Volumes of polyhedra in spaces of constant curvature

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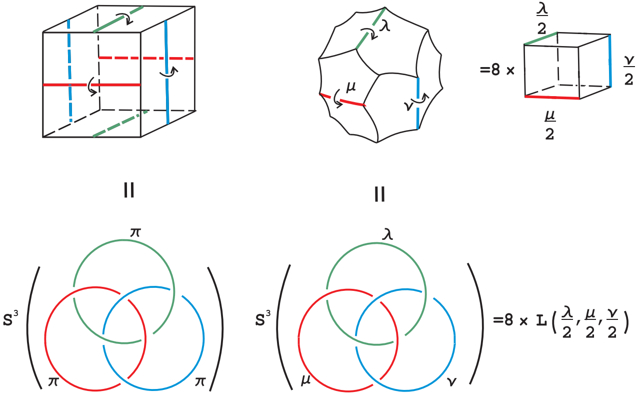
From polyhedra to knots and links

- Borromean Rings cone–manifold and Lambert cube

We start with a simple geometrical construction done by W. Thurston, D. Sullivan and J. M. Montesinos.



From the above consideration we get

$$Vol\, B(\lambda, \mu, \nu) = 8\, Vol\, L(\frac{\lambda}{2}, \frac{\mu}{2}, \frac{\nu}{2}).$$

Recall that $B(\lambda, \mu, \nu)$ is

i) hyperbolic if $0 < \lambda, \mu, \nu < \pi$ **(E. M. Andreev)**

ii) Euclidean if $\lambda = \mu = \nu = \pi$

iii) spherical if $\pi < \lambda, \mu, \nu < 3\pi, \quad \lambda, \mu, \nu \neq 2\pi$
(R. Diaz, D. Derevnin and M.)

- Volume calculation for $L(\alpha, \beta, \gamma)$. The main idea.

0. Existence

$$L(\alpha,\beta,\gamma):\begin{cases} 0<\alpha,\beta,\gamma<\pi/2, & H^3 \\ \alpha=\beta=\gamma=\pi/2, & E^3 \\ \pi/2<\alpha,\beta,\gamma<\pi, & S^3. \end{cases}$$

1. Schläfli formula for $V = Vol\, L(\alpha, \beta, \gamma)$

$$k\mathrm{d}V=\frac{1}{2}(\ell_\alpha d\alpha+\ell_\beta d\beta+\ell_\gamma d\gamma), \quad k=\pm1,0$$

In particular in hyperbolic case:

$$\begin{cases} \frac{\partial V}{\partial\alpha}=-\frac{\ell_\alpha}{2}, \frac{\partial V}{\partial\beta}=-\frac{\ell_\beta}{2}, \frac{\partial V}{\partial\gamma}=-\frac{\ell_\gamma}{2} & (*) \\ V(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})=0. & (**) \end{cases}$$

2. Trigonometrical and algebraic identities

(i) Tangent Rule

$$\frac{\tan\alpha}{\tanh\ell_\alpha}=\frac{\tan\beta}{\tanh\ell_\beta}=\frac{\tan\gamma}{\tanh\ell_\gamma}=T\quad(\textbf{R.Kellerhals})$$

(ii) Sine-Cosine Rule (3 different cases)

$$\frac{\sin\alpha}{\sinh\ell_\alpha}\frac{\sin\beta}{\sinh\ell_\beta}\frac{\cos\gamma}{\cosh\ell_\gamma}=1\quad(\textbf{Derevnin}-\textbf{Mednykh})$$

(iii)

$$\frac{T^2-A^2}{1+A^2}\,\frac{T^2-B^2}{1+B^2}\,\frac{T^2-C^2}{1+C^2}\,\frac{1}{T^2}=1,\quad(\textbf{HLM},\textbf{Topology}'\textbf{90})$$

where

$A=\tan\alpha, B=\tan\beta, C=\tan\gamma$. Equivalently,

$(T^2+1)(T^4-(A^2+B^2+C^2+1)T^2+A^2B^2C^2)=0.$

Remark. (ii) $\Rightarrow$(i) and (i) & (ii) $\Rightarrow$ (iii).

3. Integral formula for volume

Hyperbolic volume of $L(\alpha, \beta, \gamma)$ is given by

$$W = \frac{1}{4} \int\limits_{T}^{\infty} \log \left(\frac{t^2 - A^2}{1 + A^2} \, \frac{t^2 - B^2}{1 + B^2} \, \frac{t^2 - C^2}{1 + C^2} \, \frac{1}{t^2} \right) \frac{\mathrm{d}t}{1 + t^2},$$

where T is a positive root of the integrant equation (iii).

Proof. By direct calculation and Tangent Rule (i) we have:

$$\frac{\partial W}{\partial \alpha} = \frac{\partial W}{\partial A} \frac{\partial A}{\partial \alpha} = -\frac{1}{2} \arctan \frac{A}{T} = -\frac{\ell_\alpha}{2}.$$

In a similar way

$$\frac{\partial W}{\partial \beta} = -\frac{\ell_\beta}{2} \quad \text{and} \quad \frac{\partial \mathrm{W}}{\partial \gamma} = -\frac{\ell_\gamma}{2}.$$

By convergence of the integral $W(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = 0$. Hence,
$W = V = Vol\, L(\alpha, \beta, \gamma)$.

Lambert cube: spherical volume

Theorem 21 (D. A. Derevnin and M., 2009)

The volume of a spherical Lambert cube $Q(\alpha,\beta,\gamma)$, $\frac{\pi}{2}<\alpha,\beta,\gamma<\pi$ *is given by the formula*

$$V(\alpha,\beta,\gamma)=\frac{1}{4}(\delta(\alpha,\Theta)+\delta(\beta,\Theta)+\delta(\gamma,\Theta)-2\delta(\frac{\pi}{2},\Theta)-\delta(0,\Theta)),$$

where

$$\delta(\alpha,\Theta)=\int\limits_{\Theta}^{\frac{\pi}{2}}\log(1-\cos 2\alpha\cos 2\tau)\frac{d\tau}{\cos 2\tau}$$

and Θ, $\frac{\pi}{2}<\Theta<\pi$ *is defined by*

$$\tan^2\Theta=-K+\sqrt{K^2+L^2M^2N^2},\ K=(L^2+M^2+N^2+1)/2,$$

$L=\tan\alpha,\ M=\tan\beta,\ N=\tan\gamma.$

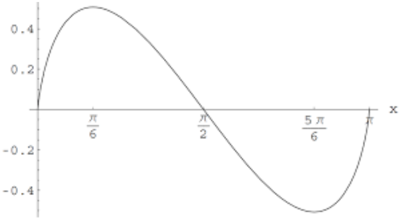
Remark. The function $\delta(\alpha, \Theta)$ can be considered as a spherical analog of the function

$$\Delta(\alpha, \Theta) = \Lambda(\alpha + \Theta) - \Lambda(\alpha - \Theta).$$

The main result of R. Kellerhals (1989) for hyperbolic volume can be obtained from the above theorem by replacing $\delta(\alpha, \Theta)$ to $\Delta(\alpha, \Theta)$ and K to $-K$.

Recall that the Lobachevsky function is defined by the integral

$$\Lambda(x) = -\int_0^x \log|2\sin t|\,dt.$$



Lambert cube: hyperbolic volume

As a consequence of the above mentioned volume formula for Lambert cube we obtain

Proposition 1 (D. A. Derevnin and M., 2009)

Let $L(\alpha, \beta, \gamma)$ *be a spherical Lambert cube such that* $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. *Then*

$$Vol\,L(\alpha, \beta, \gamma) = \frac{1}{4}\left(\frac{\pi^2}{2} - (\pi - \alpha)^2 - (\pi - \beta)^2 - (\pi - \gamma)^2\right).$$

1. Since $\cos^2\frac{2\pi}{3} + \cos^2\frac{2\pi}{3} + \cos^2\frac{3\pi}{4} = 1$, we have
$$Vol\,L\left(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}\right) = \frac{31}{576}\pi^2 \text{ (D. Derevnin, A. Mednykh).}$$
2. Also, since $\cos^2\frac{2\pi}{3} + \cos^2\frac{3\pi}{5} + \cos^2\frac{4\pi}{5} = 1$, we get
$$Vol\,L\left(\frac{2\pi}{3}, \frac{3\pi}{5}, \frac{4\pi}{5}\right) = \frac{17}{360}\pi^2 \text{ (A. Kolpakov and S. Robins).}$$

Rational Volume Problem

The following problem is widely known and still open.

Rational Volume Problem. Let P be a spherical polyhedron whose dihedral angles are in $\pi\mathbb{Q}$. Then $Vol\,(P) \in \pi^2\mathbb{Q}$.

One of the reasons for the problem to be true is the following observation.

Let P be a Coxeter polyhedron in S^3 (that is all dihedral angles of P are $\frac{\pi}{n}$ for some $n \in \mathbb{N}$). Then the Coxeter group $\Delta(P)$ generated by reflections in faces of P is finite and

$$Vol\,(P) = \frac{Vol\,(S^3)}{|\Delta(P)|} = \frac{2\pi^2}{|\Delta(P)|} \in \pi^2\mathbb{Q}.$$

History

Heron of Alexandria (60 B.C.) left us the following remarkable formula that relates the area $\mathcal{A}$ of a triangle to its side lengths a, b and c

$$A = \sqrt{(s-a)(s-b)(s-c)s},$$

where $s = (a+b+c)/2$ is the semiperimeter.
Brahmagupta (XVII century) gave the analogues formula for a convex cyclic (= inscribed in a circle) quadrilateral with side lengths a, b, c and d

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = (a+b+c+d)/2$.
D.P. Robbins (1994) found a way to generalize these formulas.

History

The general result is the following

Theorem 1

For each $n \geq 3$ there is a unique irreducible homogeneous polynomial α_n with integer coefficients, such that

$$\alpha_n(16A^2, a_1^2, \ldots, a_n^2) = 0,$$

whenever $a_1, \ldots, a_n$ are side lengths of a cyclic $n-$gon and A is its area.

The polynomials α_n are known in the literature as generalized Heron polynomials. Certainly, the Heron's and Brahmagupta's theorems are the partial cases of the above theorem. The properties of polynomials α_n were investigated by V.V. Varfolomeev (2003) and M. Fedorchuk and I. Pak (2005). Related results are also obtained by Ren Guo and Nilgün Sönmez (2010).

History

A three dimensional version of the Heron formula belongs to Tartaglia (1499-1557) who found a formula for the volume of Euclidean tetrahedron. More precisely, let be an Euclidean tetrahedron with edge lengths $d_{ij},\ 1 \leq i < j \leq 4$. Then $V = \mathrm{Vol}(T)$ is given by

$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix}.$$

Note that V is a root of quadratic equation whose coefficients are integer polynomials in $d_{ij},\ 1 \leq i < j \leq 4$. High dimensional generalization of this result is known as the Cayley-Menger formula.

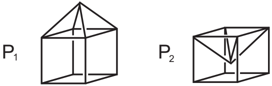
History

The result of Tartaglia can be generalized in the following way.

Theorem 2 (I. Kh. Sabitov, 1996)

Let P be a simplicial Euclidean polyhedron. Then $V = Vol(P)$ is a root of an even degree algebraic equation whose coefficients are integer polynomials in edge lengths of P depending on combinatorial type of P only.

Example



(All edge lengths are taken to be 1)

The volumes $V_1 = Vol(P_1)$ and $V_2 = Vol(P_2)$ are roots of the same algebraic equation $a_0 V^{2n} + a_1 V^{2n-2} + \ldots + a_n V^0 = 0.$
Recently, A.A. Gaifullin (2015) proved a muti–dimensional dimensional version of the Sabitov's theorem.
Also, he explained a situation in the hyperbolic and spherical spaces.

History

The volume of non-Euclidean tetrahedron was investigated by many authors. Very shortly, the history is the following.
A formula the volume of an arbitrary hyperbolic tetrahedron has been unknown until recently. The general algorithm for obtaining such a formula was indicated by W.–Y. Hsiang (1988) and the complete solution of the problem was given by Yu. Cho and H. Kim (1999), J. Murakami, M. Yano (2001) and A. Ushijima (2002).
An excellent exposition of these results and a complete geometric proof of the volume formula was given by Y. Mohanty (2003) in her Ph.D. thesis.
A simple integral formula was obtained in our joint paper D. Derevnin and A. Mednykh (2005).
More than a century ago, in 1906, the Italian mathematician G. Sforza found the formula for the volume of a non-Euclidean tetrahedron. It was discovered during a discussion of the author with J. M. Montesinos in August 2006.

Non-Euclidean geometry

We start with the following well-known results from non-Euclidean geometry. The area of a triangle with angles α, β and γ is given by the formulas

- $A = \pi - \alpha - \beta - \gamma,$ $(\mathbb{H}^2)$
- $A = \alpha + \beta + \gamma - \pi,$ $(\mathbb{S}^2)$
- $A = s^2 \tan(\alpha/2) \tan(\beta/2) \tan(\gamma/2).$ $(\mathbb{E}^2)$

In the later formula the semiperimeter s plays a role of scale on the Euclidean plane $\mathbb{E}^2$.

Non-Euclidean geometry

There are three non-Euclidean version of the Heron formula on the hyperbolic plane. The area A of a hyperbolic triangle with side lengths $a, b,$ and c is given by each of the following formulas

- Sine of 1/2 Area Formula

$$\sin^2\frac{A}{2}=\frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)\sinh(s)}{4\cosh^2\left(\frac{a}{2}\right)\cosh^2\left(\frac{b}{2}\right)\cosh^2\left(\frac{c}{2}\right)},$$

- Tangent of 1/4 Area Formula

$$\tan^2\frac{A}{4}=\tanh\left(\frac{s-a}{2}\right)\tanh\left(\frac{s-b}{2}\right)\tanh\left(\frac{s-c}{2}\right)\tanh\left(\frac{s}{2}\right),$$

- Sine of 1/4 Area Formula

$$\sin^2\frac{A}{4}=\frac{\sinh\left(\frac{s-a}{2}\right)\sinh\left(\frac{s-b}{2}\right)\sinh\left(\frac{s-c}{2}\right)\sinh\left(\frac{s}{2}\right)}{\cosh\left(\frac{a}{2}\right)\cosh\left(\frac{b}{2}\right)\cosh\left(\frac{c}{2}\right)}.$$

The third formula can be obtained by the squaring of the product of the first two.

Brahmagupta's theorem for cyclic non-Euclidean quadrilateral

Theorem 3 (M., 2013)

The area A of a cyclic hyperbolic quadrilateral with side lengths a, b, c and d can be found by the formula

$$\sin^2\frac{A}{2}=\frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)\sinh(s-d)}{4\cosh^2\left(\frac{a}{2}\right)\cosh^2\left(\frac{b}{2}\right)\cosh^2\left(\frac{c}{2}\right)\cosh^2\left(\frac{d}{2}\right)}(1-\varepsilon),$$

where

$$\varepsilon=\frac{\sinh(\frac{a}{2})\sinh(\frac{b}{2})\sinh(\frac{c}{2})\sinh(\frac{d}{2})}{\cosh(\frac{s-a}{2})\cosh(\frac{s-b}{2})\cosh(\frac{s-c}{2})\cosh(\frac{s-d}{2})}$$

and $s=(a+b+c+d)/2.$

We note that if $d=0$ then $\varepsilon=0$ and the theorem reduces to the correspondent theorem for a hyperbolic triangle.

Brahmagupta's theorem for cyclic non-Euclidean quadrilateral

Theorem 4 (M., 2013)

The area A of a cyclic hyperbolic quadrilateral with side lengths a, b, c and d can be found by the formula

$$\tan^2\frac{A}{4} = \frac{1}{1-\varepsilon}\tanh(\frac{s-a}{2})\tanh(\frac{s-b}{2})\tanh(\frac{s-c}{2})\tanh(\frac{s-d}{2}),$$

where

$$\varepsilon = \frac{\sinh(\frac{a}{2})\sinh(\frac{b}{2})\sinh(\frac{c}{2})\sinh(\frac{d}{2})}{\cosh(\frac{s-a}{2})\cosh(\frac{s-b}{2})\cosh(\frac{s-c}{2})\cosh(\frac{s-d}{2})}$$

and $s = (a+b+c+d)/2$.

If $d = 0$ then $\varepsilon = 0$ and the theorem reduces to the theorem for a hyperbolic triangle.

Brahmagupta's theorem for cyclic non-Euclidean quadrilateral

By squaring the product of the two previous area formulas we obtain

Theorem 5

The area A of a cyclic hyperbolic quadrilateral with side lengths a, b, c and d can be found by the formula

$$\sin^2\frac{A}{4}=\frac{\sinh(\frac{s-a}{2})\sinh(\frac{s-b}{2})\sinh(\frac{s-c}{2})\sinh(\frac{s-d}{2})}{\cosh\left(\frac{a}{2}\right)\cosh\left(\frac{b}{2}\right)\cosh\left(\frac{c}{2}\right)\cosh\left(\frac{d}{2}\right)},$$

where $s=(a+b+c+d)/2$.

Brahmagupta's theorem for inscribed and circumscribed quadrilateral

Corollary

The area A of an inscribed and circumscribed hyperbolic quadrilateral with side lengths a, b, c and d can be found by the formula

$$\sin^2\frac{A}{4}=\tanh(\frac{a}{2})\tanh(\frac{b}{2})\tanh(\frac{c}{2})\tanh(\frac{d}{2}).$$

An Euclidean version of this result is known for a long time. See for example (Ivanoff, 1960). In this case

$$A^2=a\,b\,c\,d.$$

Sketch of the proof

The proof is based on the following two observations.

1° The necessary and sufficient condition for hyperbolic quadrilateral to be inscribed into circle, horocycle or one branch of an equidistant curve were suggested by J. E. Valentine (1970) who were influenced by H. S. M. Coxeter. In terms of side lengths it can be given by the following the non-Euclidean version of Ptolemy's theorem.

$$s(a)s(c) + s(b)s(d) = s(e)s(f),$$

where $s(x) = \sinh(\frac{x}{2})$, and e and f are lengths of the diagonals.

2° The necessary and sufficient condition for hyperbolic quadrilateral to be inscribed into circle, horocycle or one branch of an equidistant curve were given by F.V. Petrov (2009) in terms of angles. They are just

$$A + C = B + D.$$

The Bretschneider theorem for area of an arbitrary quadrilateral

In 1842 Carl Bretschneider related the area of an arbitrary Euclidean quadrilateral to its side lengths and the sum of two opposite angles. The area S of an Euclidean quadrilateral with side lengths a, b, c, d and opposite angles A and C is given by the formula

$$S^2 = (s-a)(s-b)(s-c)(s-d) - a\,b\,c\,d\cos^2\frac{A+C}{2},$$

where $s = (a+b+c+d)/2$ is the semiperimeter. The statement of the theorem remains valid if one substitutes $A+C$ with the sum of another pair of opposite angles $B+D$. By making use of the identity $A+B+C+D=2\pi$ for any Euclidean quadrilateral we can rewrite the Bretschneider theorem in the following more symmetric way

$$S^2 = (s-a)(s-b)(s-c)(s-d) - a\,b\,c\,d\sin^2\frac{A-B+C-D}{4}.$$

The Bretschneider theorem

Th latter statement allows a generalization for the case of non-Euclidean quadrilateral. In the particular case of inscribed quadrilateral (when $A + C = B + D$) we get a Brahmagupta formula. Recall that the sum of angles is not equal to 2π anymore.
The following theorem gives a hyperbolic version of Bretschneider formula.

Theorem (Baigonakova, Mednykh, 2012)

The area S of a hyperbolic quadrilateral with side lengths a, b, c, d and angles A, B, C, D is given by the formula

$$\sin^2 \frac{S}{4} = \frac{\sinh \frac{s-a}{2} \sinh \frac{s-b}{2} \sinh \frac{s-c}{2} \sinh \frac{s-d}{2}}{\cosh \frac{a}{2} \cosh \frac{b}{2} \cosh \frac{c}{2} \cosh \frac{d}{2}} - \tanh \frac{a}{2} \tanh \frac{b}{2} \tanh \frac{c}{2} \tanh \frac{d}{2} \sin^2 \frac{A - B + C - D}{4},$$

where $s = (a + b + c + d)/2$ is the semiperimeter.

For $K = A - B + C - D = 0$, as a consequence, we have the Brahmahupta formula again

$$\sin^2 \frac{S}{4} = \frac{\sinh \frac{p-a}{2} \sinh \frac{p-b}{2} \sinh \frac{p-c}{2} \sinh \frac{p-d}{2}}{\cosh \frac{a}{2} \cosh \frac{b}{2} \cosh \frac{c}{2} \cosh \frac{d}{2}}.$$

The Bretschneider theorem for a circumscribed quadrilateral

For a circumscribed quadrilateral, like in the Euclidean case, we have $a + c = b + d$, hence $p - a = c, p - b = d, p - c = a, p - d = b$. By the evident identity $1 - \sin^2 \frac{K}{4} = \cos^2 \frac{K}{4}$, from the Bretschneider theorem we obtain

Theorem

The area S of a circumscribed hyperbolic quadrilateral is given by the formula

$$\sin^2 \frac{S}{4} = \tanh \frac{a}{2} \tanh \frac{b}{2} \tanh \frac{c}{2} \tanh \frac{d}{2} \cos^2 \frac{A - B + C - D}{4}.$$

Area formula for trapezoid

Following F.V. Petrov (2009) we define a trapezoid to be a quadrilateral $ABCD$ whose angles satisfy $A+B=C+D$. We note that $A+B=C+D=\pi$, $<\pi$ or $>\pi$ in the Euclidean, hyperbolic and spherical geometries, respectively.

Theorem 6 (Dasha Sokolova and M., 2004)

The area S of a hyperbolic trapezoid $ABCD$ with side lengths $a=AB$, $b=BC$, $c=CD$, $d=DA$ can be found by the formula

$$\sin^2\frac{S}{4}=\frac{\sinh^2\frac{b+d}{2}\sinh\frac{a+b-c-d}{4}\sinh\frac{a+b+c-d}{4}\sinh\frac{-a+b+c-d}{4}\sinh\frac{a-b+c+d}{4}}{\sinh^2\frac{b-d}{2}\cosh\frac{a-b-c-d}{4}\cosh\frac{a-b+c-d}{4}\cosh\frac{a+b-c+d}{4}\cosh\frac{a+b+c+d}{4}}.$$

Recall that the area S of an Euclidean trapezoid satisfies the equation $S^2=$
$$\frac{(a+b-c-d)(a+b+c-d)(-a+b+c-d)(a-b+c+d)(b+d)^2}{16(b-d)^2}.$$

Pythagorean Theorem on the plane

Everybody knows the classical Pythagorean Theorem $a^2 + b^2 = c^2$. Its non-Euclidean versions are also well-known. On the hyperbolic and spherical planes they are respectively

$$\cosh a \cosh b = \cosh c \text{ and } \cos a \cos b = \cos c.$$

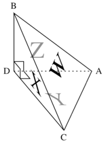
The are just the consequences of more general cosine rules

$$\cosh c = \cosh a \cosh b + \sinh a \sinh b \cos \gamma$$

and

$$\cos c = \cos a \cos b - \sin a \sin b \cos \gamma.$$

Pythagorean Theorem for tetrahedra in the space



Right angled tetrahedron

By elementary calculation one can easily check that

$$W^2 = X^2 + Y^2 + Z^2,$$

where X, Y, Z, W are the respective face areas of the tetrahedron.

Pythagorean Theorem for tetrahedra in Non-Euclidean 3-space

Now, the Pythagorean Theorem has a form:

Theorem

$\cos\frac{W}{2} = \cos\frac{X}{2}\cos\frac{Y}{2}\cos\frac{Z}{2} \pm \sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}$,
where, throughout, $\pm$ *is* $+$ *in hyperbolic space, and* $-$ *in spherical space.*

For details of the proof, see numerous internet papers written by B. D. S. McConnell on his website Blue's Blog: The Bloog!
http://daylateanddollarshort.com/bloog/category/hedronometry/