2_\Sigma(M)$ and $j = 1, \dots, d$
 $\varphi_j \in D(f_j(A))$, (2.2)

(ii) the linear transformation $\tilde{W} : L^2_\Sigma(M) \rightarrow \mathcal{H}$ given by

$$\tilde{W}f := \sum_{j=1}^d f_j(A) \varphi_j, \quad f \in L^2_\Sigma(M) \quad (2.3)$$

satisfies

$$\forall f \in L^2_\Sigma(M) \quad \|\tilde{W}f\| = \|f\|_M, \quad (2.4)$$

(iii) the transformation $W : L^2_\Sigma(M) \rightarrow \mathcal{H}$ given by

$$W[f] = \tilde{W}f, \quad f \in L^2_\Sigma(M) \quad (2.5)$$

is properly defined (i.e. $\tilde{W}f = 0$ for $f \in L^2_\Sigma(M) \cap L^2_0(M)$) and is a linear isometry into \mathcal{H} .

We shall need some abstract results concerning spectral complex measures and functional calculus for each selfadjoint operator to prove this theorem. The first is a generalization of a well-known result*)

Lemma IV.7

Let A be a s.a. operator in \mathcal{H} , E - its projection valued spectral measure, $E = E_A$, $f: \mathbb{R} \rightarrow \mathbb{C}$ - a borel function (= measurable with respect to $\text{Borel}(\mathbb{R})$) and $x \in \mathcal{H}$, $z \in D(f(A))$, then $\bar{f} \in L^1(E_{x,z})$ **) and

$$E_{x,f(A)z} = \bar{f} dE_{x,z} . \quad (2.6)$$

The proof is placed in Appendix.

The next result generalizes the above one. It will be an important tool for us, because it "hides" in some sense

*) See [Rudin AF], Lemma 3.23 (3) concerning bounded functions f .

**) Recall that for a complex measure μ $L^1(\mu) = L^1(|\mu|)$ ($|\mu|$ - the variation of μ); of course $\bar{f} \in L^1 \Leftrightarrow f \in L^1$.

the result (2.4), being in fact the isometricity part of the assertion (iii) of our theorem.

Lemma IV.8

Let A be a s.a. operator in \mathcal{H} , E - its projection-valued spectral measure, $f, g: \mathbb{R} \rightarrow \mathbb{C}$ - borel functions and $x \in D(f(A))$, $y \in D(g(A))$. Then

$$f\bar{g} \in L^1(E_{x,y}) \text{ and } E_{f(A)x, g(A)y} = f\bar{g} dE_{x,y}. \quad (2.7)$$

Proof.

By previous lemma (for f, g, x) $\bar{f} \in L^1(E_{y,x})$

and $E_{y, f(A)x} = \bar{f} dE_{y,x}$, hence

$$E_{f(A)x, y} = \overline{\bar{f} dE_{y,x}} = f dE_{x,y}. \quad (2.8)$$

Now, by the same lemma (for g), $f(A)x, y$)

$\bar{g} \in L^1(E_{f(A)x, y})$ and

$$E_{f(A)x, g(A)y} = \bar{g} dE_{f(A)x, y}$$

so, by (2.8) we get (2.7), using also:

$$\begin{aligned} \bar{g} \in L^1(E_{f(A)x, y}) &\Leftrightarrow |\bar{g}| \in L^1(|E_{f(A)x, y}|) = L^1(|f dE_{x,y}|) = \\ &= L^1(|f| d|E_{x,y}|) \Leftrightarrow |f| \cdot |\bar{g}| \in L^1(|E_{x,y}|) \Leftrightarrow f \cdot \bar{g} \in L^1(E_{x,y}). \end{aligned}$$



IV.2.6-

Proof of Theorem IV.6

Let $E = E_A$, so by the definition of the spectral matrix measure M for A and φ we have

$$M_{ij} = E_{\varphi_j, \varphi_i} \quad \text{for any } i, j = 1, \dots, d. \quad (2.9)$$

If $f \in L^2(M)$ then we have (definition of L^2_M):

$$\forall_{j=1, \dots, d} \quad f_j \in L^2(M_{jj}) = L^2(E_{\varphi_j})$$

But, by "the functional calculus" for s.a. operator we know that $x \in D(h(A))$ iff $h \in L^2(E_x)$ for borel h and $x \in \mathbb{H}$. Hence $\varphi_j \in D(f_j(A))$ for any $j=1, \dots, d$, i.e. (i) holds. Moreover we have

$$\|f\|_M^2 = \langle f, f \rangle_M = \langle f, f \rangle_{\Sigma} = \sum_{i,j=1}^d \int_{\mathbb{R}} f_j \bar{f}_i dM_{ij}$$

by Fact III... (p. III.4.9). Now, by (2.9) and Lemma IV.8 used for f_j, f_i, φ_j and φ_i for any $i, j = 1, \dots, d$

we obtain:

$$\begin{aligned} \|f\|_M^2 &= \sum_{i,j=1}^d \int_{\mathbb{R}} f_j \bar{f}_i dE_{\varphi_j, \varphi_i} = \sum_{i,j=1}^d E_{f_j(A)\varphi_j, f_i(A)\varphi_i} (R) = \\ &\sum_{i,j=1}^d \underbrace{\left\langle E(\mathbb{R}) f_j(A) \varphi_j, f_i(A) \varphi_i \right\rangle}_{\text{I}} = \left\langle \sum_{j=1}^d f_j(A) \varphi_j, \sum_{i=1}^d f_i(A) \varphi_i \right\rangle \\ &= \|\tilde{W}(f)\|^2 \end{aligned}$$

— so (ii) holds..

If moreover $f \in L^2_0(M)$ then $\|f\|_M = 0$
and by (2.4) (just proved) $\tilde{W}f = 0$. So
 W is properly defined on $L^2_\Sigma(M)$, and for any $f \in L^2_\Sigma$
 $\|[f]\|_M = \|f\|_M = \|\tilde{W}f\| = \|W[f]\|$.
So W is a ^{linear} isometry.

□

-IV.2.8-

IV. 3 The canonical spectral transformation ("CST") for A and $\vec{\varphi}$

Consider again a s.a. operator in a Hilbert space A and an ordered system of vectors $\vec{\varphi} \in \mathcal{H}^d$ ($d \in \mathbb{N}$). In the previous subsection we constructed the spectral matrix measure $E_{A, \vec{\varphi}}$ for A and $\vec{\varphi}$ on $\text{Borel}(\mathbb{R})$ σ -algebra of subsets of \mathbb{R} . Moreover we constructed a transformation $W: L^2(E_{A, \vec{\varphi}}) \rightarrow \mathcal{H}$ being a linear isometry (see Theorem IV. 6 (iii)). Now, we shall extend W to the whole $L^2(E_{A, \vec{\varphi}})$ to the so-called (here...) canonical spectral transformation (CST for short) for A and $\vec{\varphi}$.

Our main goal in section IV is to prove that just this CST states a unitary equivalence between A in \mathcal{H} and $T_{\vec{\varphi}}$ - the multiplication

by the identity (" $x \mapsto x$ ") function on \mathbb{R} in the space $L^2(E_{A,\vec{\varphi}})$, if $\vec{\varphi}$ is cyclic for A .
 (being just ~~the~~ MVE theorem for the finitely cyclic case).
 We shall reach this goal in the last subsection of IV, but here we do not need any cyclicity property.

We start from the extension of W .

Theorem IV. 9

Let $M := E_{A,\vec{\varphi}}$. There exists exactly one operator $U \in B(L^2(M), \mathcal{H})$ satisfying

$$\forall f \in \sum L^2(M) \quad U([f]) = \sum_{j=1}^d f_j(A) \varphi_j. \quad (3.1)$$

The above unique U is an isometry from $L^2(M)$ into \mathcal{H} ("onto the image of U ").

Proof

The unicity of such U is obvious from the continuity requirement and from the density of $L^2_{\Sigma}(M)$ in $L^2(M)$. The existence of such U is the direct consequence of ^{(Theorem IV.6 (iii) and)} the abstract result on extension of a bounded linear map from a subspace X_0 of X into Y to the bounded linear map from the closure \overline{X}_0 into Y for X, Y - Banach spaces. It remains only to prove that U is an isometry (which is also standard and abstract...). Using the continuity of U and the density of $L^2_{\Sigma}(M)$, for any $[f] \in L^2(M)$ we choose $\{f_n\}_{n \geq 1}$ in $L^2_{\Sigma}(M)$ such that

$[f_n] \xrightarrow[L^2(M)]{} [f]$, so $\|U[f_n]\| \rightarrow \|U[f]\|$, but U was the extension of W from Theorem IV.6 (iii), so also

$$\|U[f_n]\| = \|W[f_n]\| = \|[f_n]\|_M \rightarrow \|[f]\|_M, \text{ hence}$$

$$\|U[f]\| = \|f\|_M.$$



Note here, that this step — the extension from $L^2_{\Sigma}(M)$ onto $L^2(M)$ was not necessary in the case $d=1$, because then we simply have $L^2_{\Sigma}(M) = L^2(M)$!

Now we are ready to formulate the announced definition.

Definition IV.10

Let A be a s.a. operator in \mathcal{H} and $\vec{\varphi} \in \mathcal{H}^d$. Then the unique $U \in B(L^2(E_{A, \vec{\varphi}}), \mathcal{H})$ satisfying (3.1) is called the canonical spectral transformation (CST) for A and $\vec{\varphi}$. We denote it by

$$U_{A, \vec{\varphi}}. \quad (3.2)$$

IV.4 Vector polynomials and x MUE Theorem

Recall that the construction of the unitary transformation U stating the unitary equivalence for the $d=1$ case of x MUE Theorem was based on polynomials. — We considered the subspace $\text{Pol}_\mu(\mathbb{R}) \subset L^2(\mu)$ for the spectral measure $\mu = E_{A,\varphi}$ ($= E_{A,(\varphi)}$, with " $\vec{\varphi} = (\varphi)$ ", adopting our d -dim. notation to the case $d=1$), and this space played quite an important role. $\text{Pol}_\mu(\mathbb{R})$ was just the space of classes $[f]$ of polynomials $f \in \text{Pol}(\mathbb{R})$, $f: \mathbb{R} \rightarrow \mathbb{C}$. Now, for general $d \in \mathbb{N}$, we should consider the appropriate "vector polynomials" $f: \mathbb{R} \rightarrow \mathbb{C}^d$, so we shall try to introduce here a +/- convenient notation for them:

for $n \in \mathbb{N}_0$ and $j = 1, \dots, d$
 $\mathbb{X}_j^n : \mathbb{R} \rightarrow \mathbb{R}^d$ is the "monomial" \mathbb{X}^n on the
 j -th coordinate, and 0 on the remaining one, i.e.
 $\mathbb{X}_j^n = \mathbb{X}^n \cdot e_j \quad (4.1)$

where e_j is the j -th vector of the canonical
base (e_1, \dots, e_d) of \mathbb{C}^d , $(e_j)_k = \begin{cases} 1 & k=j \\ 0 & k \neq j \end{cases}$.

$\text{Pol}_d(\mathbb{R}) := (\text{Pol}(\mathbb{R}))^d$, so obviously we have

$$\text{Pol}_d(\mathbb{R}) = \text{lin}\{\mathbb{X}_j^n : n \in \mathbb{N}_0, j = 1, \dots, d\} \quad (4.2)$$

if M is a $(d \times d)$ matrix measure on $\text{Borel}(\mathbb{R})$, such
that $\text{Pol}_d(\mathbb{R}) \subset L^2(M)$, then

$$\text{Pol}(M) := \{[f] \in L^2(M) : f \in \text{Pol}_d(\mathbb{R})\} \quad (4.3)$$

* Recall that $\mathbb{X}^n : \mathbb{R} \rightarrow \mathbb{R}$ is given by $\mathbb{X}^n(f) := t^n$
for $t \in \mathbb{R}$ ($\mathbb{X}^0 = 1$).

Before we formulate our main theorem - x MUE Th. we need a result which gives even more than the inclusion $\text{Pol}_d(\mathbb{R}) \subset L^2(M)$ for the most important choice of M .

So, we consider again a Hilbert space \mathcal{H} .

Lemma IV.11.

Suppose that A is s.a. in \mathcal{H} and $\vec{\varphi} \in \mathcal{H}^d$. If $\varphi_1, \dots, \varphi_d \in D(A^\infty)$, then

$$\text{Pol}_d(\mathbb{R}) \subset L_\Sigma^2(E_{A, \vec{\varphi}}).$$

Proof

Denote $M = E_{A, \vec{\varphi}}$. By (4.2) it suffices to check that for any $j = 1, \dots, d$ and $n \in \mathbb{N}_0$, $X_j^n \in L_\Sigma^2(M)$, so - by (4.3) and by the definition of $L_\Sigma^2(M)$ (see p. III.4.9) we should only check, that $X^n \in L^2(M_{jj})$ for any $n \in \mathbb{N}_0$ and $j = 1, \dots, d$. Fix j . By the definition of the spectral matrix measure $E_{A, \vec{\varphi}}$ we have

$M_{jj} = E_{\varphi_j}$ where $E = E_A$ (the proj. valued. spectral measure for A).

So, by STh + FCTh, from $\varphi_j \in D(A^\infty)$
for any $n \in \mathbb{N}$, we have $\varphi_j \in D(\mathbb{X}^n(A))$, which means
that $\mathbb{X}^n \in L^2(E_{\varphi_j}) = L^2(M_{jj})$. □

So, we see now that assuming only that
each term of $\vec{\varphi}$ is in $D(A^\infty)$ we have
the properly defined $\overset{(by (4.3))}{\text{subspace}}$ $\text{Pol}(M)$ of
 $L^2(M)$ for $M = E_{A, \vec{\varphi}}$. In particular, we
have this if $\vec{\varphi}$ is cyclic for A .

Theorem (\times MUE - the general finitely cyclic case)

Suppose that A is a s.a. finitely cyclic operator in \mathcal{H} . If $\vec{\varphi} \in \mathcal{H}^d$ is a cyclic system for A , $M = E_{A, \vec{\varphi}}$ and $U = U_{A, \vec{\varphi}}$, then

(1) $\text{Pol}_d(\mathbb{R}) \subset L^2_{\Sigma}(M)$ and $\text{Pol}(M)$ is dense in $L^2(M)$;

 (3) U is a unitary transformation from $L^2(M)$ onto \mathcal{H} ;

 (2) U is the unique operator from the set of such $U' \in B(L^2(M), \mathcal{H})$, that

$$\forall_{\substack{n \in \mathbb{N}_0 \\ j=1, \dots, d}} U'[x_j^n] = A^n \varphi_j; \quad (4.4)$$

(4) $A = UT_{\mathbb{X}}U^{-1}$,

where $T_{\mathbb{X}}$ is the multiplication by \mathbb{X} operator in $L^2(M)$.

Remark 4.12

Note, that we do not assume, that $\vec{\varphi}$ is a linearly independent system! So, we can choose such $\vec{\varphi}$, that $d > \dim \text{space of cyclicity}!$

Proof

We have already proved " \subset " from (1) in Lemma IV.11. It is also proved that U satisfies the conditions for U' in (2) (still without the unicity), because by Theorem IV.9, using (3.1) for $f = \mathbb{X}_j^n$ we get $U[\mathbb{X}_j^n] = \mathbb{X}^n(A)\varphi_j$, but $\mathbb{X}^n(A) = A^n$ by functional calculus (STh + FCTh), and

$$\bigvee_{\substack{n \in \mathbb{N}_0 \\ j=1,\dots,d}} U[\mathbb{X}_j^n] = A^n \varphi_j. \quad (4.5)$$

By Theorem IV.9 we also know, that U is an isometry from $L^2(M)$ onto $\tilde{\mathcal{H}} \subset \mathcal{H}$, so $\tilde{\mathcal{H}}$ is a closed subspace of \mathcal{H} as an isometric image of the complete space $L^2(M)$. So, we shall use now the cyclicity of $\tilde{\varphi}$ (see Definition IV.1 and Fact IV.3) - it guarantees that $\text{lin } \text{Orb}_A(\tilde{\varphi})$ is dense in \mathcal{H} . But $\text{lin}(\text{Orb}_A(\tilde{\varphi})) \subset \tilde{\mathcal{H}}$ by (4.4) used for $U' = U$, hence $\tilde{\mathcal{H}}$ is closed and dense in \mathcal{H} , i.e. $\tilde{\mathcal{H}} = \mathcal{H}$. So, we just proved (3). Now, knowing that U is unitary from $L^2(M)$ onto \mathcal{H} , we also get easily the density from (1) and the unicity from (2).

Indeed: we already know, that

$\text{lin}(\text{Orb}_A(\vec{\varphi}))$ is dense in \mathcal{H} , so - by the unitarity of U - also $U^{-1}(\text{lin}(\text{Orb}_A(\vec{\varphi})))$ is dense in $L^2(M)$, but using (4.5) we get

$$U^{-1}(\text{lin}(\text{Orb}_A(\vec{\varphi}))) = \text{lin}(\{[x_j^n] : n \in \mathbb{N}_0, j=1, \dots, d\}) = \text{Pol}(M).$$

Hence $\text{Pol}(M)$ is dense, and the unicity of U from (2) follows directly from the "linearity + continuity + density" argument. It remains only to prove (4).

The proof of (4) is very similar to the proof of the analogical part for the cyclic ($d=1$) case of Theorem (see XMVE theorem in Section II.1).

This similarity is mainly related to the similar "form" of spectral properties, including spectral projections, for the operators of multiplication by functions to its appropriate form in the case $d=1$ (see Section III.5).

By the unicity of the (proj. valued) spectral measure for s.a. operator we should only prove that for any $\omega \in \text{Borel}(\mathbb{R})$

$$E_A(\omega) = \cup_{T_X} E_{T_X}(\omega) U^{-1}. \quad (4.6)$$

Let us fix $\omega \in \text{Borel}(\mathbb{R})$ and recall, that

$$E_{T_X}(\omega) = T_{X_\omega} \quad (\text{see Theorem III.} \dots (10))$$

Using the boundedness of U , $E_A(\omega)$ and E_{T_X} and the linear density of the set $\{\mathbb{X}_j^n : n \in \mathbb{N}_0, j=1\dots d\}$ in $L^2(M)$, to get (4.6) it suffices to prove

$$E_A(\omega) U [\mathbb{X}_j^n] = U [X_\omega \mathbb{X}_j^n] \quad (4.7)$$

for any $n \in \mathbb{N}_0$ and $j=1, \dots, d$, because

$$E_{T_X}(\omega) [\mathbb{X}_j^n] = T_{X_\omega} [\mathbb{X}_j^n] = [X_\omega \mathbb{X}_j^n].$$

But (fortunately...) $X_\omega \mathbb{X}_j^n \in L^2_\Sigma(M)$. Observe first that the unique nonzero term of $X_\omega \mathbb{X}_j^n$ can be only the j -th term equal to $X_\omega \mathbb{X}_j^n$, but $|X_\omega \mathbb{X}_j^n| \leq |\mathbb{X}_j^n|$ and $\mathbb{X}_j^n \in L^2(M_{jj})$ because $\mathbb{X}^n \in L^2_\Sigma(M)$, by Lemma 4.11 (or by (1) - already proved). So also $X_\omega \mathbb{X}_j^n \in L^2(M_{jj})$, which proves that $X_\omega \mathbb{X}_j^n \in L^2_\Sigma(M)$.

So, we can easily prove (4.7) using the explicit formula for $U = U_{A, \vec{\varphi}}$ on $L^2_\Sigma(M)$, which is provided by Theorem IV.9 and Definition IV.10. — Namely, for any $n \in \mathbb{N}_0$ and $j=1\dots d$ we have:

$$U[X_\omega X_j^n] = (X_\omega X^n)(A) \varphi_j, \quad (4.8)$$

so by (4.5), we will get (4.7), if we check

$$E_A(\omega) A^n \varphi_j = (X_\omega X^n)(A) \varphi_j. \quad (4.9)$$

Let us repeat here the argument from Section II, which uses the standard FCTh fact on multiplication of two functions of the s.a. operator:

$$E_A(\omega) A^n = X_\omega(A) X^n(A) \subset (X_\omega X^n)(A)$$

but φ_j belongs to the domains of both sides, so (4.8) holds. □

* We know this simply from the formulae (4.5) and (4.8),
+ the boundedness of $E_A(\omega)$, but the deeper reason for it is just

the property (i) from Theorem IV.6 for any $f \in L^2_\Sigma(M)$, which allowed us to

properly define $U = U_{A, \vec{\varphi}}$.

A. Appendix

{ Appendix ... - it will be
also in the near future -
even before the introduction (see
p. I. o) }

- A.O -