Не всякое зацепление изотопно гладкому

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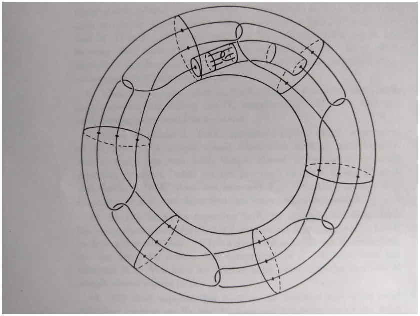

The Bing sling (R. H. Bing, 1956)

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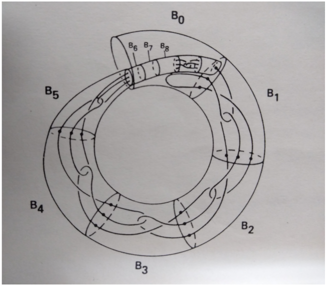

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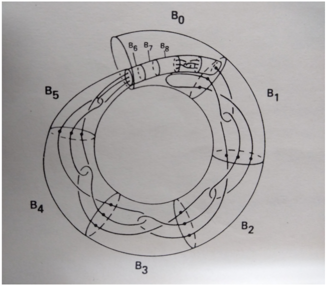

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Theorem 1. Not all links are isotopic to PL links.

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Update: No progress...

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F need not be a true Seifert surface (even if K is PL or smooth) because $\bar{F}$ may fail to be a manifold with boundary

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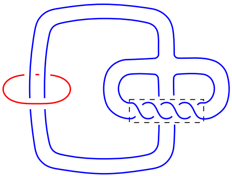
Theorem. *β is well-defined and is an invariant of I-equivalence.*

Example: twisted Whitehead link



W_n for $n = 2$

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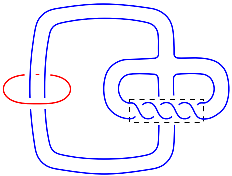

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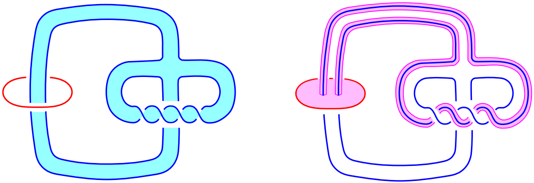

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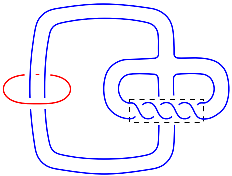

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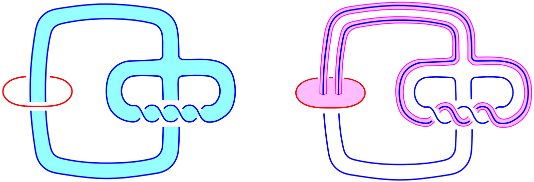

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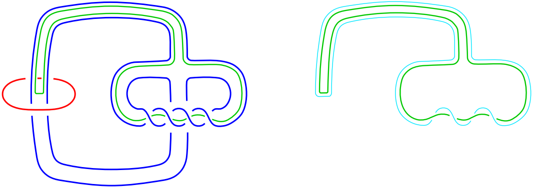

$F = \Sigma \cap \Sigma'$ and its pushoff F^{+0} within Σ'

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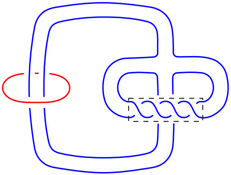

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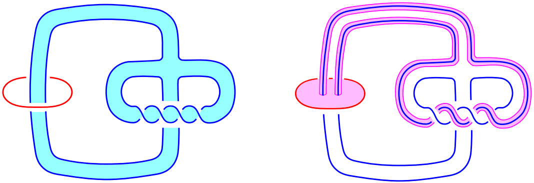

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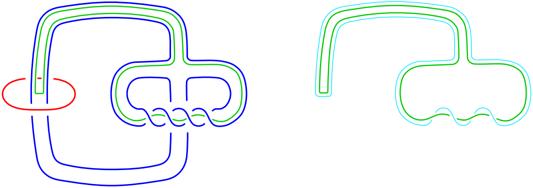

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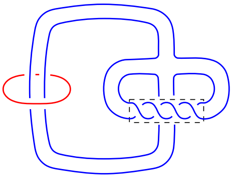

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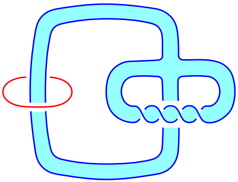

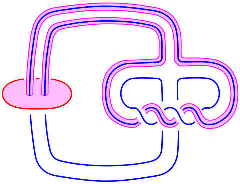

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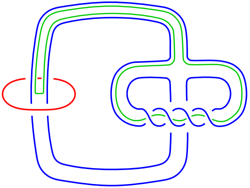

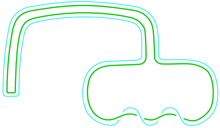

$F = \Sigma \cap \Sigma'$ and its pushoff F^{+0} within Σ'

$$\beta(W_2) = \mathrm{lk}(F, F^{++}) = \mathrm{lk}(F, F^{+0}) = 2 \text{ and similarly } \beta(W_n) = n.$$

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$\beta'_n(K', K) := \beta_n(K, K') = \beta(\underbrace{\partial_2 \dots \partial_2}_{n-1} L)$, where $\partial_2 L := (K, F^{++})$.

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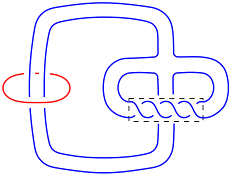

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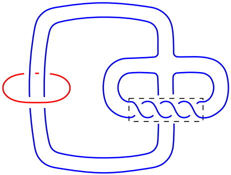

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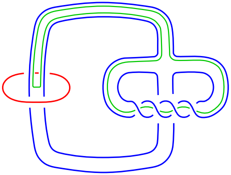

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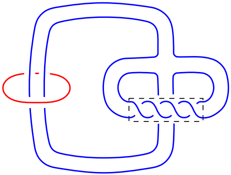

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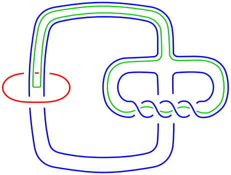

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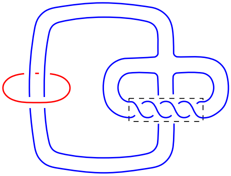

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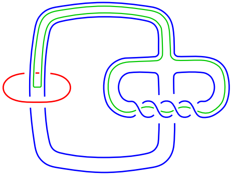

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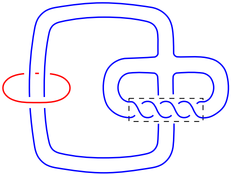

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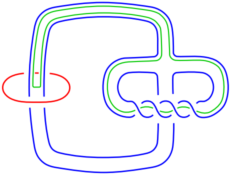

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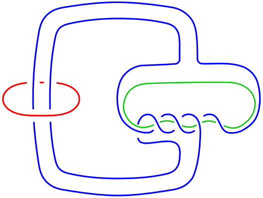
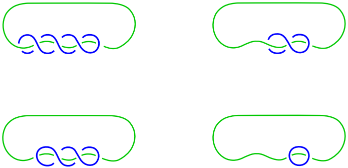
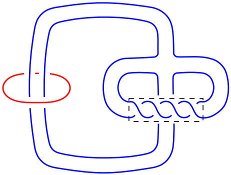




Example: twisted Whitehead link



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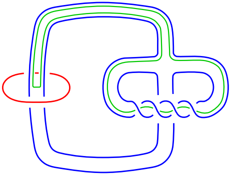

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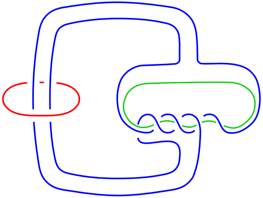

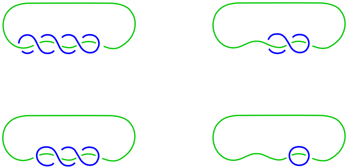

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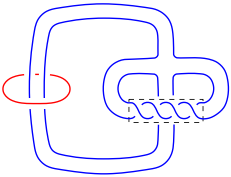

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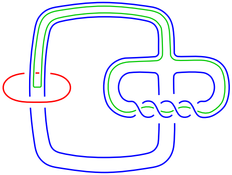

${\color{green}F} = {\color{magenta}\Sigma} \cap {\color{cyan}\Sigma'}$

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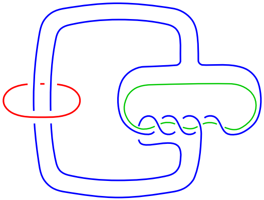

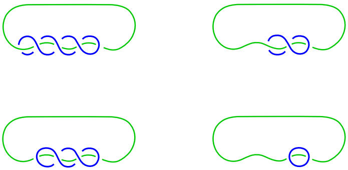

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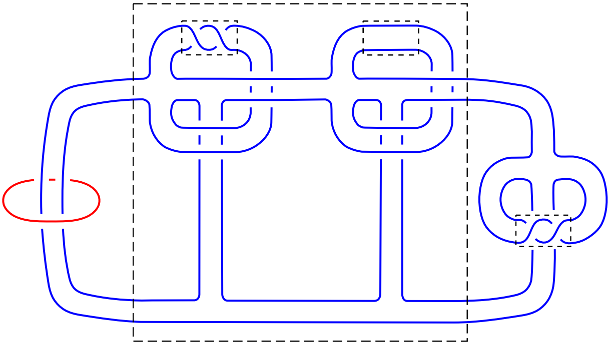

$W_{n_1,\ldots,n_m}$ for $m = 3$ and $(n_1, n_2, n_3) = (1, 0, -1)$.

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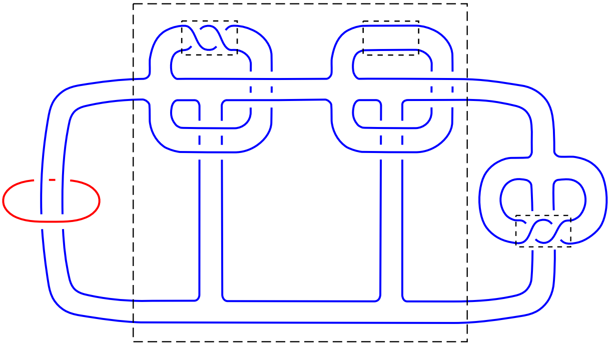

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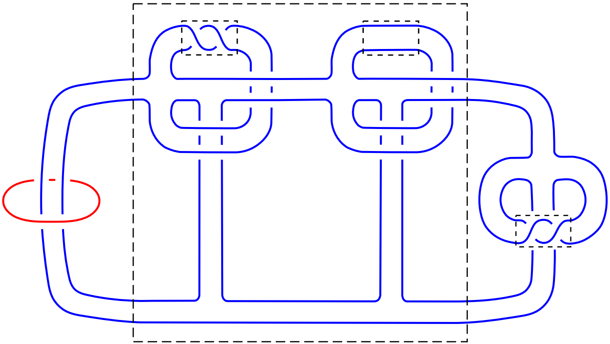

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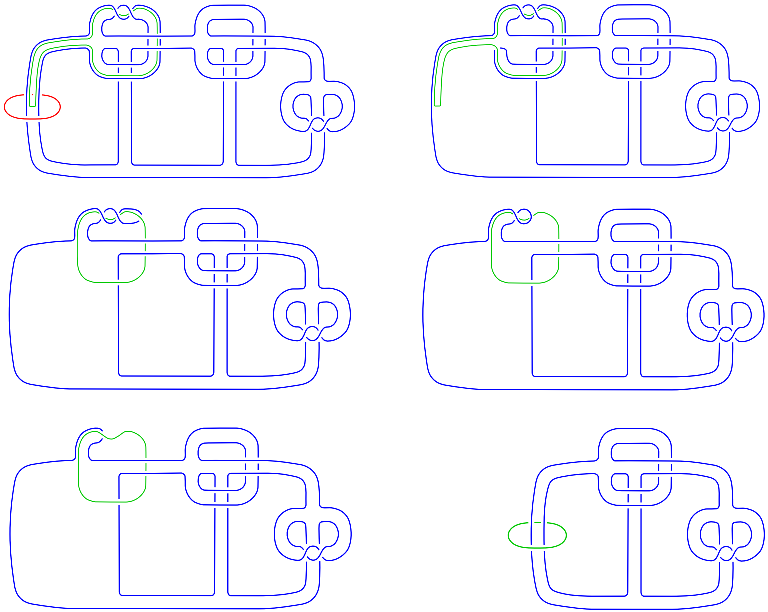

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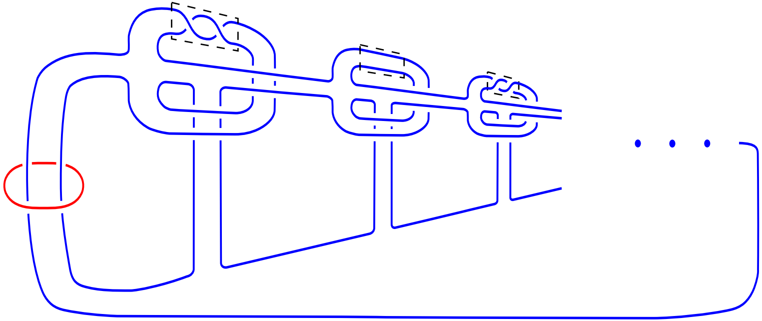

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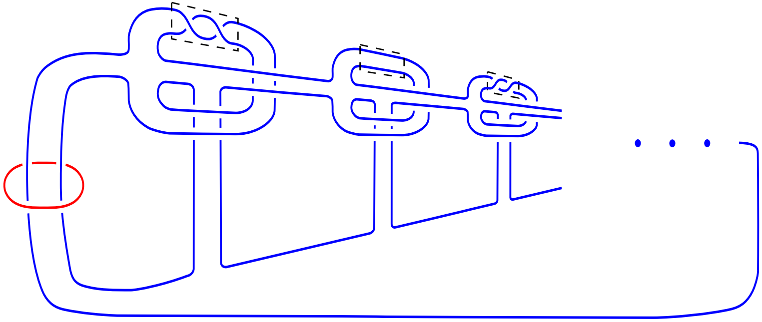

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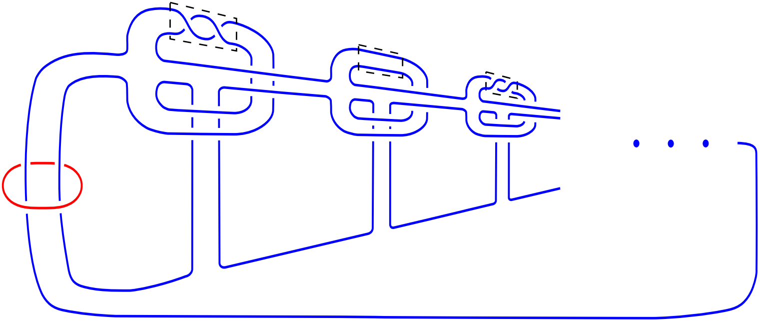

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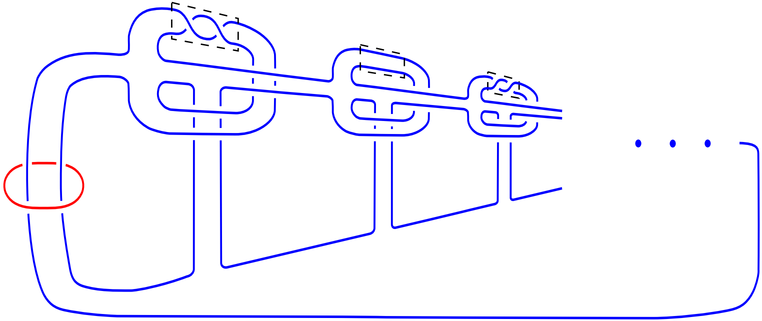

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Realization Theorem. For every formal power series $P \in \mathbb{Z}[[x]]$ there exists a link $W(P)$ such that $C_{W(P)} = P$.

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Example. $W_{1!,2!,3!,\ldots}$ is not I-equivalent to any PL link.

Alternative links $M_{n_1,n_2,\ldots}$ such that

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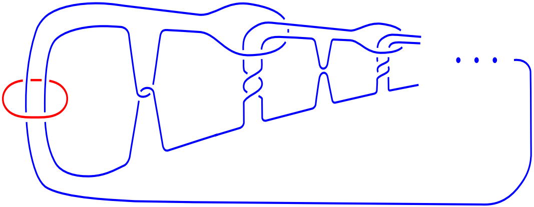

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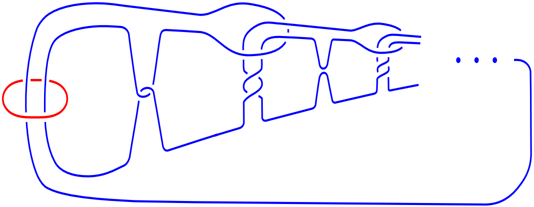

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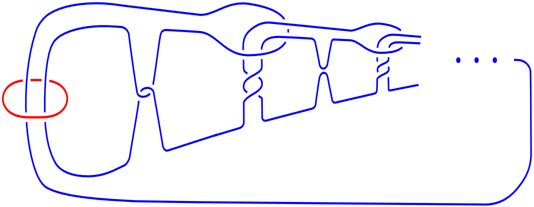

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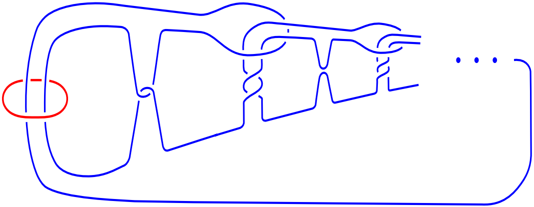

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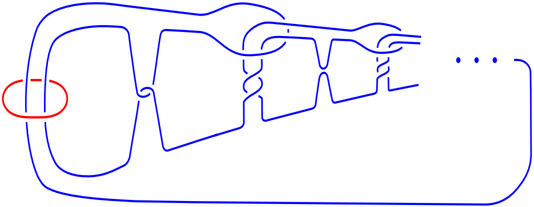

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Kojima and Yamasaki (1979): λ-polynomial — defined for wild links, but non-invariant under isotopy.

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When $\mathrm{lk} = 1$, $\bar{\beta}_i$ of PL links is not even a concordance invariant.