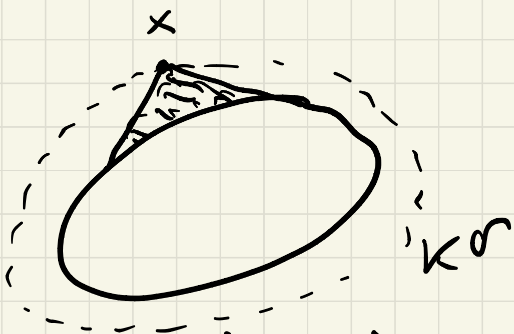
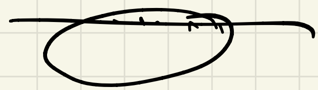
dual concept to floating body

illumination body (W)



$$|[x,K]\setminus K| \le \delta$$



$$K^\delta = \{x : |[x,K]\setminus K| \le \delta\}$$

K^δ is convex

<u>W</u>

$$\lim_{\delta\to 0} \frac{|K^\delta| - |K|}{\delta^{\frac{2}{n+1}}} = d_n \, as(K)$$

L_p- affine surface $\quad -\infty \le p \le +\infty$
$p \ne -n$

$$as_p(K) = \int_{\partial K} \frac{\kappa^{\frac{p}{n+p}}}{\left(\langle x, N(x)\rangle\right)^{\frac{n(p-1)}{n+p}}} \, d\mu_K \qquad 0 \in \mathrm{int}(K)$$

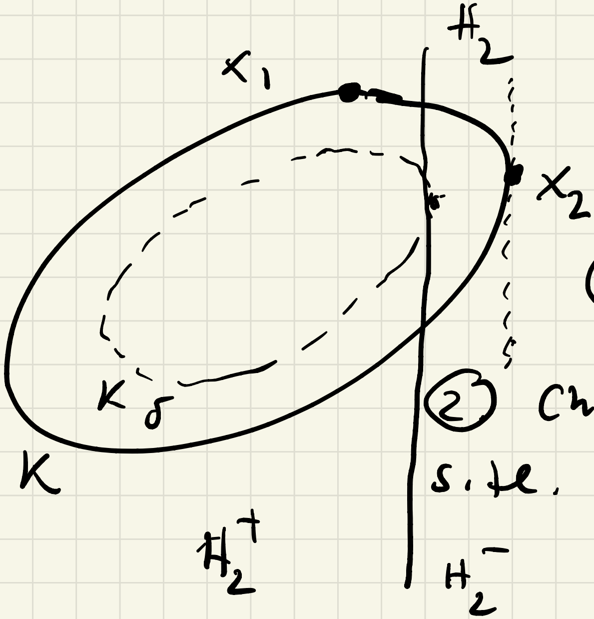
$p=1 \longrightarrow as_1(K) = as(K)$

$$p=0: \quad as_0(K) = \int_{\partial K} \langle x, N(x)\rangle \, d\mu_K$$
$$= n\,|K|$$

$$p=\pm\infty: \quad as_{\pm\infty}(K) = n\,|K^\circ|$$

II. Approximation of convex bodies by polytopes

II.1 The floating body algorithm (Schütt)



K is given

K_δ is given

① Choose $x_1 \in \partial K$

② Choose a hyperplane H_2 s.th. $x_1 \in \operatorname{int}(H_2^+)$

$K_\delta \subseteq H_2^+$

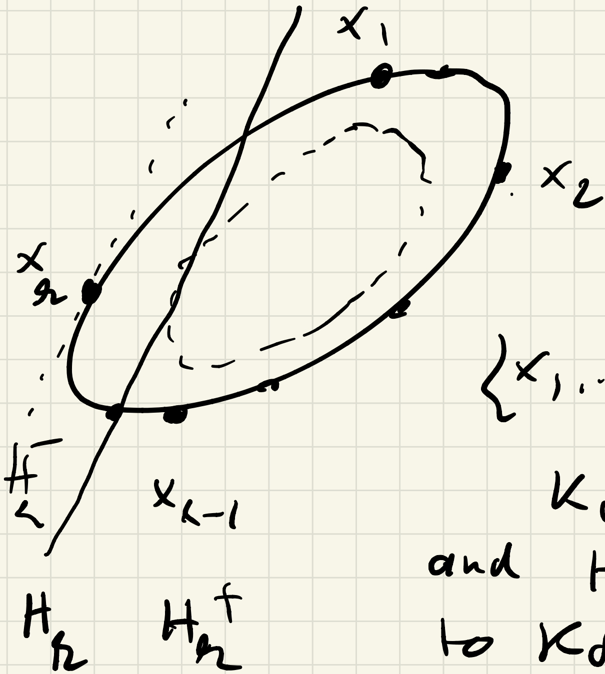
③ Choose $x_2 \in \partial K \cap H_2^-$ s.th. $N_K(x_2)$ is normal to H_2

By choice of H_2: $|H_2^- \cap K| \geq \delta$

⋮

Suppose we have chosen $x_1, \dots x_{k-1}$.

Now chose x_k



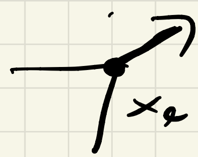
we choose a hyperplane H_k s.t.

$$\{x_1, \dots x_{k-1}\} \subseteq H_k^+$$

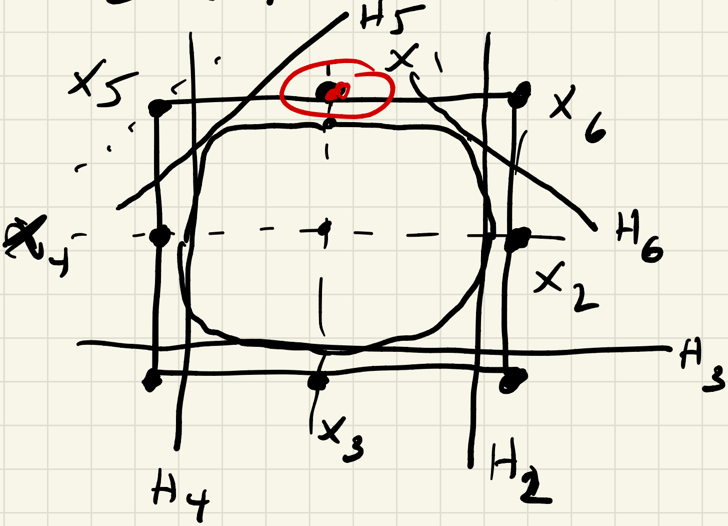
$$K_\delta \subseteq H_k^+$$

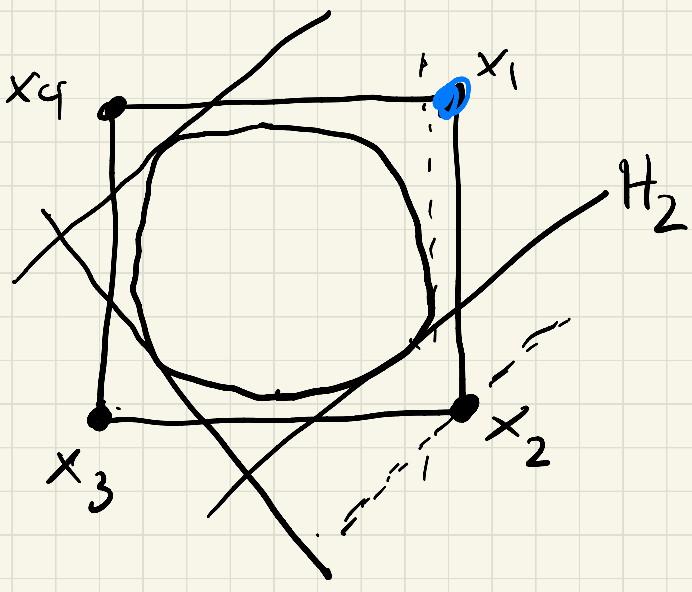
and H_k is a support hyperplane to K_δ

and then we choose a point $x_k \in \partial K \cap H_k^-$ s.t. $\underline{N_K(x_k)}$ is a normal to H_k

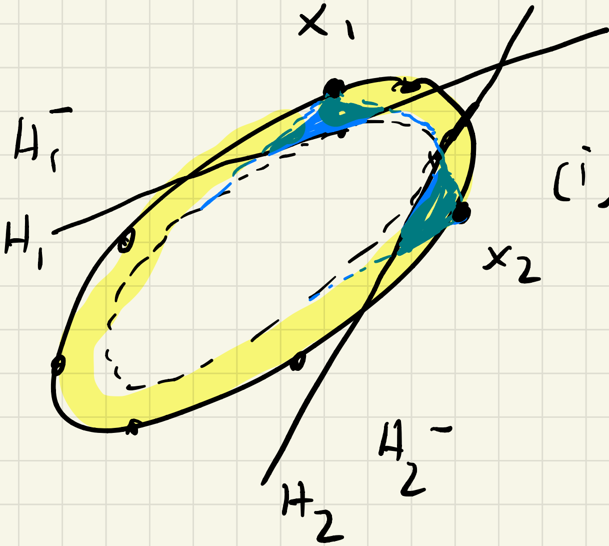


Example of square $= B^2_\infty$





How many points can we choose?



Note

(i) $[x_k, K_\delta] \cap H_k^- \cap$

$[x_{k+1}, K_\delta] \cap H_{k+1}^-$

$= \phi$

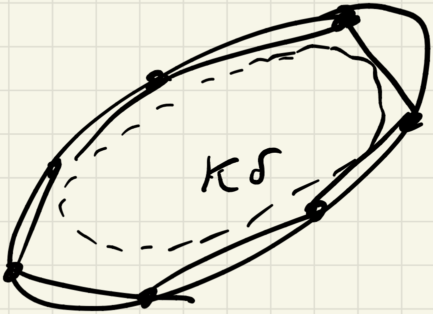
(ii) $\left(|[x_k, K_r] \cap H_k^-| \right) \approx \delta$

How many points can we choose?

$$N \cdot \delta = \sum_{k=1}^{N} |[x_k, K_r] \cap H_k^-|$$

$$= \left| \bigcup_{k=1}^{N} \left([x_k, K_\delta] \cap H_k^- \right) \right|$$

$$\approx |K \setminus K_\delta| \Rightarrow N \approx \frac{|K \setminus K_\delta|}{\delta}$$



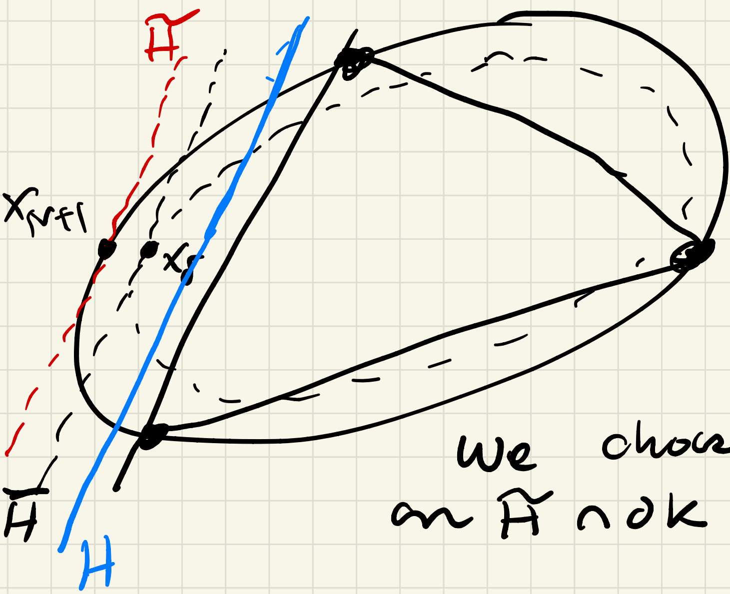
$P_N = [x_1, \dots x_N]$

P_N "approximates" K

Note $\boxed{K_\delta \subseteq P_N} \subseteq K$

We show: $K_\delta \subseteq P_N$.

Suppose not. Then there is $x_\delta \in K_\delta$, $x_\delta \notin P_N$



By separation theorem $\exists$ a hyperplane H s.t.

$x_\delta \in H^-$ and $P_N \subseteq H^+$

We choose a point x_{N+1} on $\tilde{H} \cap K$ $\Downarrow$

Theorem (Schütt)

For all $\delta \leq \frac{|K|}{4e^4}$ there is a $N \in \mathbb{N}$

s.th.

$$N \leq e^{16n} \frac{|K \setminus K_\delta|}{\delta |B_2^n|}$$

and there is a polytope P_N with at most N vertices s.th.

$$\boxed{K_\delta \subseteq P_N \subseteq K}$$

We want to see how good this approximation is:

Let K be s.th. $as(K) > 0$

$$\frac{|K \setminus K_\delta|}{\delta^{\frac{2}{n+1}}} \xrightarrow{\delta \to 0} \frac{1}{2}\left(\frac{n+1}{|B_2^{n-1}|}\right)^{\frac{2}{n+1}} \underline{as(K)}$$

$$\approx \boxed{c \cdot n} \; as(K)$$

By the theorem there $N \approx \dfrac{e^{16n}\,|K\setminus K_\delta|}{\delta\,|B_2^n|}$

and a polytope P_N with at most N vertices s.t. $K_\delta \subseteq P_N \subseteq K \implies$

$$\boxed{\frac{|K\setminus P_N|}{N^{-\frac{2}{n-1}}}} \le \frac{|K\setminus K_\delta|}{N^{-\frac{2}{n-1}}} \approx \frac{|K\setminus K_\delta|^{\frac{n+1}{n-1}}\; e^{32\frac{n}{n-1}}}{\delta^{\frac{2}{n-1}}\,|B_2^n|^{\frac{2}{n-1}}}$$

$$\simeq \left(\frac{|K\setminus K_\delta|}{\delta^{\frac{2}{n+1}}}\right)^{\frac{n+1}{n-1}} \cdot c \cdot n$$

$$\le c\, n^2\, \mathrm{as}(K)^{\frac{n+1}{n-1}}$$

How good is this result?

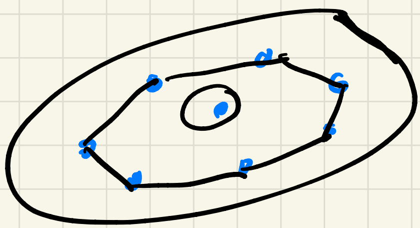
II.1 Random polytopes

a random polytope is the convex hull of N randomly i.i.d chosen points in $\mathbb{R}^n$ with respect to a probability measure

$$P_N = [X_1, \dots X_N]$$

typical settings

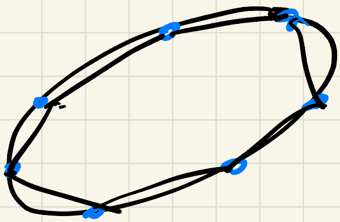
(i) N points are chosen at random w.r.t. to $\frac{m}{|K|}$ in a convex body K



→ not all points chosen become vertices of P_N

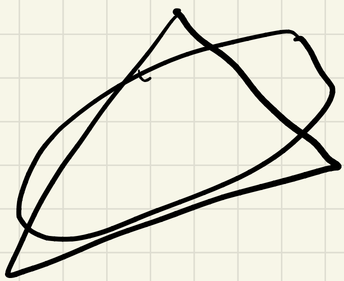
we have wasted points

(ii) Choose the points on ∂K instead



The setting (i), (ii) produces approximating polygons inside K

(iii) arbitrary positioned polygons



Case (ii)

We will choose N points on the boundary of a convex body K with respect to $\mathbb{P}_f = f \cdot d\mu$ where f is a continuous strictly positive function s.t. $\int_{\partial K} f \, d\mu_K = 1$

The expected volume of such a random polytope is

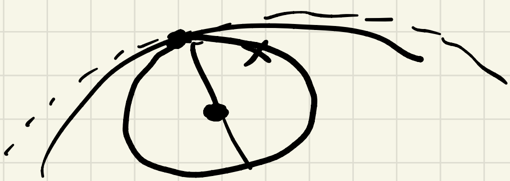
$$\underline{\mathbb{E}_{\underline{N}}(\partial K, \mathbb{P}_f)} = \int_{\partial K} \cdots \int_{\partial K} \left| [x_1 \dots x_N] \right| \, d\mathbb{P}_f(x_1) \cdots d\mathbb{P}_f(x_N)$$

Theorem (Schütt, W; Reitzner; K unit ball J. Müller)

Let K be a convex body in $\mathbb{R}^n$ s.t.

$\exists\ 0 < r \le R < \infty$ s.t. $\forall x \in \partial K$

$$B_2^n(x - rN(x), r) \subseteq K \subseteq B_2^n(x - RN(x), R)$$



Then

$$\lim_{N \to \infty} \frac{|K| - \mathbb{E}_N(\partial K, \mathbb{P}_f)}{N^{-\frac{2}{n-1}}} = \overset{\sim n}{d_n} \int_{\partial K} \frac{\kappa^{\frac{1}{n-1}}}{f^{\frac{2}{n-1}}} \, d\mu_K$$

Remarks

- we know d_n explicitly
- When the points are chosen in K

$$|K| - \mathbb{E}_N\left(K, \frac{m}{|K|}\right) \overset{N \to \infty}{\sim} \beta_n \, as(K) \left(\frac{|K|}{N}\right)^{\frac{2}{n+1}}$$

Q: For which f is the RHS minimal

A: This happens for $\underline{f_a}(x) = \dfrac{\kappa^{\frac{1}{n+1}}(x)}{as(K)}$

$$\lim_{N\to\infty} \frac{|K| - \mathbb{E}_N(\partial K, \mathbb{P}_{f_a})}{N^{-\frac{2}{n-1}}} = d_n$$

$$\int \frac{\kappa^{\frac{1}{n-1}} \, d\mu_K}{\left(\kappa^{\frac{1}{n+1}}\right)^{\frac{2}{n-1}}} \quad as(K)^{\frac{2}{n-1}}$$

$$= d_n \, as(K)^{\frac{n+1}{n-1}}$$

<u>floating body algorithm:</u>

$$\frac{|K \setminus P_N|}{N^{-\frac{2}{n-1}}} \le c \cdot \underline{n^2} \, \underline{as(K)^{\frac{n+1}{n-1}}}$$

One can show: $d_n \simeq n$

Random polytopes

$$\frac{|K| - \mathbb{E}_N(\partial K, \mathbb{P}_{f_a})}{N^{-\frac{2}{n-1}}} \xrightarrow{N\to\infty} \approx \underline{d_n}\, \underline{as(K)^{\frac{n+1}{n-1}}}$$

n

Compare this to the best approximating polytope

Theorem (McClure, Vitale $n=2$
Gruber $n \geq 3$

K is C^2_+

$$\frac{|K| - |P_N^{best}|}{N^{-\frac{2}{n-1}}} \xrightarrow{N\to\infty} \approx \underline{\alpha_n}\, \underline{as(K)^{\frac{n+1}{n-1}}}$$

$$\alpha_n \leq d_n \leq \alpha_n \left(1 + \frac{\log n}{n}\right)$$

Mankiewicz, Schütt $\alpha_n \approx n$

Reiter, Schütt, W

What happens when we choose points at random on the boundary of polytope

Thank you!