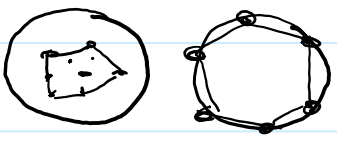
Lecture 4. Beta polytopes.

Def. Beta polytope: $P_{n,d}^{\beta} = [X_1, \dots, X_n]$, where $X_1, X_2, \dots$ are iid points in $\mathbb{R}^d$ with density:

$$f_{d,\beta}(x) = c_{d,\beta} \cdot (1 - \|x\|^2)^{\beta}, \quad \|x\| < 1, \quad \beta > -1.$$

Special cases:



- $\beta = 0$: $\Rightarrow$ uniform on $\mathbb{B}_d$.
- $\beta \downarrow -1$: $f_{d,\beta} \xrightarrow[\beta \downarrow -1]{d} \mathrm{Unif}(\mathbb{S}^{d-1})$.

  Proof. • Helly's compactness: $\exists$ weak subseq. limits.
  - $\forall$ subseq. limit is concentrated on $\mathbb{S}^{d-1}$.
  - $\forall$ subseq. limit is rot. invar.
- $\beta \to +\infty$: If $X_1 \sim f_{d,\beta} \Rightarrow \sqrt{2\beta}\, X_1 \xrightarrow[\beta \to \infty]{d}$ Normal on $\mathbb{R}^d$.

  Proof. Scheffe's lemma:

$$\left(1 - \frac{\|x\|^2}{2\beta}\right)^{\beta} \longrightarrow e^{-\|x\|^2/2}. \qquad \square$$

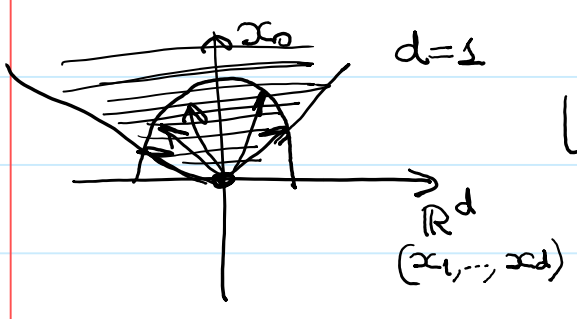
Def. Beta'-polytopes: $\tilde{P}_{n,d}^{\beta} = [\tilde{X}_1, \dots, \tilde{X}_n]$, where $\tilde{X}_1, \tilde{X}_2, \dots$ are iid random points in $\mathbb{R}^d$ with density

$$\tilde{f}_{d,\beta}(x) = \tilde{c}_{d,\beta} \cdot (1 + \|x\|^2)^{-\beta}, \quad x \in \mathbb{R}^d, \quad \beta > \frac{d}{2}.$$

Q. $\mathbb{E} f_k(P_{n,d}^{\beta})$, $\mathbb{E} f_k(\tilde{P}_{n,d}^{\beta}) = ?$

Ex. Random cones in a half-space.

Half-sphere $\mathbb{S}_+^d = \{(x_0, x_1, \dots, x_d) : \sum_{i=0}^{d} x_i^2 = 1, \ x_0 \geq 0\} \subset \mathbb{R}^{d+1}$.



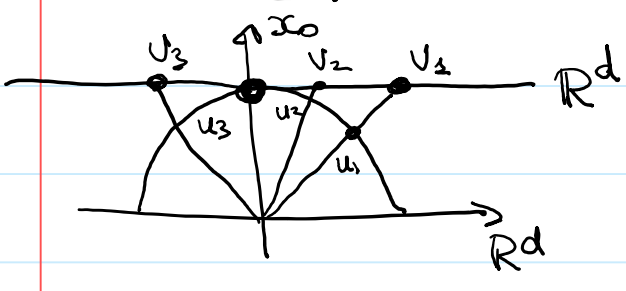
Let $U_1, U_2, \dots$ be iid $\mathrm{Unif}(\mathbb{S}_+^d)$

Consider:

$$C_n = \mathrm{pos}(U_1, \dots, U_n) = \left\{\sum_{i=1}^{n} \lambda_i U_i : \lambda_i \geq 0\right\}.$$

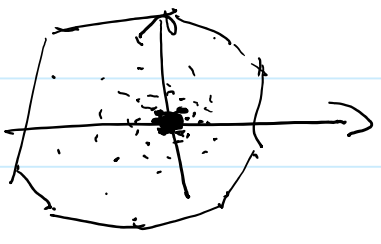
Claim. $\lim_{n \to \infty} \mathbb{E} f_k(C_n)$ exists and is finite.

Explanation.



$V_1, V_2, \dots$ are iid in $\mathbb{R}^d$ with density $\dfrac{c}{(1 + \|x\|^2)^{\frac{d+1}{2}}}$, that is $\tilde{f}_{d,\frac{d+1}{2}}$.

Section of $C_n \longleftrightarrow$ Beta' polytope.



Extreme-value theory.

Consider $V_1/n, V_2/n, \dots$. Density:

$$c \cdot n^d \left(1 + \|nx\|^2\right)^{-\frac{d+1}{2}} \underset{\substack{x \text{ fixed} \\ n \to \infty}}{\sim} \frac{1}{n} \frac{c}{\|x\|^{d+1}}. \quad (x \neq 0).$$

For a Borel set $B \subset \mathbb{R}^d \setminus \{0\}$:

$$\sum_{i=1}^{n} \mathbb{1}_{\frac{V_i}{n} \in B} \sim \mathrm{Bin}\left(n, \frac{1}{n} \int_B \frac{c}{\|x\|^{d+1}} dx\right)$$

$$\xrightarrow[n \to \infty]{d} \mathrm{Poi}\left(\int_B \frac{c}{\|x\|^{d+1}} dx\right).$$

Hence, $\displaystyle\sum_{i=1}^{n} \delta_{V_i/n} \xrightarrow[n \to \infty]{w} \mathrm{PPP}\left(\frac{c}{\|x\|^{d+1}}\right)$ on $\mathbb{R}^d \setminus \{0\}$.

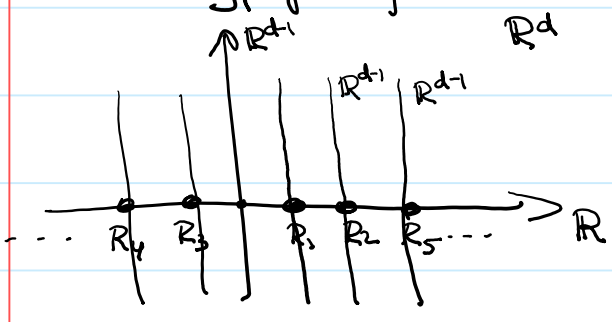
$$\frac{1}{n}[V_1, \dots, V_n] \xrightarrow[n \to \infty]{w} \text{convex hull of } \mathrm{PPP}\left(\frac{c}{\|x\|^{d+1}}\right).$$

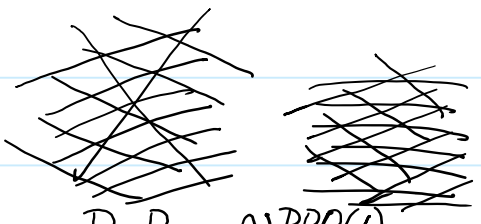
Def. Poisson polyhedron $\Pi_{d,\alpha}$ = convex hull of $\mathrm{PPP}\left(\frac{1}{\|x\|^{d+\alpha}}\right)$, $\alpha > 0$.

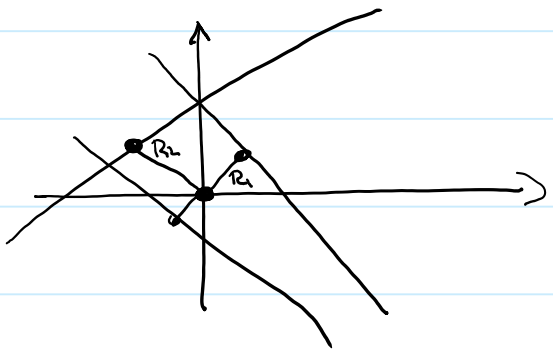
Claim. $f_k(C_n \cap \mathbb{S}_+^d) \xrightarrow[n \to \infty]{d} f_k(\Pi_{d,1})$.

Ex. Poisson hyperplane tessellation.

Poisson hyperplane process. Construction:







① Distances $R_1, R_2, \dots \sim PPP(1)$ on $\mathbb{R}$

② Directions: $\varphi_1, \varphi_2, \dots \sim \mathrm{Unif}(S^{d-1})$.

Everything independent.

Hyperplanes:

$H_i = \{x \in \mathbb{R}^d : \langle x, \varphi_i \rangle = R_i\}$.

Poisson hyperplane process (isotropic).

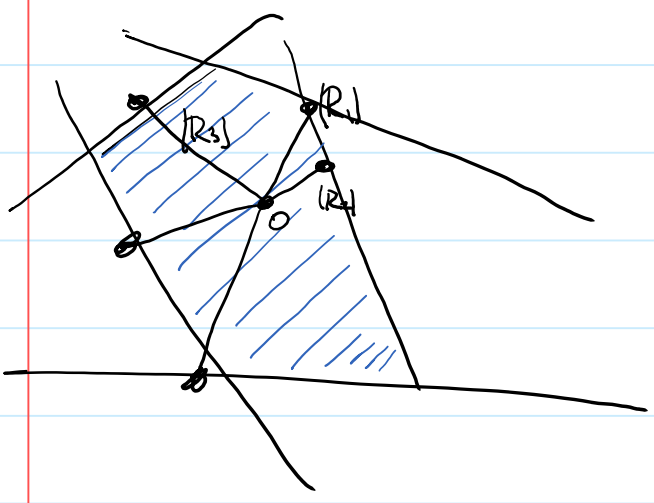
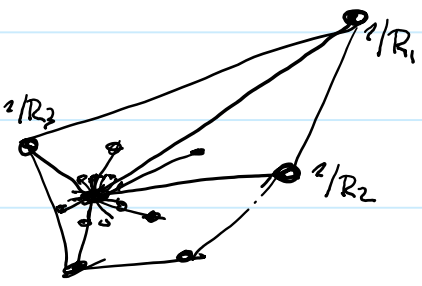
$\{H_i\}_{i=1}^{\infty}$.

Rem. Translation invariance: $\{H_i + x\}_{i=1}^{\infty} \overset{d}{=} \{H_i\}_{i=1}^{\infty}$.

Hyperplanes dissect $\mathbb{R}^d$ into countably many polytopes:

Let Z_d be the polytope that contains 0. (Poisson zero cell)

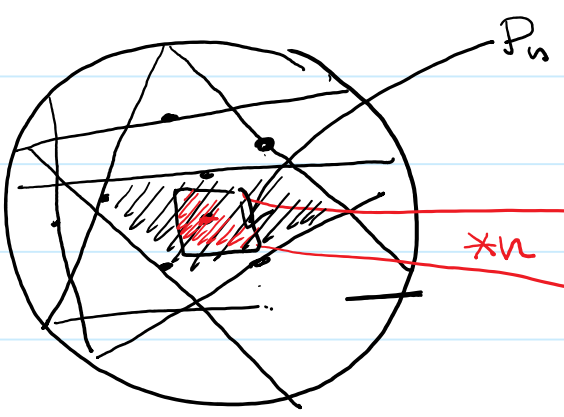
Claim. Z_d is dual to convex hull of $PPP\left(\frac{c}{\|x\|^{d+1}}\right)$.

Polytope P, $0 \in \mathrm{Int}\, P$	Dual P^0,
$P = \{x \in \mathbb{R}^d : \langle x, y_i \rangle \le 1 \ \forall i\}$. $y_1, \dots, y_m \in \mathbb{R}^d \setminus \{0\}$.	$P^0 = \{y \in \mathbb{R}^d : \langle x, y \rangle \le 1 \ \forall x \in P\}$. $y_1, \dots, y_m \in P^0$. $P^0 = [y_1, \dots, y_m]$.
Poisson zero cell: $Z_d = \{x \in \mathbb{R}^d : \langle x, \frac{\varphi_i}{R_i} \rangle \le 1\}$. $\lvert R_1 \rvert, \lvert R_2 \rvert, \dots \sim PPP(2)$ on $\mathbb{R}_+$	$Z_d^0 = \left[\frac{\varphi_i}{R_i}, i = 1, 2, \dots\right]$.
	 $\frac{1}{\lvert R_1 \rvert}, \frac{1}{\lvert R_2 \rvert}, \dots \sim PPP\left(\frac{c}{s^2}\right)$ on $\mathbb{R}_+$. $\frac{\varphi_1}{R_1}, \frac{\varphi_3}{R_2}, \dots \sim PPP\left(\frac{c}{\|x\|^{d+1}}\right)$ $\mathbb{R}^d \setminus \{0\}$

Bijection $\mathcal{F}_k(P) \longleftrightarrow \mathcal{F}_{d-k-1}(P^0)$.

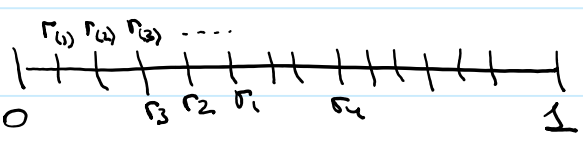
facet $\longleftrightarrow$ vertex.

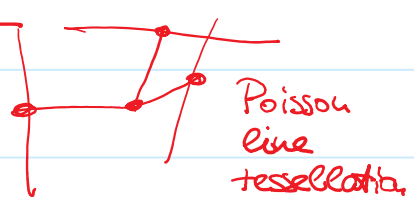
Special case: Renyi-Sulanke (1962).



Observed:

$\mathbb{E} f_0(P_n) \xrightarrow[n \to \infty]{} \pi^2/2$.





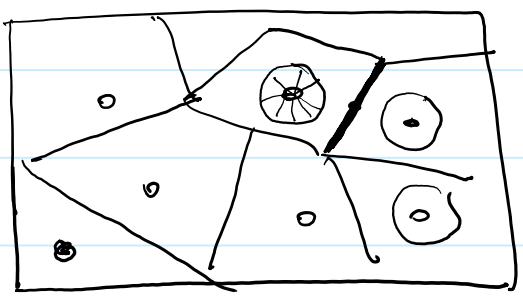
Known:

$\sum_{i=1}^{n} \delta_{r_{(i)} \cdot n} \xrightarrow{d} PPP(1)$ on $\mathbb{R}_+$.

$n P_n \xrightarrow{d} Z_2$. $\qquad \mathbb{E} f_0(Z_2) = \pi^2/2$.

Ex, Poisson-Voronoi tessellation

Consider PPP(1) on $\mathbb{R}^d$. Points: $P_1, P_2, \dots$



Crystal of P_i

$C_i = \{x \in \mathbb{R}^d : \|x - P_i\| \le \|x - P_j\| \ \forall j \in \mathbb{N}\}$.

Typical cell of the Poi-Vor. tess:

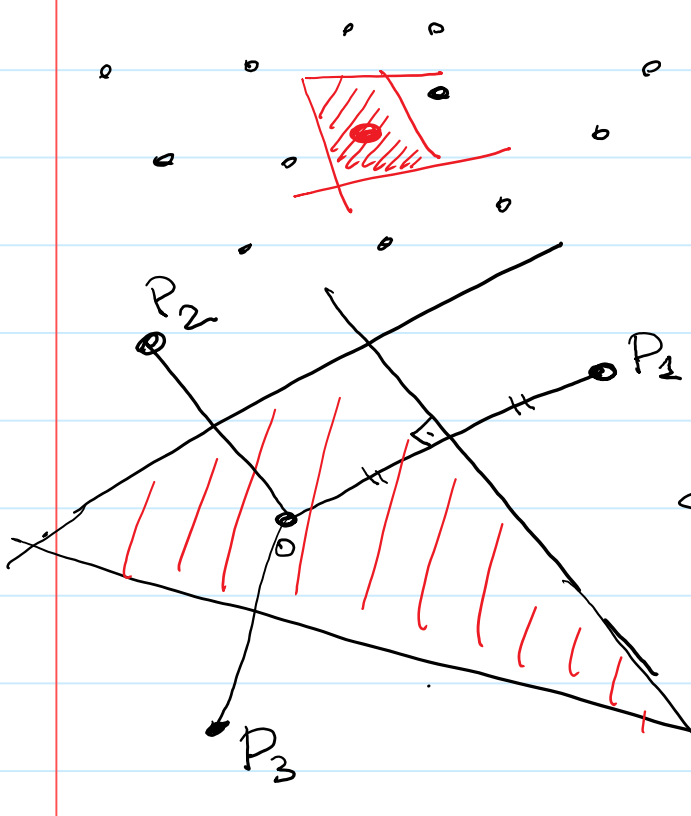
Def. (of typical cell using Palm approach).

Typical cell is the random polytope V_d:

$$\mathbb{P}[V_d \in A] = \frac{1}{\mathrm{Vol}_d(Q)} \mathbb{E} \sum_{\substack{i \in \mathbb{N} \\ P_i \in Q}}^{!} \mathbb{1}_{C_i - P_i \in A}.$$

$Q \subset \mathbb{R}^d$ Borel, $\mathrm{Vol}_d(Q) > 0$.

In our case: $V_d \overset{d}{=} \{x \in \mathbb{R}^d : \|x\| \le \|x - P_i\| \ \forall i\}$.



$\xleftrightarrow{\text{dual}}$ $\mathrm{PPP}\left(\frac{c}{\|x\|^{2d}}\right)$

$\Pi_{d,d}$.

Results on expected face numbers.

For the Poisson hyperplane tess:

- $\mathbb{E} f_k(\text{typical cell}) = f_k([0,1]^d)$. Miles, 1960's
- $\mathbb{E} f_{d-k}(\text{zero cell}) = \frac{\pi^k}{k!} A[d,k]$

$$A[d,k] = [x^k]\left(1 + (d-1)^2 x^2\right)\left(1 + (d-3)^2 x^2\right)\left(1 + (d-5)^2 x^2\right)\dots$$

if k is even.

In the Poisson-Voronoi tess:

- $\mathbb{E} f_k(\text{zero cell}) = ?$
- $\mathbb{E} f_{d-k}(V_d) = d^d \binom{d}{k} \left(\frac{\sqrt{\pi}\,\Gamma(d/2)}{\Gamma\left(\frac{d+1}{2}\right)}\right)^k \cdot$

$$\cdot [x^{-1}] \frac{\left(\int_0^x (\sin y)^{d-1} dy\right)^{d-k}}{(\sin x)^{d^2+1}}, \text{ if } dk \text{ is even.}$$

- $\mathbb{E} f_k(P^{\beta}_{n,d})$, $\mathbb{E} f_k(\tilde{P}^{\beta}_{n,d})$.

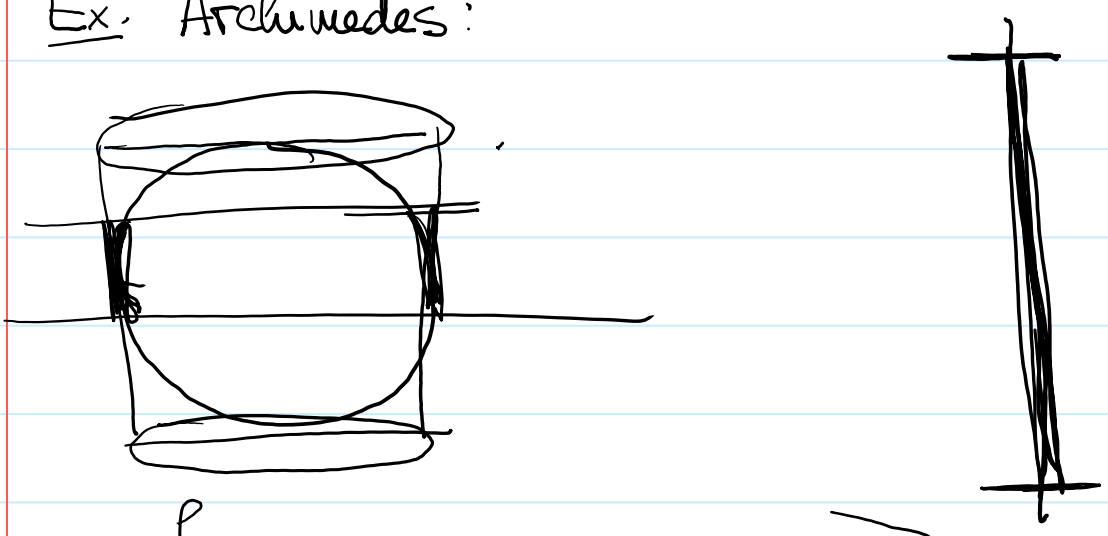
Properties of Beta-distributions.

① Invariance under projections:

$\Pi : \mathbb{R}^d \to \mathbb{R}^k$ orthogonal projection.

$X_1 \sim f_{d,\beta} \Rightarrow \Pi X_1 \sim f_{k, \beta + \frac{d-k}{2}}$

Ex. Archimedes:

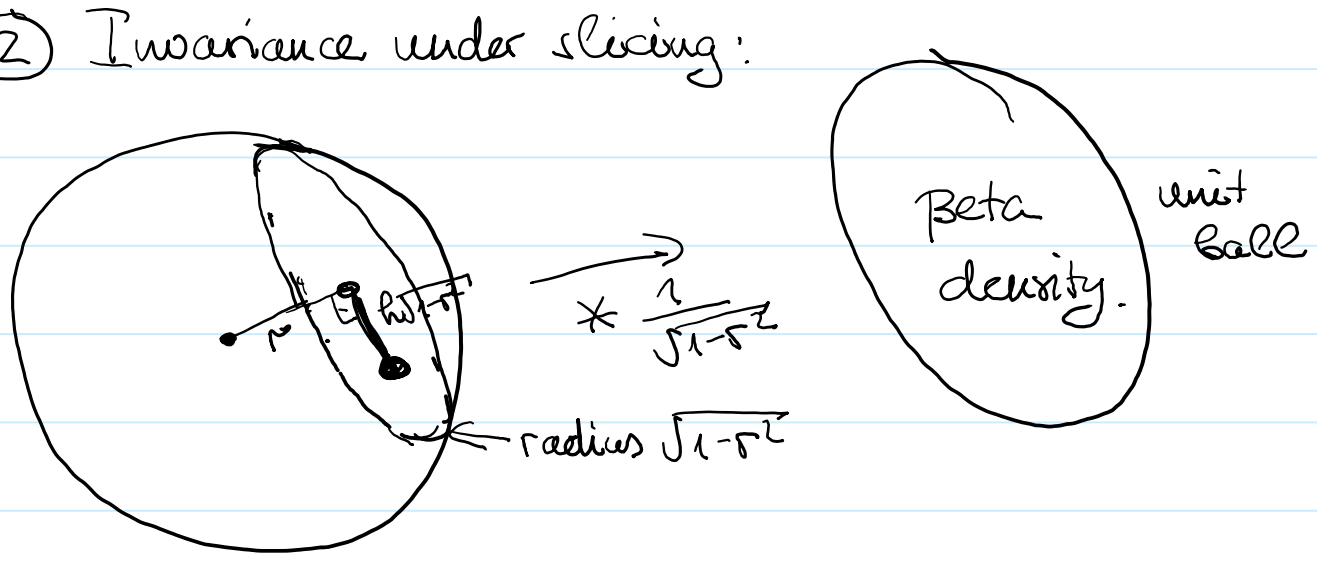


$f_{3,-1} \longrightarrow f_{1,0}$

2 dimensions are lost

Unif $(S^2) \longrightarrow$ Unif (segment).

② Invariance under slicing:



$$p(r) = (1-r^2)^\beta$$

$$p\left(\sqrt{r^2 + h^2(1-r^2)}\right) = \left(1 - r^2 - h^2(1-r^2)\right)^\beta$$

$$= \underbrace{(1-r^2)^\beta}_{\text{const}} \cdot (1-h^2)^\beta.$$