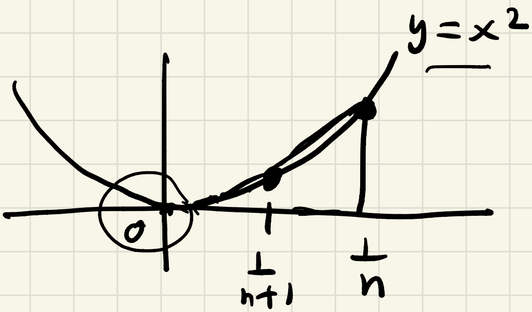
We want to show

$$\lim_{\delta \to 0} \frac{|K| - |K_\delta|}{\delta^{\frac{2}{n+1}}} \overset{!}{=} c_n \int_{\partial K} \kappa^{\frac{1}{n+1}} \, d\mu_K$$

κ generalized Gauss curvature

We need generalized Gauss curvature



∂K given by

$\text{co}\{(\pm\frac{1}{n}, \frac{1}{n^2}) : n \in \mathbb{N}\}$

f describes ∂K locally around 0

- f not twice diff. in the usual sense
- but f twice diff. in the generalized sense

$\forall x \in [\frac{1}{n+1}, \frac{1}{n}]$:

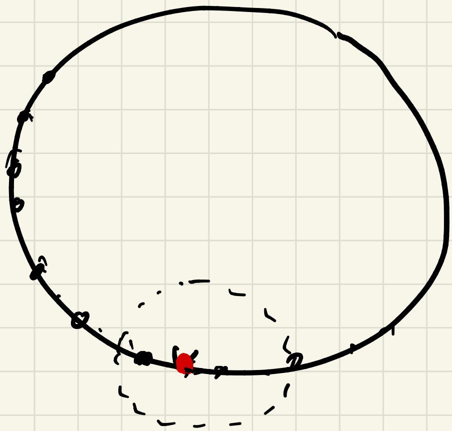
$$|df(x) - df(0) - d^2f(0)\, x|$$

$$= \left| \frac{2n+1}{n(n+1)} - 2x \right| \overset{!}{\leq} \theta(|x|)|x|$$

We can take $\theta(|x|) = 2|x|$

One can construct a convex body

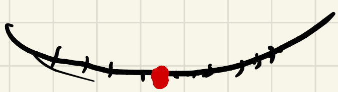
K s.t. ∂K is not diff. on a dense countable set



$\Downarrow$

2^{nd} derivative in the classical sense does not exist at any $x \in \partial K$

zoom in



set situation as above

Alexandrov, Bonnesen + Fenchel

∂K is twice diff. in the generalized sense a.e. $\Rightarrow$ generalized $\mathcal{K}$ exists a.e.

$$\lim_{\delta\to 0} \frac{|K|-|K_\delta|}{\delta^{\frac{2}{n+1}}} = \lim_{\delta\to 0} \int_{\partial K} \underbrace{\frac{\langle x, N(x)\rangle}{n\,\delta^{\frac{2}{n+1}}} \left(1-\left(\frac{\|x_\delta\|}{\|x\|}\right)^n\right)}\, d\mu_K$$

Candidate for dominating integrable function is the rolling function

$r : \partial K \longrightarrow \mathbb{R}$

$$r(x) = \begin{cases} \sup\{\rho : B_2^n(x-\rho N(x), \rho) \subseteq K\} & \text{if } N(x) \text{ is unique} \\ 0 & \text{if } N(x) \text{ is not unique} \end{cases}$$

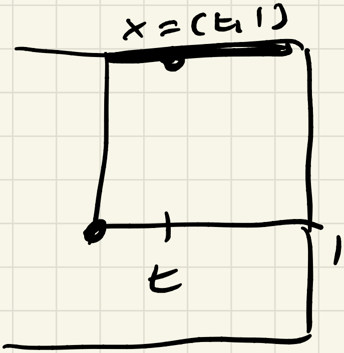
<u>Thm</u> K convex body. Then for all $0 \le t \le 1$

- $\{x \in \partial K : r(x) \ge t\}$ is closed $\Rightarrow$ r is measurable
- $\mu_K(\{x\in \partial K : r(x) \ge t\}) \ge (1-t)^{n-1} |\partial K|$

$\Downarrow$

- The function $r^{-\alpha} : \partial K \to \mathbb{R}$ is integrable for all α, $0 \le \alpha < 1$.

$C = B_\infty^2$



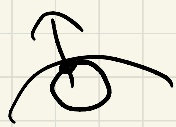
$\alpha = 1$ is not possible in general

$$r(x) = 1-t$$

$$\int_0^1 (1-t)^{-1}\, dt = -\ln(1-t)\Big|_{t=0}^{1}$$

Corollary (Rademacher)

$$\mu_K(\{x \in \partial K : N(x) \text{ is unique}\}) = |\partial K|$$
$$\parallel$$
$$\mu_K(\{x \in \partial K : r(x) > 0\})$$



Proof $\{x \in \partial K : r(x) > 0\} \subseteq \{x : N(x) \text{ is unique}\}$

$\supseteq$ No: $y = |x|^{\frac{4}{3}}$

$$|\partial K| \geq \mu_K(\{x : N(x) \text{ is unique}\})$$
$$\geq \mu_K(\{x : r(x) > 0\}) = \mu_K\left(\bigcup_{k=1}^{\infty} \{x : r(x) \geq \tfrac{1}{k}\}\right)$$

$$= \mu_K\left(\bigcup_{k=1}^{\infty} \{x : r(x) > \tfrac{1}{k}\}\right)$$

cont. of μ_K

$$= \lim_{k\to\infty} \mu_K\left(\{x : r(x) > \tfrac{1}{k}\}\right)$$

inequality of Thm.

$$\geq \lim_{k\to\infty} \left(1-\tfrac{1}{k}\right)^{n-1} |\partial K| = \underline{|\partial K|}$$

Now we show: $r^{-\alpha}$ is integrable on ∂K

for all $0 \leq \alpha < 1$

The case $\alpha = 0$ ✓

r is measurable $\Rightarrow$ $r^{-\alpha}$ is measurable

$$\underline{(1-t)^{n-1}} |\partial K| \leq \left|\{x : r(x) \geq t\}\right|$$

$$= \left|\{x : \underline{r(x)^{-\alpha} \leq t^{-\alpha}}\}\right|$$

Put $\frac{1}{t^\alpha} = s$

$$(1 - s^{-\frac{1}{\alpha}})^{n-1} |\partial K| \le |\{x : r(x)^{-\alpha} \le s\}|$$

$$= |\partial K| - |\{x : r(x)^{-\alpha} > s\}|$$

$$(1 - s^{-\frac{1}{\alpha}})^{n-1} \ge 1 - (n-1) s^{-\frac{1}{\alpha}}$$

$$\Rightarrow$$

$$|\{x : r(x)^{-\alpha} > s\}| \le |\partial K| \left[1 - (1 - s^{-\frac{1}{\alpha}})^{n-1} \right]$$

$$\le \underline{|\partial K| \, (n-1) \, s^{-\frac{1}{\alpha}}}$$

$$\int_{\partial K} r(x)^{-\alpha} \, d\mu_K(x) = \int_{s=0}^{\infty} |\{x : r(x)^{-\alpha} > s\}| \, ds$$

$$= \int_{s=0}^{1} |\underbrace{\{x : r(x)^{-\alpha} > s\}}| \, ds + \int_{s=1}^{\infty} |\{x : r(x)^{-\alpha} > s\}| \, ds$$

$$\le |\partial K| \cdot 1 + |\partial K| (n-1) \int_{s=1}^{\infty} s^{-\frac{1}{\alpha}} \, ds$$

$$= |\partial K| \left(1 + (n-1) \frac{\alpha}{1-\alpha} \right)$$

$$\frac{|K|-|K_\delta|}{\delta^{\frac{2}{n+1}}} = \int_{\partial K} \frac{\langle x, N(x)\rangle}{n\,\delta^{\frac{2}{n+1}}}\left(1-\frac{\|x_\delta\|^n}{\|x\|^n}\right) d\mu_K(x)$$

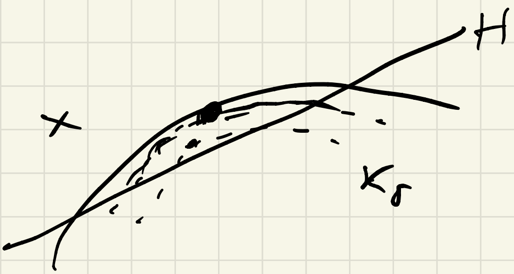
Lemma 1 $\exists\, c>0$, and δ_0 s.t. $\forall x \in \partial K$ with $r(x)>0$, $\forall \delta$, $0<\delta<\delta_0$

$$\left|\frac{\langle x, N(x)\rangle}{n\,\delta^{\frac{2}{n+1}}}\left(1-\frac{\|x_\delta\|^n}{\|x\|^n}\right)\right| \le c\, r(x)^{-\frac{n-1}{n+1}}$$

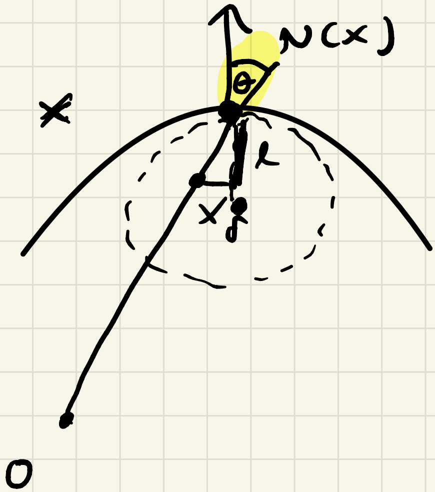
$$\lim_{\delta\to 0}\frac{|K|-|K_\delta|}{\delta^{\frac{2}{n+1}}} = \int_{\partial K} \boxed{\lim_{\delta\to 0}\frac{\langle x, N(x)\rangle}{n\,\delta^{\frac{2}{n+1}}}\left(1-\frac{\|x_\delta\|^n}{\|x\|^n}\right)} d\mu_K(x)$$

$$\| \quad \text{Lemma 2}$$

$$c_n \cdot K(x)^{\frac{1}{n+1}}$$



Sketch of Proof of the Lemma



x s.t. $r(x) > 0$

Case when x_δ

$$\cos\theta = \frac{\ell}{\|x - x_\delta\|}$$

$$\rightarrow \boxed{\ell = \cos\theta \, \|x - x_\delta\| \leq \frac{r(x)}{2}}$$

x and x_δ are colinear

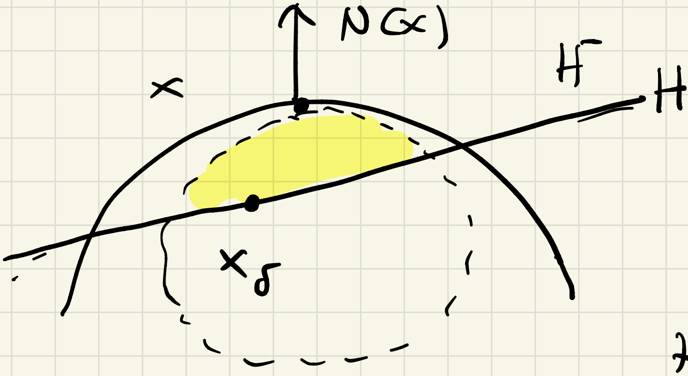
$$\|x\| = \|x_\delta\| + \|x - x_\delta\| \iff$$

$$1 = \frac{\|x_\delta\|}{\|x\|} + \frac{\|x - x_\delta\|}{\|x\|}$$

$$\left(\frac{\|x_\delta\|}{\|x\|}\right)^n = \left(1 - \frac{\|x - x_\delta\|}{\|x\|}\right)^n \geq 1 - n\frac{\|x - x_\delta\|}{\|x\|}$$

$$1 - \left(\frac{\|x_\delta\|}{\|x\|}\right)^n \leq n\frac{\|x - x_\delta\|}{\|x\|}$$

$$\boxed{\frac{\langle x, N(x)\rangle}{n}\left(1 - \left(\frac{\|x_\delta\|}{\|x\|}\right)^n\right)} \leq \overbrace{\underbrace{\left\langle \frac{x}{\|x\|}, N(x)\right\rangle}_{\cos\theta} \|x - x_\delta\|}^{= \ell}$$



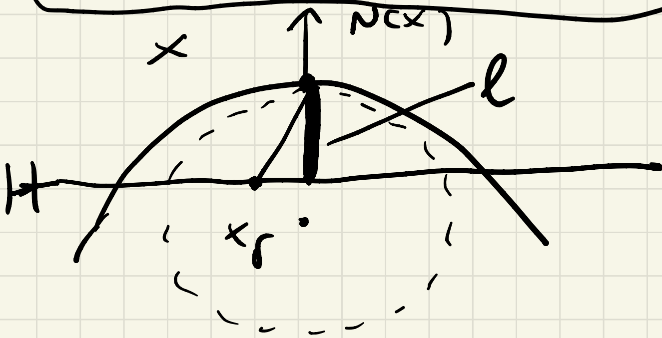
We know:

as $x_\delta \in \partial K_\delta$

$\exists$ a hyperplane H that cuts off exactly δ from K

$|H^- \cap K| = \delta$

Simplified situation



We assume that the hyperplane that cuts off δ has outer normal N(x)

$$\delta = |H^- \cap K| \geq |\underbrace{H^- \cap B_2^n(x - r(x)N(x), r(x))}_{\shortparallel}|$$

$$|\mathrm{Cap}(r(x)B_2^n, \ell)|$$

$$|\mathrm{Cap}(rB_2^n, \ell)| \approx C \cdot r^{\frac{n-1}{2}} |B_2^{n-1}| \ell^{\frac{n+1}{2}}$$

$$C = \frac{2^{\frac{n+1}{2}}}{n+1}$$

$$\delta \geq c\, r(x)^{\frac{n-1}{2}} |B_2^{n-1}|\, \ell^{\frac{n+1}{2}} \Longrightarrow$$

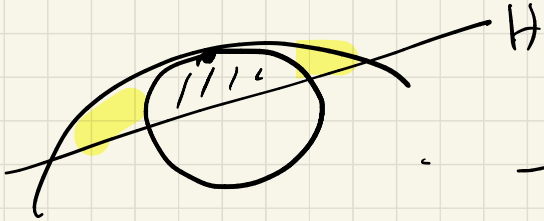
$$\ell \leq \left(\frac{1}{c\,|B_2^{n-1}|}\right)^{\frac{2}{n+1}} r(x)^{-\frac{n-1}{n+1}} \underline{\underline{\delta^{\frac{2}{n+1}}}}$$

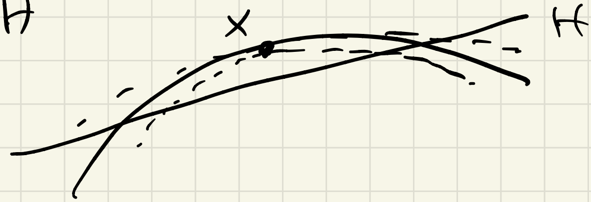
$$\wedge|$$

$$\frac{\langle x, N(x)\rangle}{n}\left(1 - \frac{\|x_r\|^n}{\|x\|^n}\right)$$

$$\Downarrow$$

$$\frac{\langle x, N(x)\rangle}{n\,\delta^{\frac{2}{n+1}}}\left(1 - \frac{\|x_\delta\|^n}{\|x\|^n}\right) \leq C \cdot r(x)^{-\frac{n-1}{n+1}}$$



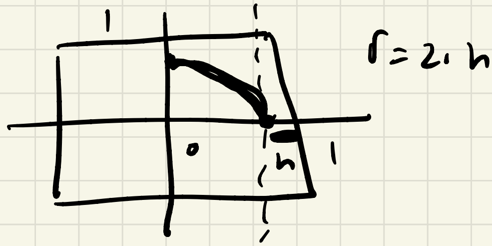


I. 3 other directions, open questions

$$\lim_{\delta \to 0} \frac{|K \setminus K_\delta|}{\delta^{\frac{2}{n+1}}} = c_n \int_{\partial K} x^{\frac{1}{n+1}} \, d\mu_K$$

$\underbrace{\qquad\qquad}_{as(K)}$

(i) $as(P) = 0$



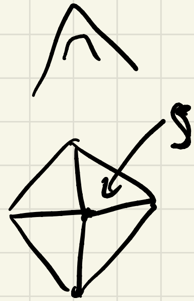
let $P = B^2_\infty$

$$y = 1 - \frac{\delta}{2(1-x)} \qquad 0 \le x \le 1 - \frac{\delta}{2}$$

$$\frac{|B^2_\infty| - |(B^2_\infty)_\delta|}{\delta^{\frac{2}{3}}} =$$

$$= \left(2\delta(1+\log 2) + 2\delta \log \frac{1}{\delta}\right) \Big/ \delta^{\frac{2}{3}} = 0$$

$$\frac{|B^2_\infty| - |(B^2_\infty)_\delta|}{\delta \log \frac{1}{\delta}} \xrightarrow{\delta \to 0} 2$$



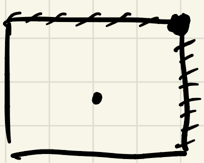
<u>Theorem</u> (Schütt) P polytope in $\mathbb{R}^n$

$$\lim_{\delta \to 0} \frac{|P| - |P_\delta|}{\delta \left(\log \frac{1}{\delta}\right)^{n-1}} = \frac{\#\,\text{flags}(P)}{n!\; n^{n-1}}$$

where a flag of P is an n-tuple $(f_0, f_1, \dots, f_{n-1})$ where each f_i is an i-dim. face of P and

$f_0 \subseteq f_1 \subseteq \dots f_{n-1}$

$\#(\text{flags}) = 4 \cdot 2 = 8$



|| extended the result to spherical + hyperbolic space (Besau, Schütt, W)

(ii) $$|K \setminus K_\delta| \underset{\delta \text{ small}}{\sim} \begin{cases} \delta^{\frac{2}{n+1}} & K \text{ is smooth} \\ \delta \left(\ln \frac{1}{\delta}\right)^{n-1} & K \text{ is a polytope} \end{cases}$$

Q: Suppose f is given, f is concave t...

s.th. $\delta \left(\ln \frac{1}{\delta}\right)^{n-1} \leq f(\delta) \leq \delta^{\frac{2}{n+1}}$

Does there exit a convex body $K = K(f)$

s.th. $|K \setminus K_\delta| \approx f(\delta)$

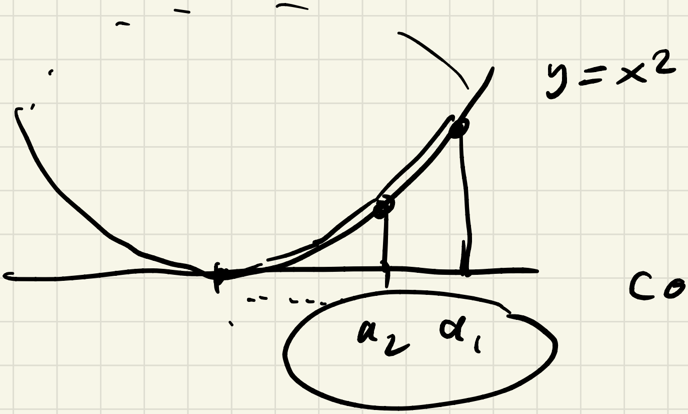
We only know the answer in dimension 2 $\rightarrow$ Schütt + W

almost polygonal bodies

works for $\boxed{f(\delta) = \delta^\beta}$ $\frac{2}{3} \leq \beta < 1$

$\downarrow$

$K(f)$



$$\mathrm{co}\left[(a_i, a_i^2) : i \in \mathbb{N}\right]$$