Geometry of knots and links

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Low-dimensional topology
Saint Petersburg
April 25, 2022

Three geometries of constant curvature

1. Euclidean geometry $\mathbb{E}^3$, $K = 0$ (Euclid)
2. Spherical geometry $\mathbb{S}^3$, $K > 0$ (before Euclid ?) Studed by Riemann
3. Hyperbolic geometry $\mathbb{H}^3$, $K < 0$ (N.I. Lobachevsky and Janos Bolyai)
4. Eight geometries by William Thurston:

$$\mathbb{E}^3,\ \mathbb{S}^3,\ \mathbb{H}^3,\ \mathbb{S}^2 \times \mathbb{R},\ \mathbb{H}^2 \times \mathbb{R},\ \mathrm{Nil},\ \mathrm{Solv}\ \text{and}\ \widetilde{\mathrm{PSL}}(2,\mathbb{R}).$$

Thurston's geometrization conjecture: Any three dimensional manifold can be decomposed into pieces, each modeled in one of the eight above mentioned geometries.

This conjecture was proved by Grigori Perelman in 2003. As as consequence, he proved the famous **Poincaré conjecture**:
Any closed three dimensional manifold with the trivial fundamental group is the sphere.

Three dimensional hyperelliptic manifolds

The notion of *hyperelliptic surface* originally comes from complex analysis. Riemann surface is said to be *hyperelliptic* if it can be repesented as a two fold branched covering of the sphere. To create its three dimensional counterpart consider a 3-manifold M and suppose that there exists an involution $\tau : M \to M$ such that the factor space $M/\langle\tau\rangle$ is homeomarphic to a three dimensional sphere S^3.

In this case, τ is called a *hyperelliptic involution*, and M is called a *hyperelliptic manifold*. If M is endowed by geometrical structure, we suppose that τ is an isometry. The factor space $M/\langle\tau\rangle$ is the 3-sphere with singular locus formed by knot of link.

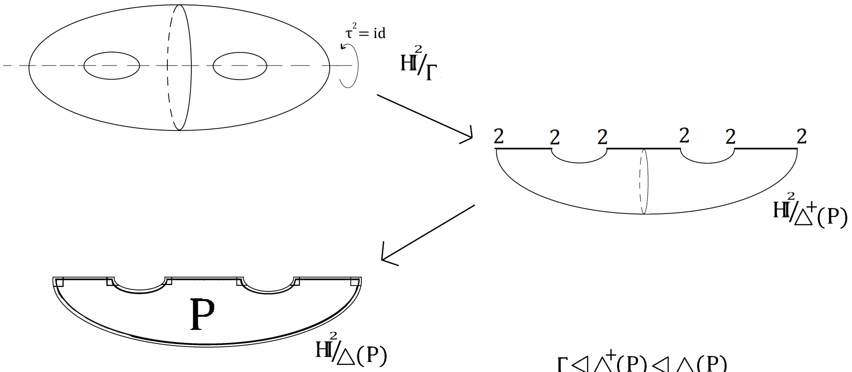
We claim that there exist three dimensional hyperelliptic manifolds in all the eight Thurston geometries. The general construction was described in my paper (M., 1990).

Theorem

There exists a hyperelliptic manifold in each of the eight Thurston geometries $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \oplus \mathbb{R}^1$, $\mathbb{H}^2 \oplus \mathbb{R}^1$, Nil, Solv, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

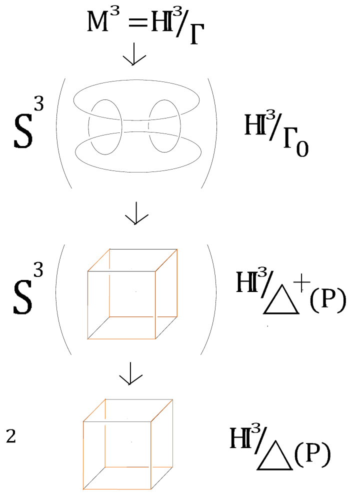
In the recent paper A. Mednykh and B. Vuong "On hyperelliptic Euclidean 3-manifolds"(2020) we proved that up to homeomorphism there are exactly five hyperelliptic manifolds modelled in the Eucliden geometry $\mathbb{E}^3$.

Three dimensional hyperelliptic manifolds



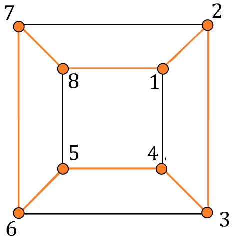
$$\Gamma \underset{2}{\lhd} \Delta^+(P) \lhd \Delta(P)$$

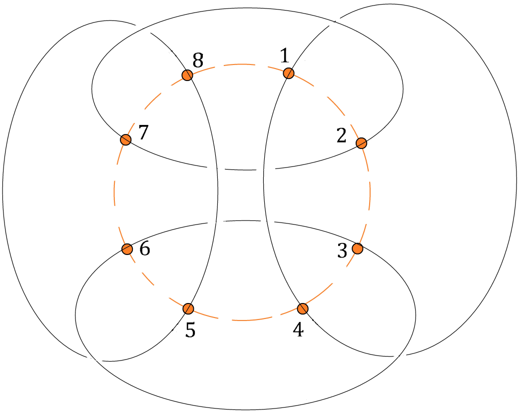
Three dimensional hyperelliptic manifolds



$$\Gamma \mathrel{\underset{2}{\lhd}} \Gamma_0 \mathrel{\underset{2}{\lhd}} \Delta^+(P) \mathrel{\underset{2}{\lhd}} \Delta(P)$$

Three dimensional hyperelliptic manifolds





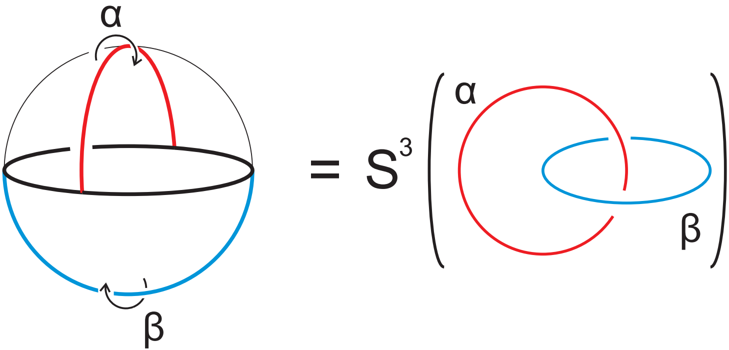
Cone–manifold and geometry knots

Following (D. Cooper, C.D. Hodgson and S.P. Kerckhoff, 2000) we introduce the basic definitions from the cone-manifold theory.

Definition

An *n-dimensional cone-manifold* is a manifold, M, which can be triangulated so that the link of each simplex is piecewise linear homeomorphic to a standard sphere and M is equipped with a complete path metric such that the restriction of the metric to each simplex is isometric to a geodesic simplex of constant curvature K. The cone-manifold is hyperbolic, Euclidean or spherical if K is $-1, 0$, or $+1$ respectively.

The *singular set* Σ of a cone-manifold M consists of the points with no neighbourhood isometric to a ball in a Riemannian manifold.
In the present paper, we will deal only with cone-manifolds whose underlying space M is the three dimensional sphere $\mathbb{S}^3$ and singular set Σ is a knot or link.



Hopf link cone-manifold $2_1^2(\alpha, \beta)$.

Schläfli formula

The main tool for volume calculation is the following Schläfli formula. Let M be a 3-dimensional cone–manifold of constant curvature $K = \pm 1$. Then its volume V is a solution of the differential equation

$$KdV = \frac{1}{2}\sum_i \ell_{\alpha_i} d\alpha_i,$$

where the sum is taken over all components of the singular set Σ with lengths l_{α_i} and cone-angles α_i.

- ⋆ In the above case of Hopf link we have $K = +1$, $\ell_\alpha = \beta, \ell_\beta = \alpha$. Hence $dV = \frac{1}{2}(\beta d\alpha + \alpha d\beta)$ and $V = \frac{\alpha\beta}{2}$.

Geometry of two bridge knots. Trefoil knot.

Let $\mathcal{T}(\alpha) = 3_1(\alpha)$ be a cone–manifold whose underlying space is the three-dimensional sphere $\mathcal{S}^3$ and singular set is trefoil knot $\mathcal{T}$ with cone angle α. See Figure below.

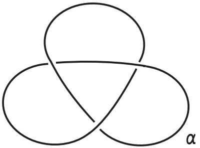


Рис.: Cone-manifold $3_1(\alpha)$

Since $\mathcal{T}$ is a toric knot by the Thurston theorem its complement $\mathcal{T}(0) = \mathcal{S}^3 \setminus \mathcal{T}$ in the $\mathcal{S}^3$ does not admit a hyperbolic structure. However, the trefoil knot admits othrer geometric structures. By H. Seifert and C. Weber (1933) the spherical space of dodecahedron (also known as the Poincaré homology 3-sphere) is a cyclic 5-fold covering of $\mathcal{S}^3$ branched over $\mathcal{T}$. This means that cone–manifold $3_1(\frac{2\pi}{5})$ has a spherical structure.

Geometry of two bridge knots. Trefoil knot.

Note that $\mathcal{T}(2\pi/n)$, $n \in \mathbb{N}$ is a geometric orbifold, so it can be represented in the form $\mathbb{X}^3/\Gamma$, where $\mathbb{X}^3$ is one of the eight three-dimensional homogeneous geometries and Γ is a discrete group of isometries of $\mathbb{X}^3$. By Dunbar classification (1983) of non-hyperbolic orbifolds, $\mathcal{T}(2\pi/n)$ has a spherical structure for $n \leq 5$, *Nil* for $n = 6$ and $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ for $n \geq 7$. Quite surprising situation appears in the case of the trefoil knot complement $\mathcal{T}(0)$. By P. Norbury (see Appendix A in the lecture notes by W. P. Neumann), the manifold $\mathcal{T}(0)$ admits two geometrical structures $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$.

Geometry of two bridge knots. Trefoil knot.

The following theorem describes a spherical structure on the trefoil cone-manifold.

Theorem

The trefoil cone-manifold $\mathcal{T}(\alpha)$ is spherical for $\frac{\pi}{3} < \alpha < \frac{5\pi}{3}$. The spherical volume of $\mathcal{T}(\alpha)$ is given by the formula

$$\mathrm{Vol}\,(\mathcal{T}(\alpha)) = \frac{(3\alpha - \pi)^2}{12}.$$

For the proof consider $\mathbb{S}^3$ as the unite sphere in the complex space $\mathbb{C}^2$ endowed by the Riemannian metric

$$\mathrm{d}s_\lambda^2 = |\mathrm{dz}_1|^2 + |\mathrm{dz}_2|^2 + \lambda(\mathrm{dz}_1\mathrm{d}\bar{\mathrm{z}}_2 + \mathrm{d}\bar{\mathrm{z}}_1\mathrm{dz}_2),$$

where $\lambda = (2\sin\frac{\alpha}{2})^{-1}$. Then $\mathbb{S}^3 = (\mathbb{S}^3, \mathrm{d}s_\lambda^2)$ is the spherical space of constant curvature $+1$.

The fundamental set for $\mathcal{T}(\alpha)$ is given by the following polyhedron

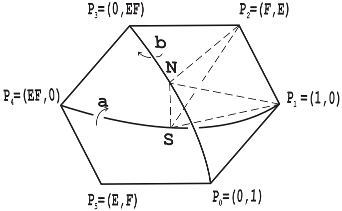


Рис.: Fundamental set for $\mathcal{T}(\alpha)$

where $E = e^{i\,\alpha}$ and $F = e^{i\,\frac{\alpha-\pi}{2}}$ (see Figure 2). The length ℓ_α of singular geodesic of $\mathcal{T}(\alpha)$ is given by $\ell_\alpha = |P_0P_3| + |P_1P_4| = 3\alpha - \pi$.
By the Schläfli formula $\mathrm{dVol}\,\mathcal{T}(\alpha) = \frac{\ell_\alpha}{2}\mathrm{d}\alpha = \frac{3\alpha-\pi}{2}\mathrm{d}\alpha$. So,
$\mathrm{Vol}\,\mathcal{T}(\alpha) = \frac{(3\alpha-\pi)^2}{12} + C$, where C is a constant of integration.
Recall that a 2-fold cover of orbifold $3_1(\pi)$ is the lens space $L(3,1)$ which, in turn, is thrice covered by the three dimensional sphere $\mathbb{S}^3$. Since the spherical volume of $\mathbb{S}^3$ is $2\pi^2$, we have $\mathrm{Vol}\,\mathcal{T}(\pi) = 2\pi^2 : 6 = \frac{\pi^2}{3}$.
Therefore, $C = 0$ and $\mathrm{Vol}\,\mathcal{T}(\alpha) = \frac{(3\alpha-\pi)^2}{12}$.

Geometry of two bridge knots. 4_1– knot.

The figure eight knot or 4_1 knot is the unique prime knot of four crossings.

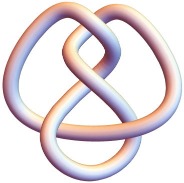


Рис.: Figure eight knot 4_1

Geometry of two bridge knots. 4_1– knot.

It was shown in Thurston lecture notes (1990) that the figure eight knot compliment $\mathbb{S}^3 \setminus 4_1$ can be obtained by gluing two copies of a regular ideal tetrahedron. Thus, $\mathbb{S}^3 \setminus 4_1$ admits a complete hyperbolic structure. Independently, the existence of the complete hyperbolic structure on the complement of the figure eight knot was proved by R. Riley in his unpublished manuscript. Later, it was discovered by A.C. Kim, H. Helling and J. Mennicke (1998) that the *n*-fold cyclic coverings of the 3-sphere branched over 4_1 produce beautiful examples of the hyperbolic Fibonacci manifolds.

Geometry of two bridge knots. 4_1– knot.

The following result takes a place due to W.P. Thurston, H.M. Hilden, M.T. Lozano, J.M. Montesinos (1998), S. Kojima (1998), A.A. Rasskazov and A.D. Mednykh (2006).

Theorem

A cone-manifold $4_1(\alpha)$ is hyperbolic for $0 \leq \alpha < \alpha_0 = 2\pi/3$, Euclidean for $\alpha = \alpha_0$ and spherical for $\alpha_0 < \alpha < 2\pi - \alpha_0$.

Other geometries on the figure eight cone-manifold were studied by C. Hodgson, W. Dunbar, E. Molnár, J. Szirmai and A. Vesnin (2009).

The volume of the figure eight cone-manifold in the spaces of constant curvature is given by the following theorem.

Theorem 6 (A. Rasskazov and M., 2006)

Let $\mathrm{V}(\alpha) = \mathrm{Vol}\ 4_1(\alpha)$ *and* ℓ_α *is the length of singular geodesic of* $4_1(\alpha)$. *Then*

$(\mathbb{H}^3)$ $\mathrm{V}(\alpha) = \int_\alpha^{\alpha_0} \mathrm{arccosh}\,(1 + \cos\theta - \cos 2\theta)\mathrm{d}\theta,\ 0 \le \alpha < \alpha_0 = \frac{2\pi}{3},$

$(\mathbb{E}^3)$ $\mathrm{V}(\alpha_0) = \frac{\sqrt{3}}{108}\,\ell_{\alpha_0}^3,$

$(\mathbb{S}^3)$ $\mathrm{V}(\alpha) = \int_{\alpha_0}^{\alpha} \mathrm{arccos}\,(1 + \cos\theta - \cos 2\theta)\mathrm{d}\theta,\ \alpha_0 < \alpha \le \pi,\quad \mathrm{V}(\pi) = \frac{\pi^2}{5},$

$\mathrm{V}(\alpha) = 2\mathrm{V}(\pi) + \pi(\alpha - \pi) - \mathrm{V}(2\pi - \alpha),\ \pi \le \alpha < 2\pi - \alpha_0.$

The following fundamental polyhedron can be realized in each of three spaces of constant curvature $\mathbb{H}^3$, $\mathbb{S}^3$, and $\mathbb{E}^3$.

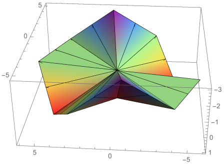
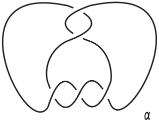


Рис.: Fundamental set for a two bridge knot

- The three twist knot 5_2

The knot 5_2 is a rational knot of a slope $7/2$.



Historically, it was the first knot which was related with hyperbolic geometry. Indeed, it has appeared as a singular set of the hyperbolic orbifold constructed by L.A. Best (1971) from a few copies of Lannér tetrahedra with Coxeter scheme $\circ \equiv \circ - \circ = \circ$. The fundamental set of this orbifold is a regular hyperbolic cube with dihedral angle $2\pi/5$. Later, R. Riley (1979) discovered the existence of a complete hyperbolic structure on the complement of 5_2. In his time, it was one of the nine known examples of knots with hyperbolic complement.

A few years later, it has been proved by W. Thurston that all non-satellite, non-toric prime knots possess this property. Just recently it became known (2007) that the Weeks-Fomenko-Matveev manifold $\mathcal{M}_1$ of volume 0.9427... is the smallest among all closed orientable hyperbolic three manifolds. We note that $\mathcal{M}_1$ was independently found by J. Przytycki and his collaborators (1986). It was proved by A. Vesnin and M. (1998) that manifold $\mathcal{M}_1$ is a cyclic three fold covering of the sphere $\mathbb{S}^3$ branched over the knot 5_2. It was shown by J. Weeks computer program Snappea and proved by Moto-O Takahahsi (1989) that the complement $\mathbb{S}^3 \setminus 5_2$ is a union of three congruent ideal hyperbolic tetrahedra.

Geometry of two bridge knots. 5_2- knot.

The next theorem has been proved by A. Rasskazov and M. (2002), R. Shmatkov (2003) and J. Porti (2004) for hyperbolic, Euclidean and spherical cases, respectively.

Theorem

A cone manifold $5_2(\alpha)$ *is hyperbolic for* $0 \leq \alpha < \alpha_0$, *Euclidean for* $\alpha = \alpha_0$, *and spherical for* $\alpha_0 < \alpha < 2\pi - \alpha_0$, *where* $\alpha_0 \simeq 2.40717...$ *and* $A_0 = \cot(\frac{\alpha_0}{2})$ *is given by the formula*

$$A_0 = \sqrt{1/23(-17 - 8\sqrt{2} + 2\sqrt{-235 + 344\sqrt{2}})}.$$

Geometry of two bridge knots. 5_2- knot.

Theorem 8 (A. Mednykh, 2009)

Let $5_2(\alpha),\ 0 \le \alpha < \alpha_0$ *be a hyperbolic cone-manifold. Then the volume of* $5_2(\alpha)$ *is given by the formula*

$$\mathrm{Vol}\,(5_2(\alpha)) = i \int_{\bar{z}}^{z} \log\left[\frac{8(\zeta^2 + A^2)}{(1 + A^2)(1 - \zeta)(1 + \zeta)^2}\right] \frac{d\zeta}{\zeta^2 - 1},$$

where $A = \cot\frac{\alpha}{2}$ *and* $z,\ \Im z > 0$ *is a root of equation*

$$8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2.$$

A completely different approach to find volume of the above cone-manifold is contained in our resent paper (Ji-Young Ham, Alexander Mednykh, Vladimir Petrov, 2014).

Geometry of two bridge knots. 5_2- knot.

Spherical volume of the 5_2- knot is given by the following theorem.

Theorem

Let $5_2(\alpha),\ \alpha_0 < \alpha < 2\pi - \alpha_0$ *be a spherical cone-manifold. Then for any* $\alpha,\ \alpha_0 < \alpha < \pi$, *the volume* $V(\alpha)$ *of* $5_2(\alpha)$ *is given by the formula*

$$V(\alpha) = \int_{z_1}^{z_2} \log\left(\frac{8(\zeta^2 + A^2)}{(1 + A^2)(1 - \zeta)(1 + \zeta)^2}\right) \frac{d\zeta}{\zeta^2 - 1},$$

where $A = \cot\frac{\alpha}{2}$ *and* $z_1, z_2, (-1 < z_1 < z_2)$ *are roots of the cubic equation*

$$8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2.$$

Also, $V(\pi) = \pi^2/7$ *and*

$$V(\alpha) = 2\,V(\pi) + \pi(\alpha - \pi) - V(2\pi - \alpha) \text{ for } \pi < \alpha < 2\pi - \alpha_0.$$

How to get a hyperbolic structure?

Let M be a hyperbolic 3-dimensional cone-manifold whose singular set Σ_α is a knot with cone angle α, $0 < \alpha \leq 2\pi$. Choose the canonical longitude-meridian pair (l, m) in the fundamental group $\pi_1(M \setminus \Sigma_\alpha)$ in such a way that m is an oriented boundary of meridian disc of Σ_α and a longitude curve l is nullhomologous outside of Σ_α. Let $h : \pi_1(M \setminus \Sigma_\alpha) \to PSL(2, \mathbb{C})$ be the holonomy map of $M \setminus \Sigma_\alpha$. Then (see F. Gonzalez-Acuña, J. M. Montesinos-Amilibia, 1993) h admits two liftings to $SL(2, \mathbb{C})$. The image of l in $SL(2, \mathbb{C})$ under these two liftings is the same since l is nullhomologous outside the singular set. Thus up to conjugation in $SL(2, \mathbb{C})$,

$$h(m) = \pm \begin{bmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{bmatrix}, h(l) = \begin{bmatrix} e^{\frac{\gamma_\alpha}{2}} & 0 \\ 0 & e^{-\frac{\gamma_\alpha}{2}} \end{bmatrix},$$

where $\gamma_\alpha = \ell_\alpha + i\,\varphi_\alpha$, ℓ_α is the length of Σ_α, and φ_α, $-2\pi \leq \varphi_\alpha < 2\pi$, is the angle of the lifted holonomy of Σ_α. For the sake of simplicity, we will refer to $\gamma_\alpha = \ell_\alpha + i\,\varphi_\alpha$ as a *complex length* of the singular geodesics Σ_α.

Let M be a spherical cone-manifold and Σ_α be it singular set formed by a knot. Let (l, m) be the canonical longitude-meridian pair in the fundamental group $\pi_1(M \setminus \Sigma_\alpha)$. Following HLM (1996), we note that the holonomy map $h : \pi_1(M \setminus \Sigma_\alpha) \to SO(4)$ has two lifts into $SU(2) \times SU(2)$. Up to conjugation in $SU(2) \times SU(2)$, they are given by the formulas

$$h(m) = \left(\pm \begin{bmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{bmatrix}, \pm \begin{bmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{bmatrix}\right),$$

$$h(l) = \left(\begin{bmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{bmatrix}, \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}\right).$$

In this case, $\ell_\alpha = \gamma - \phi$ is the length of knot Σ_α, and $\varphi_\alpha = \gamma + \phi$, $-2\pi \le \varphi_\alpha < 2\pi$, is the angle of the lifted holonomy of Σ_α. We have the following important relations

$$\gamma = \frac{1}{2}(\varphi_\alpha + \ell_\alpha),\ \phi = \frac{1}{2}(\varphi_\alpha - \ell_\alpha). \tag{1}$$

A-polynomial equation

In this report, we contribute a notion of A-polynomial for $M \setminus \Sigma_\alpha$ given by (D. Cooper, M. Culler, H. Gillet, D.D. Long and P.B. Shalen, 1994).
In the hyperbolic case, cone angle α and complex length $\gamma_\alpha = \ell_\alpha + i\,\varphi_\alpha$ of knot Σ_α are related by the equation

$$A(L, M) = 0, \text{ where } L = e^{\frac{\gamma_\alpha}{2}} \text{ and } M = e^{i\frac{\alpha}{2}}. \qquad (2)$$

Also, by the basic properties of A-polynomial we have
$A(L, M) = A(L^{-1}, M)$ and $A(L, M) = A(L, -M)$.
Up to our knowledge, A-polynomials never used before in spherical geometry.
Form the above observation, in the spherical geometry, A-polynomial equation has the form

$$A(L, M) = 0, \text{ where } L = e^{\frac{i}{2}(\varphi_\alpha \pm \ell_\alpha)}, \text{ and } M = e^{i\frac{\alpha}{2}}. \qquad (3)$$

Geometry of two bridge knots. 5_2– knot.

The proof of the spherical volume formula is based on the following Cotangent Rule. Indeed, this is a trigonometrical version of the A-polynomial equation.

Theorem

Let $5_2(\alpha),\ \alpha_0 < \alpha < 2\pi - \alpha_0$ be a spherical cone-manifold. Denote by ℓ_α the length of the longitude of $4_1(\alpha)$ and by φ_α the angle of its lifted holonomy. Then

$$\cot\Big(\frac{4\alpha + \varphi_\alpha \pm \ell_\alpha}{4}\Big)\cot\Big(\frac{\alpha}{2}\Big) = z_{1,2},$$

where z_1 and is z_2 are roots of the equation
$8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2$ *and* $A = \cot(\frac{\alpha}{2})$.

Proof of the Cotangent Rule for 5_2– knot.

The A-polynomial of knot 5_2 is given by

$$A_{52}(L,M) = L^3M^{14} + L^2\left(-M^{14} + 2M^{12} + 2M^{10} - M^6 + M^4\right) + L\left(M^{10} - M^8 + 2M^4 + 2M^2 - 1\right) + 1.$$

We set $L = e^{\frac{i}{2}(\varphi_\alpha \pm \ell_\alpha)}$ and $M = e^{i\frac{\alpha}{2}}$. Then, by the spherical version of A-polynomial equation we have $A_{52}(L,M) = 0$. To find its trigonometrical version we set $z = \frac{(LM^4+1)(M^2+1)}{(LM^4-1)(M^2-1)}$ and $A = i\frac{M^2+1}{M^2-1} = \cot(\frac{\alpha}{2})$. Eliminating L and M from the obtained equations, we derive that $\cot(\frac{4\alpha+\varphi_\alpha\pm\ell_\alpha}{4})\cot(\frac{\alpha}{2}) = z$ and equation $A_{52}(L,M) = 0$ is equivalent to $8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2$.

Proof of the spherical volume formula

Suppose that $\alpha_0 < \alpha < \pi$. Let ℓ_α be the length of the longitude for $5_2(\alpha)$ and φ_α be the angle of its lifted holonomy. By the Cotangent Rule, there are real roots z_1 and z_2 of the equation $8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2$ such that $z_1 = \cot(\frac{4\alpha+\varphi_\alpha-\ell_\alpha}{4})\cot(\frac{\alpha}{2})$ and $z_2 = \cot(\frac{4\alpha+\varphi_\alpha+\ell_\alpha}{4})\cot(\frac{\alpha}{2})$. Consider the function

$$V(\alpha) = \int_{z_1}^{z_2} \frac{\log F(A, \zeta)}{\zeta^2 - 1}\, d\zeta,$$

where $F(A, \zeta) = \frac{8(\zeta^2+A^2)}{(1+A^2)(1-\zeta)(1+\zeta)^2}$ and $A = \cot(\frac{\alpha}{2})$. To prove the integral volume formula, one has to show that $V(\alpha)$ satisfies the Schläfli equation $V'(\alpha) = \frac{\ell_\alpha}{2}$ with initial data $V(\alpha_0) = 0$. Taking into account that z_1 and z_2 are roots of the integrant, we obtain

Proof of the spherical volume formula

$$\frac{dV(\alpha)}{d\alpha} = \frac{\log F(A,z_2)}{z_2^2-1}\frac{dz_2}{d\alpha} - \frac{\log F(A,z_1)}{z_1^2-1}\frac{dz_1}{d\alpha}$$

$$+\int_{z_1}^{z_2}\frac{\partial}{\partial A}\left(\frac{\log F(A,\zeta)}{\zeta^2-1}\right)\frac{dA}{d\alpha}d\zeta = \int_{z_1}^{z_2}\frac{A}{A^2+\zeta^2}d\zeta$$

$$= \operatorname{arccot}(z_2/A) - \operatorname{arccot}(z_1/A)$$

$$= \left(\frac{4\alpha+\varphi_\alpha+\ell_\alpha}{4}\right) - \left(\frac{4\alpha+\varphi_\alpha-\ell_\alpha}{4}\right) = \frac{\ell_\alpha}{2}.$$

Proof of the spherical volume formula

Note that function $A = \cot(\frac{\alpha}{2})$ is strictly increasing on the interval $\alpha_0 < \alpha < \pi$ and varies from 0 to $A_0 = \cot(\frac{\alpha_0}{2}) = 0.3846585\ldots$ For any $A \in (0, A_0)$, the cubic equation $8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2$ has three real solutions z_1, z_2, z_3 which are continuous functions of A.
Two of them, z_1, z_2 chosen such that $-1 < z_1 < z_2$, satisfy the property $z_1, z_2 \to z_0 = \sqrt{2} - \sqrt{2\sqrt{2} - 1} = 0.0620201\ldots$ as $A \to A_0$. This ensures that the initial condition $V(\alpha_0) = 0$ holds. The third one z_3, satisfies the inequality $z_3 < -8$ on $(0, A_0)$ and has no geometrical meaning.
Now let $\alpha = \pi$. Since 5_2 is a rational knot with slope $7/2$, we have

$$\mathrm{Vol}(5_2(\pi)) = \frac{1}{2}\mathrm{Vol}(L(7,2)) = \frac{1}{14}\mathrm{Vol}(\mathbb{S}^3) = \frac{\pi^2}{7}.$$

The equality $V(\alpha) = 2\,V(\pi) + \pi(\alpha - \pi) - V(2\pi - \alpha)$ for $\pi < \alpha < 2\pi - \alpha_0$ follows from the Schäfli formula and the identity $\ell_\alpha = 2\pi - \ell_{2\pi-\alpha}$.

Specific Euclidean volume of $5_2(\alpha)$

The following theorem gives the specific volume of cone-manifold $5_2(\alpha)$ in the Euclidean case. Numerically, this result was obtained earlier by R. N. Shmatkov in his Ph.D. thesis (2003).

Theorem

Let $5_2(\alpha_0)$, *where* $\alpha_0 = 2.40717\ldots$ *be an Euclidean cone-manifold. Then its specific volume* $v_0 = \frac{\mathrm{Vol}\,(5_2(\alpha_0))}{\ell^3_{\alpha_0}}$ *is given by the formula*

$$\mathrm{vol}\,(5_2(\alpha_0)) = 1/\left(6\sqrt{-6+68\sqrt{2}+4\sqrt{983+946\sqrt{2}}}\right) = 0.00990963\ldots$$

To prove the theorem, we note that $v_0 = \lim_{\alpha\to\alpha_0} \frac{\mathrm{Vol}(5_2(\alpha))}{\ell^3_\alpha}$ and $\mathrm{Vol}(5_2(\alpha)) \to 0$ and $\ell_\alpha \to 0$ as $\alpha \to \alpha_0$. Assume $0 < \alpha < \alpha_0$. Then, by making use of the Schläfli formula and L'Hôpital's rule we obtain
$v_0 = \lim_{\alpha\to\alpha_0} \frac{(\mathrm{Vol}(5_2(\alpha)))'_\alpha}{(\ell^3_\alpha)'_\alpha} = \lim_{\alpha\to\alpha_0} \frac{-\ell_\alpha/2}{3\ell^2_\alpha(\ell_\alpha)'_\alpha} = \lim_{\alpha\to\alpha_0} \frac{1}{-3(\ell^2_\alpha)'_\alpha}.$

We also have the following result for Stevedore's Knot 6_1.

Theorem

The volume of the hyperbolic cone-manifold $6_1(\alpha)$ is given by integral

$$i \int_{\overline{z}}^{z} \log \left[\frac{8(\zeta^2 + A^2)}{(1 + A^2)(1 - \zeta)(2 + \zeta + \zeta^2 - (1 - \zeta)\sqrt{2 + 2\zeta + \zeta^2})} \right] \frac{d\zeta}{\zeta^2 - 1},$$

where $A = \cot \frac{\alpha}{2}$ and z and $\overline{z}$ are complex conjugated roots of the integrand.

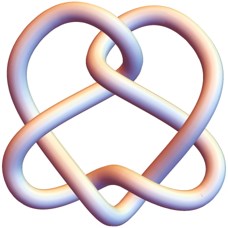


Рис.: Knot 6_1

We resume the results of numerical calculation for limit of hyperbolicity α_0 and specific Euclidean volume v_0 in the following table. The table contains all hyperbolic knots up to 7 crossings.

Knot	Slope	Limit of hyperboliciy α_0	Specific volume v_0
4_1	$5/2, 5/3$	2.094395	0.01603750
5_2	$7/2, 7/3$	2.407169	0.00990963
6_1	$9/2, 9/5$	2.574141	0.00732926
6_2	$11/3, 11/4$	2.684035	0.00538066
6_3	$13/5, 13/8$	2.757265	0.00431666
7_2	$11/2, 11/6$	2.678787	0.00585537
7_3	$13/3, 13/9$	2.755110	0.00449424
7_4	$15/4, 15/11$	2.808209	0.00376538
7_5	$17/5, 17/7$	2.848733	0.00321842
7_6	$19/8, 19/12$	2.880078	0.00283945
7_7	$21/8, 21/13$	2.905300	0.00254482