I. Floating bodies and affine surface area

I.1. Floating bodies

Dupin, Blaschke $n = 2, 3$

K is a convex body in $\mathbb{R}^n$

(a compact subset of $\mathbb{R}^n$

s. th. $\operatorname{int}(K) \neq \emptyset$)

let $\delta \geq 0$.

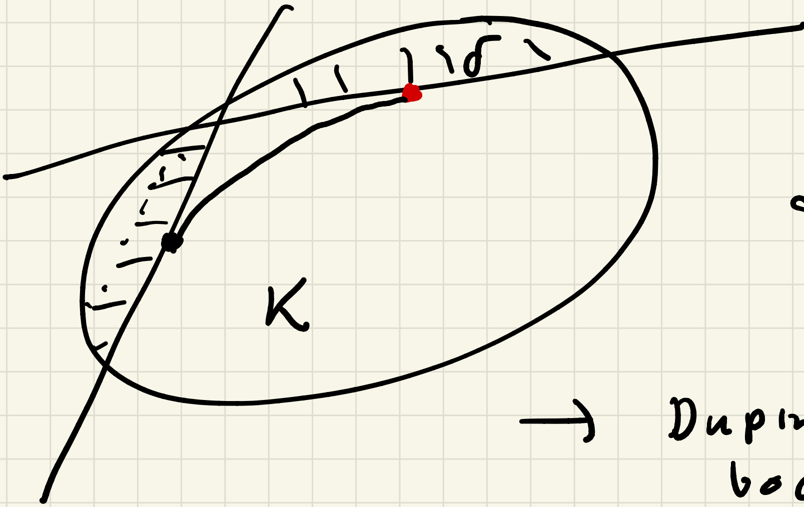
Dupin floating body $K_{[\delta]}$

is the set that has boundary

given by the centroids of

a hyperplane H that cuts

off a set of volume δ of K

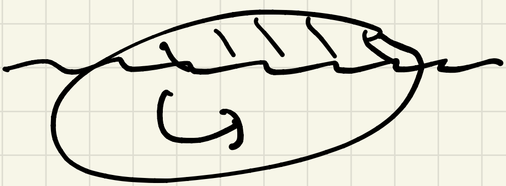


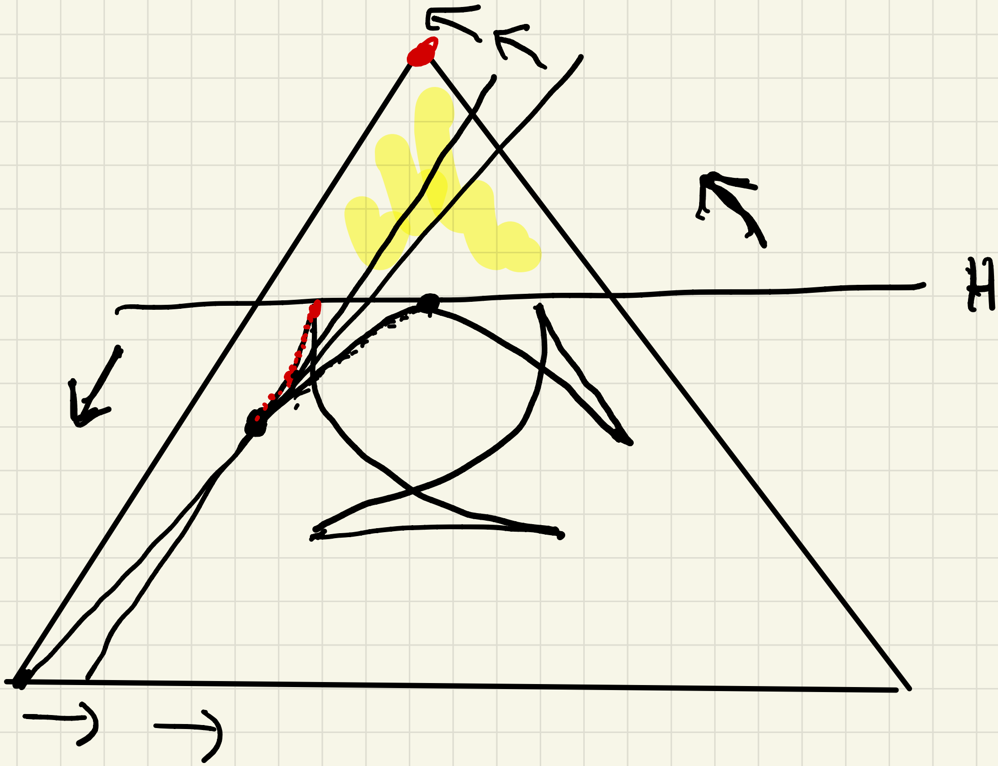
H

center of gravity of $H \cap K$

$\rightarrow$ Dupin floating body $K_{[\delta]}$

Note: name floating body comes from Archimedean principle





→ $K_{[\delta]}$ need not be convex !

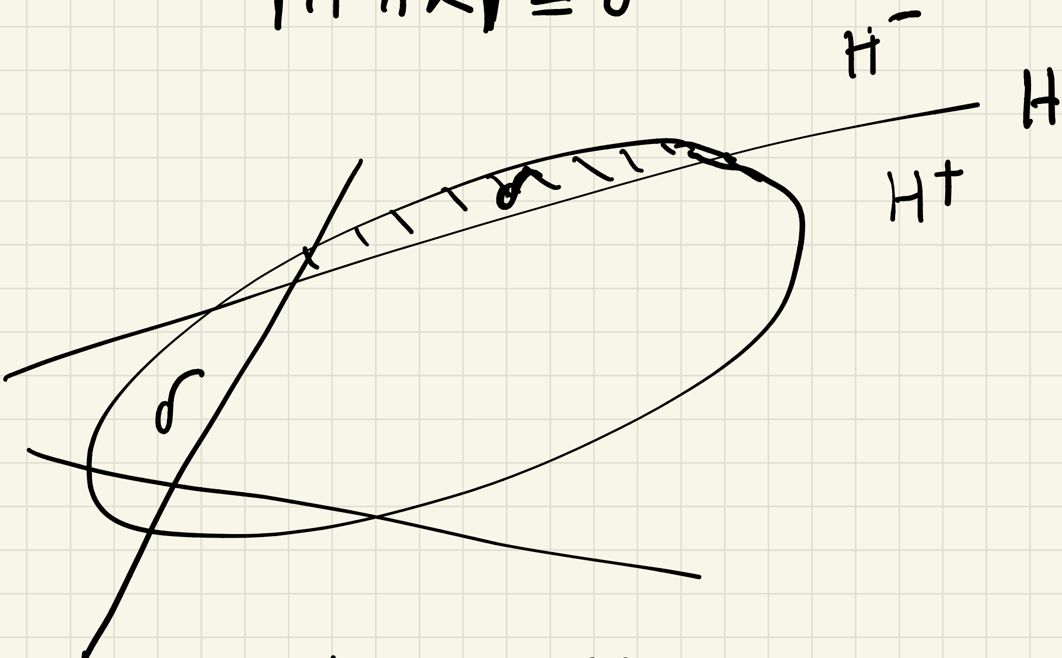
→

Q: When is the Dupin floating body convex ?

Definition (Barany + Larman; Schütt + W)

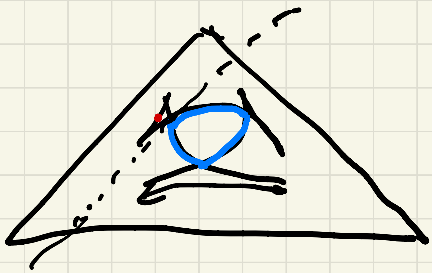
Let K be a convex body in $\mathbb{R}^n$.
Let $\delta \geq 0$.
Then the (convex) floating body K_δ

$$K_\delta = \bigcap_{|H^- \cap K| \leq \delta} H^+$$



- K_δ is convex
- $K_0 = K$, $K_\delta \subseteq K$

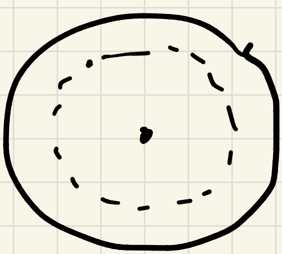
Examples

- 

we see: there are points on the boundary of K_δ where a support hyperplane cuts off more than δ

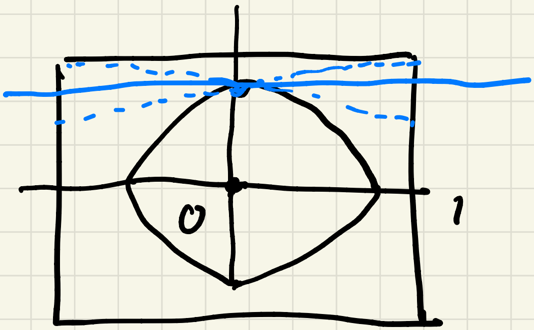
- Euclidean unit ball B_2^n

$$(B_2^n)_\delta = \left(1 - c_n \delta^{\frac{2}{n+1}}\right) B_2^n$$



$$c_n = \frac{1}{2}\left(\frac{n+1}{|B_2^{n-1}|}\right)^{\frac{2}{n+1}}$$

- square in $\mathbb{R}^2$ with side length 2 or B_∞^2.



boundary of $(B_\infty^2)_\delta$

$$f(x) = 1 - \frac{\delta}{2(1-x)}$$

Note: • $\partial(K)_\delta$ need not be C^1

- 2 Facts:

If $K_{[\delta]}$ is convex, then $K_{[\delta]} = K_\delta$

<u>Thm</u> (Meyer + Reisner)

If K is a 0-symmetric convex ($x \in K \iff -x \in K$) body, then $K_{[\delta]} = K_\delta$

Some more properties

- K_δ is strictly convex

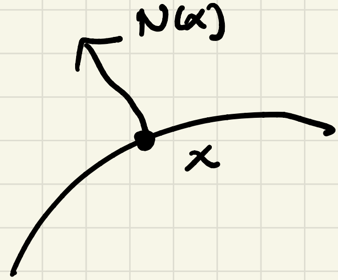
- through every point on ∂K_δ there is at least one hyperplane H that cuts off a set of volume δ of K and this hyperplane touches ∂K_δ in exactly ~~this~~ one point which is then the centroid of $H \cap K$

Q: Why consider floating bodies ?

I. 2. Affine surface area

K convex body in $\mathbb{R}^n$.

$$as(K) = \int_{\partial K} \cancel{\kappa(x)}^{\frac{1}{n+1}} \, d\mu_K(x) = \int_{S^{n-1}} \underline{\underline{f_K}}^{\frac{n}{n+1}}(u) \, d\sigma(u)$$



where κ is the Gauss-curvature at $x \in \partial K$

μ_K is the usual surface measure on ∂K : $\int_{\partial K} d\mu_K = |\partial K|$

f_K is the curvature function of K at u, i.e. $f_K(u) = \frac{1}{\kappa(x)}$ s.th. $\underline{N(x) = u}$

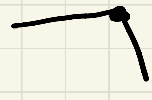
Examples

$$as(K) = \int_{\partial K} \kappa^{\frac{1}{n+1}} \, d\mu_K$$

- $as(B_2^n) = |\partial B_2^n| = n\,|B_2^n|$

- Let P be a polytope

$$\boxed{as(P) = 0} \longleftarrow$$



Properties

- affine invariant: T is an affine map, $\det T \neq 0$

$$as(TK) = |\det T|^{\frac{n-1}{n+1}} \, as(K)$$

- valuation (Schütt), K, L

$$as(K \cup L) + as(K \cap L) = as(K) + as(L)$$

s.t. $K \cup L$ is convex

- upper-semicontinuous (Lutwak)

$$K_j \xrightarrow{d_H} K \Rightarrow \limsup_j as(K_j) \le as(K)$$

as cannot be continuous for the Hausdorff metric

$$P_j \longrightarrow B_2^n \qquad as(P_j) = 0 \ \forall j$$
$$as(B_2^n) > 0$$

- affine isoperimetric inequality

$$\frac{as(K)}{as(B_2^n)} \leq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n+1}} \quad \text{with}$$

equality iff K is an ellipsoid

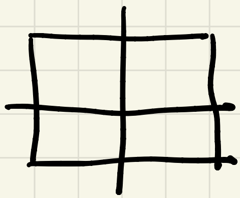
affine isoperimetric inequality

$$\Longleftrightarrow$$

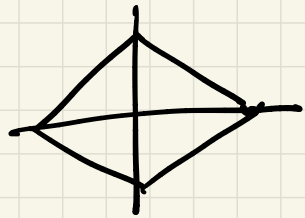
Blaschke – Santaló inequality

K convex body in $\mathbb{R}^n$. $0 \in \mathrm{int}(K)$

$$K^\circ = \{ y \in \mathbb{R}^n : \langle y, x \rangle \le 1 \ \forall x \in K \}$$



B_∞^n



$(B_\infty^n)^\circ = B_1^n$

Blaschke Santaló inequality

$\exists$ a unique $s_0 \in \mathrm{int}(K)$, w.l.o.g $s_0 = 0$, s.t.

$$|K|\,|K^\circ| \le |B_2^n|^2$$

with equality iff K is an ellipsoid

Q: what about lower bounds for

$|K|\,|K^\circ|$?

Mahler conjecture:

For 0-symmetric K

- $|K||K^\circ| \ge |B_\infty^n||B_1^n| = \frac{4^n}{n!}$
- $|K||K^\circ| \ge |S||S^\circ|$ S simplex

Mahler conjecture is open for $n \geq 4$

Symmetric 3-dim. case solved by

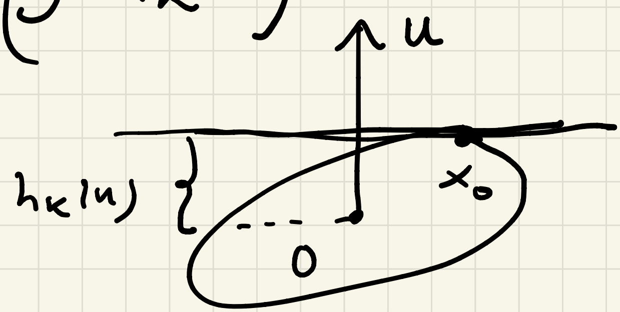
- Iriyeh + Shibata
- ......

Blaschke Santaló inequality $\Longleftrightarrow$ affine isoperimetric inequality

$$as(K) = \int_{S^{n-1}} f_K^{\frac{n}{n+1}} \, d\sigma = \int_{S^{n-1}} \left(\frac{f_K \cdot h_K}{h_K} \right)^{\frac{n}{n+1}} d\sigma$$

Hölder $p = \frac{n+1}{n} \rightarrow \frac{1}{p} = \frac{n}{n+1} \rightarrow \frac{1}{q} = \frac{1}{n+1}$ $\quad q = n+1$

$$\leq \left(\int_{S^{n-1}} f_K \cdot h_K \, d\sigma \right)^{\frac{n}{n+1}} \left(\int h_K^{-n} \, d\sigma \right)^{\frac{1}{n+1}}$$



h_K is the support function of K

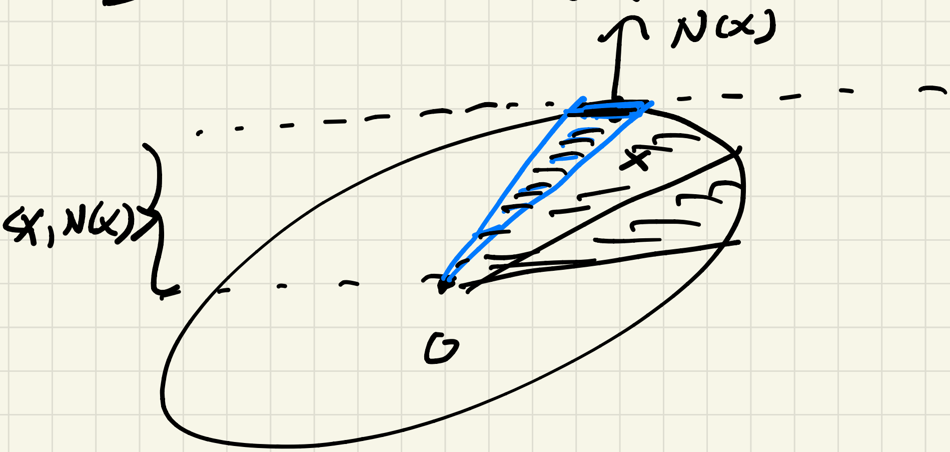
$$h_K(u) = \max_{x \in K} \langle u, x \rangle = \langle u, x_0 \rangle$$

$$\underline{\text{Claim}} \qquad \int_{S^{n-1}} f_K h_K = n\,|K| \quad \Longleftarrow$$

$$\longrightarrow \int_{S^{n-1}} h_K^{-n} = n \cdot |K^\circ|$$

$$\boxed{d\mu_K = f_K \cdot d\sigma}$$

$$\frac{1}{n}\int_{S^{n-1}} f_K\, h_K\, d\sigma = \frac{1}{n}\int_{\partial K} \boxed{\langle x, N(x)\rangle\, d\mu_K} = |K|$$



$$\langle N(x), x\rangle$$

$$\underline{h_K(u)} = \max_{y \in K} \langle u, y\rangle = \langle u, x\rangle^{*}$$

s.t. $u = N(x)$

Instead of $|K^\circ| = \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_K^n} \, d\sigma$

we show

$$|K| = \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_{K^\circ}^n} \, d\sigma$$

$$|K^\circ| = \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_{K^{\circ\circ}}^n}, \quad h_{K^{\circ\circ}} = h_K$$

$$|K| = \int_{\mathbb{R}^n} \chi_K(x)\, dx =$$

$$\int_{S^{n-1}} \int_{r=0}^{\infty} \underline{\chi_K(ru)}\, r^{n-1} \, dr \, d\sigma(u)$$

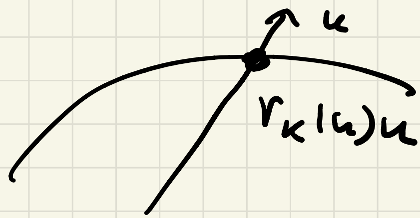
$$= \int_{S^{n-1}} \int_{r=0}^{r_K(u)} r^{n-1} \, dr \, d\sigma = \frac{1}{n} \int_{S^{n-1}} r_K^n(u) \, d\sigma$$

$$r_K^n(u) = \frac{1}{h_{K^\circ}^n(u)}$$

$$h_{K^\circ}(u) = \max_{y\in K^\circ} \langle u, y\rangle = \langle u, y_0\rangle$$

$$= \left\langle y_0, \frac{r_K(u)\,u}{r_K(u)}\right\rangle = \frac{1}{r_K(u)} \underbrace{\langle \overset{y_0\in K^\circ}{\overset{\downarrow}{y_0}}, \underbrace{r_K(u)u}_{\overset{\uparrow}{K}}\rangle}_{\le 1}$$

$$\le \frac{1}{r_K(u)}$$



$\rho_K(u)u \in \partial K$

$\to \exists\, y_0 \in K^\circ$ s.t.

$$1 = \langle y_0, r_K(u)\,u\rangle = r_K(u)\underbrace{\langle y_0, u\rangle}_{y_0\in K^\circ}$$

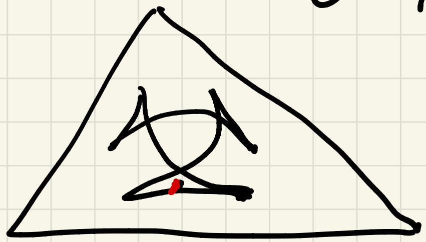
$$\le r_K(u)\, \underset{\underset{\max_{y\in K^\circ}\langle y, u\rangle}{\shortparallel}}{h_{K^\circ}(u)}$$

Petty

$f_K^{-1} \cdot h_K^{n+1} = \text{const} \iff K$ is an ellipsoid

$\Updownarrow$

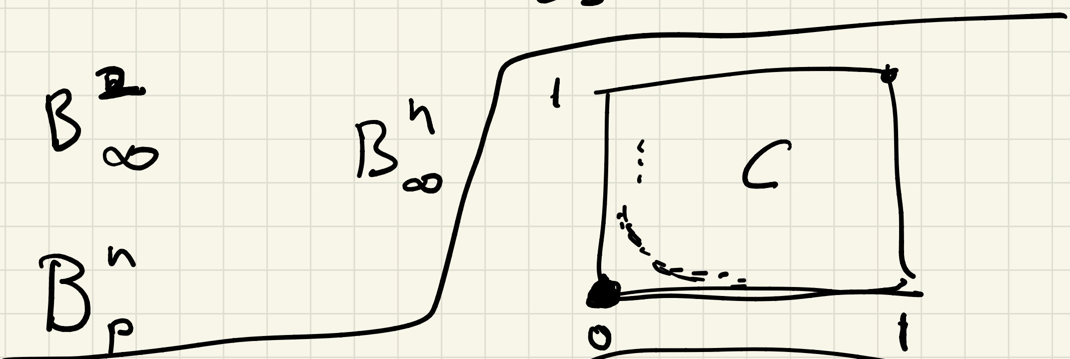
$\langle x, N(x)\rangle^{-n+1} \mathcal{K}(x) = \text{const} \iff K$ is an ellipsoid



$\partial K_{[\delta]} \geq \partial K_\delta$

Meyer + Reisner

$K = -K \implies K_{[\delta]} = K_\delta$



$$\delta = \prod_{i=1}^{n} x_i$$

For behaviour around 0 of C_δ