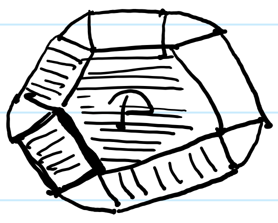
# Lecture 2. Conic intrinsic volumes.

**Usual intrinsic volumes.** $P \subset \mathbb{R}^d$ polytope.

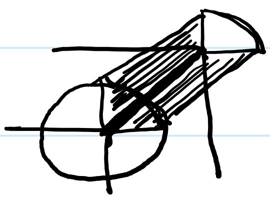
$\mathbb{B}_d$ = unit ball in $\mathbb{R}^d$. $\quad \kappa_d = \mathrm{Vol}_d(\mathbb{B}_d) = \frac{\pi^{d/2}}{(d/2)!}$

$P + r\mathbb{B}_d = \{x \in \mathbb{R}^d : d(x,P) \le r\}$

Steiner formula: $\mathrm{Vol}(P + r\mathbb{B}_d)$.



$d=2$: $\mathrm{Area}(P + r\mathbb{B}_2) = \mathrm{Area}(P) + \mathrm{Per}(P)\cdot r + \pi r^2$.



General $d$: Face $F \in \mathcal{F}_i(P)$ contributes

$$\mathrm{Vol}_i(F) \cdot r^{d-i} \kappa_{d-i} \cdot \underbrace{\gamma(F,P)}_{\text{external angle}}.$$

$$\mathrm{Vol}_d(P + r\mathbb{B}_d) = \sum_{i=0}^{d} r^{d-i}\kappa_{d-i} \cdot V_i(P),$$

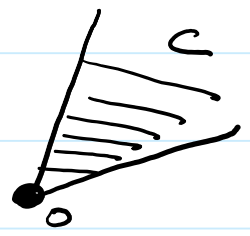
where $V_i(P) = \sum_{F \in \mathcal{F}_i(P)} \mathrm{Vol}_i(F)\cdot\gamma(F,P)$. $i$-th intrinsic volume.

$V_d(P) = \mathrm{Vol}_d(P)$, $V_{d-1}(P) = \frac{1}{2}$ surface area of $P$, ....

**Conic intrinsic volumes.**

**Def.** Cone = intersection of finitely many half-spaces

$$C = \bigcap_{i=1}^{m} H_i^+, \quad H_1, \dots, H_m \ni 0$$



**Ex.** $\{0\}$, $\mathbb{R}^d$, linear spaces, orthant

$$\mathbb{R}^d_+ = \{x : x_i \ge 0 \ \forall i\}.$$

Weyl chamber: $\{x_1 \ge \dots \ge x_d\}$.

**Def.** Let $C \subset \mathbb{R}^d$ be a cone. For $x \in \mathbb{R}^d$:

$$\Pi_C(x) = \operatorname*{argmin}_{y \in C} \|x-y\| \quad \text{metric projection.}$$

**Def.** For $k = 0, \dots, d$:

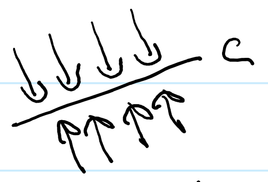
$$\upsilon_k(C) = \mathbb{P}[\Pi_C(g) \in \text{relint of some } k\text{-dim face of } C].$$

$g$ is st. Gauss. in $\mathbb{R}^d$ or $\mathrm{Unif}(\mathbb{S}^{d-1})$.
$\mathrm{Unif}(\mathbb{B}_d)$


**Rem.** $\sum_{k=0}^{d} \upsilon_k(C) = 1$, $\upsilon_k(C) \ge 0$.

**Ex.** $d=2$. $\upsilon_2(C) = \alpha$, $\upsilon_1(C) = \frac{1}{2}$, $\upsilon_0(C) = \frac{1}{2} - \alpha$.

**Ex.** $C$ = lin space of dim $j$.




$\upsilon_j(C) = 1$, $\upsilon_k(C) = 0$ for $k \ne j$.

**Ex.** $C = \mathbb{R}^d_+$. $\quad \Pi_C: (1,-2,+2,-3,-1,0,1)$

$$\downarrow$$

$(\underline{1}, 0, \underline{2}, 0, 0, 0, \underline{1})$. dim face $= 3$.
free, $\ge 0$.

$\upsilon_k(C) = \mathbb{P}[k \text{ coord are} > 0]$
$= \frac{1}{2^d}\binom{d}{k}$.

$(\upsilon_0, \dots, \upsilon_d)$ forms $\mathrm{Bin}(d, 1/2)$.

**Duality of cones.**

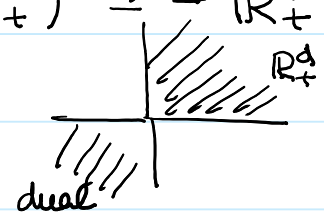
Cone: $C = \{x \in \mathbb{R}^d : \langle x, y_i\rangle \le 0 \ \forall i\}$, $y_1, \dots, y_m \in \mathbb{R}^d \setminus \{0\}$

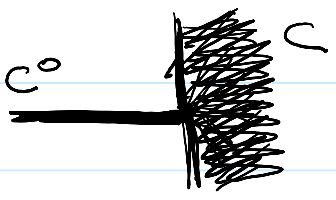
Dual cone: $C^0 = \{y \in \mathbb{R}^d : \langle x, y\rangle \le 0 \ \forall x \in C\}$.

**Rem.** $C^{00} = C$. $\quad$ **Ex.** $y_1, \dots, y_m \in C^0$.

**Ex.** $\{0\} = \mathbb{R}^d$, $L^0 = L^\perp$ (if $L$ is linear subspace).

$(\mathbb{R}^d_+)^0 = -\mathbb{R}^d_+$

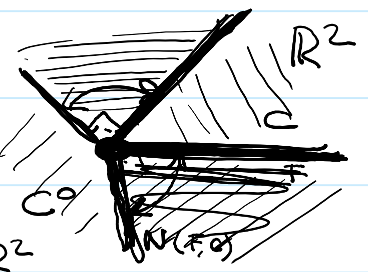


**Bijection:** $\mathcal{F}_k(C) \longleftrightarrow \mathcal{F}_{d-k}(C^\circ)$

$$F \longleftrightarrow (\operatorname{lin} F)^\perp \cap C^\circ = N(F,C)$$

(normal face of $F$)

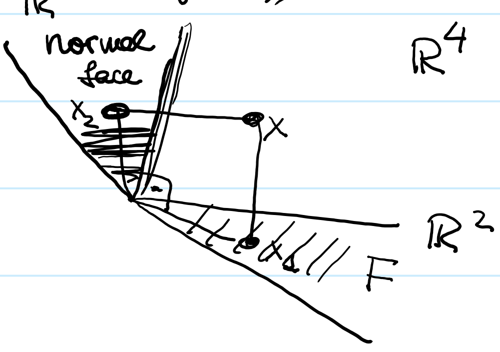



$$v_k(C) = \sum_{F \in \mathcal{F}_k(C)} \angle(F) \cdot \angle(N(F,C)),$$

where $\angle(\mathcal{D}) = \mathbb{P}[g \in \mathcal{D}]$,

where $g$ is st. normal on $\operatorname{lin}(\mathcal{D})$.

$\angle(\mathcal{D})$ = solid angle.




$\Pi_C x \in \operatorname{relint} F$

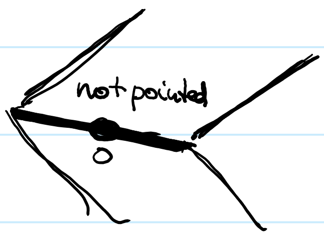
$\Longleftrightarrow x \in F \oplus N(F,C)$

**Rem.** • If $C$ is full-dim:

$$\vartheta_d(C) = \angle(C).$$

• If $C$ is pointed, that $C \cap -C = \{0\}$.

$$v_0(C) = \angle(C^\circ).$$




**Rem.** $\vartheta_k(C^\circ) = \vartheta_{d-k}(C)$.

**Thm.** • $\sum_{k=0}^{d} \vartheta_k(C) = 1 \quad \forall$ cone $C$

• $\sum_{k=0}^{d} (-1)^k \vartheta_k(C) = 0$ if $C$ is not linear space.

**Idea of proof.** • $\sum_{F \in \mathcal{F}(C)} \mathbb{1}_{\operatorname{relint} F \oplus N(F,C)} = 1$. ($\approx$ trivial)

• $\sum_{F \in \mathcal{F}(C)} \mathbb{1}_{F - N(F,C)} \cdot (-1)^{\dim F} = 0$ (non-trivial)

Integrate wrt Gauss measure. $\square$

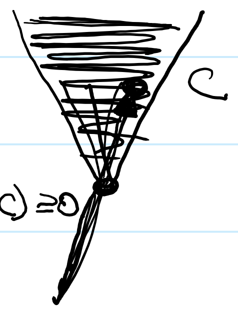
**Grassmann angles** [Grünbaum].

unif. distr. lin. subspace of codim $k$.

$C \subset \mathbb{R}^d$ cone. $\quad \gamma_k(C) = \mathbb{P}[C \cap L_{d-k} \neq \{0\}]$

**Ex.** If $\dim C = d$, $C \neq \mathbb{R}^d$:

$$\gamma_{d-1}(C) = 2\angle(C).$$




**Rem.** For $C \neq \{0\}$: $\quad 1 = \gamma_0(C) \geq \gamma_1(C) \geq \ldots \geq \gamma_d(C) = 0$

**Thm.** [Conic Crofton formula].

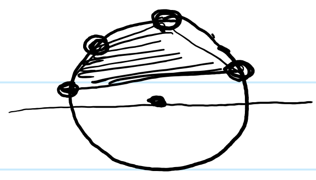
If $C$ is not linear subspace:

$$\gamma_k(C) = \mathbb{P}[C \cap L_{d-k} \neq \{0\}] = 2\left[\vartheta_{k+1}(C) + \vartheta_{k+3}(C) + \ldots\right].$$

$\square$

**Ex** [Wendel]. Let $X_1, X_2, \ldots, X_n$ be iid Unif($\mathbb{S}^{d-1}$) or standard Gaussian.

Compute $\mathbb{P}[[X_1, \ldots, X_n] \ni 0]$.



**Sol.** $[X_1, \ldots, X_n] \ni 0 \Longleftrightarrow$

$\exists \lambda_1, \ldots, \lambda_n \geq 0$ : $\lambda_1 X_1 + \ldots + \lambda_n X_n = 0$.
$\lambda_1 + \ldots + \lambda_n = 1$

$\Longleftrightarrow \exists \lambda_1, \ldots, \lambda_n \geq 0$ (not all 0) : $\lambda_1 X_1 + \ldots + \lambda_n X_n = 0$.

Let $X_1, \ldots, X_n$ be st. Gaussian.

$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_+ \setminus \{0\}$.

$A = \left( \boxed{X_1}\ \boxed{X_2} \cdots \boxed{X_n} \right) : \mathbb{R}^n \to \mathbb{R}^d$.

Condition:

$A\lambda = 0$

or $\lambda \in \operatorname{Ker} A$.

$$[X_1, \ldots, X_n] \ni 0 \Longleftrightarrow \underbrace{\operatorname{Ker} A}_{\text{unif on } G(n,n-d)} \cap \mathbb{R}^n_+ \neq \{0\}.$$

$$\mathbb{P}[[X_1, \ldots, X_n] \ni 0] = \gamma_d(\mathbb{R}^n_+) \underset{\text{Crofton}}{=} 2 \sum_{\ell = 1,3,\ldots} \vartheta_{d+\ell}(\mathbb{R}^n_+)$$

$$= \frac{2}{2^n}\left[\binom{n}{d+1} + \binom{n}{d+3} + \ldots\right]$$

$$= \frac{1}{2^{n-1}}\left[\binom{n-1}{d} + \binom{n-1}{d+1} + \binom{n-1}{d+2} + \ldots\right].$$

$$= \mathbb{P}[\operatorname{Bin}(n-1, 1/2) \geq d].$$

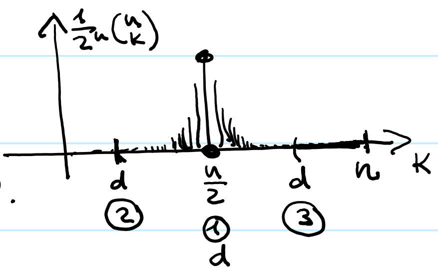
$$\mathbb{P}[[x_1,\dots,x_n]\ni 0] = \mathbb{P}[\mathrm{Bin}(\overset{\approx n}{n-1}, 1/2)\geq d].$$

Q. $n$ points, Unif($S^{d-1}$), iid. $d = 10^6$.

① $n = 2d$ points. $\mathbb{P}\approx \frac{1}{2}$

② $n=(2+\varepsilon)d$ points $\mathbb{P}\approx 1$.

③ $n=(2-\varepsilon)d$ points $\Rightarrow \mathbb{P}\approx 0$.




Ex 2. [Vysotsky, Zaporozhets, K].

Let $X_1,\dots,X_n$ iid st. Gauss in $\mathbb{R}^d$. $S_k = X_1+\dots+X_k$. random walk in $\mathbb{R}^d$.

$$\mathbb{P}[0\in[S_1,\dots,S_n]] = ?$$

Sol. $0\in[S_1,\dots,S_n] \iff \exists \lambda_1,\dots,\lambda_n\geq 0$ (not all 0) st. $\lambda_1 S_1+\dots+\lambda_n S_n = 0$.

$$\iff \lambda_1 X_1 + \lambda_2(X_1+X_2)+\lambda_3(X_1+X_2+X_3)+\dots+\lambda_n(X_1+\dots+X_n)=0.$$

$$\iff X_1\cdot\underbrace{(\lambda_1+\dots+\lambda_n)}_{y_1} + X_2\underbrace{(\lambda_2+\lambda_3+\dots+\lambda_n)}_{y_2}+\dots X_n\cdot\underbrace{\lambda_n}_{y_n} = 0.$$

Condition: $y_1\geq y_2\geq\dots\geq y_n\geq 0$. (not all are 0).

Weyl chamber of type B: $W=\{y\in\mathbb{R}^n: y_1\geq\dots\geq y_n\geq 0\}$.

$0\in[S_1,\dots,S_n]\iff \mathrm{Ker}\,A\cap W\neq\{0\}$.

Ker A ~ Unif on $G(n,n-d)$.

$$\mathbb{P}[[S_1,\dots,S_n]\ni 0] = \gamma_d(W) = 2\sum_{l=1,3,5,\dots}\upsilon_{d+l}(W)$$

Tomorrow: $\upsilon_k(W) = \frac{B(n,k)}{2^n\cdot n!}$, $(t+1)(t+3)\dots(t+2n-1) = \sum_{k=0}^{n} B(n,k)t^k$

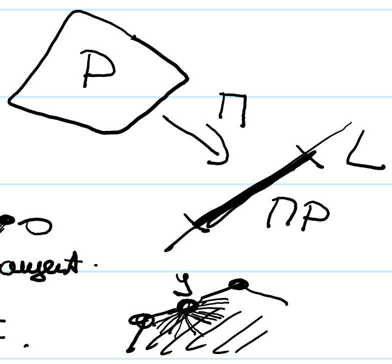
Ex. [Affentranger and Schneider].

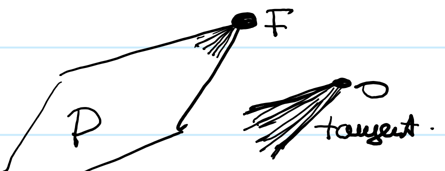
$P\subset\mathbb{R}^n$, $\dim P = n$, Polytope.

$L\sim$ Unif lin subspace in $G(n,d)$.

$\Pi$ orthogonal proj. to $L$.

$$\mathbb{E} f_k(\Pi P) = ?$$

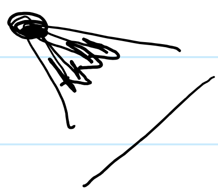



Sol. Let $F\in\mathcal{F}_k(P)$.

$y\in$ relint $F$.

Tangent cone of $F$: $T(F,P) = \{z\in\mathbb{R}^n: \exists\varepsilon>0\ \ y+\varepsilon z\in P\}$.



$$\mathbb{E} f_k(\Pi P) = \mathbb{E}\sum_{F\in\mathcal{F}_k(P)} \mathbb{1}_{\Pi F \text{ is a face of } \Pi P}$$

$$= \sum_{F\in\mathcal{F}_k(P)} \mathbb{P}[\Pi F \text{ is a face of } \Pi P]$$

$$= \sum_{F\in\mathcal{F}_k(P)} \mathbb{P}[T(F,P)\cap \underbrace{L^\perp}_{\text{Unif on } G(n,n-d)} = \{0\}]$$

$$= \sum_{F\in\mathcal{F}_k(P)} \gamma_d(T(F,P)).$$