Brunnian link maps in the 4-sphere

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A *link map* is a map $f\colon X_1 \sqcup \cdots \sqcup X_m \to Y$ such that the images of the X_i are pairwise disjoint.

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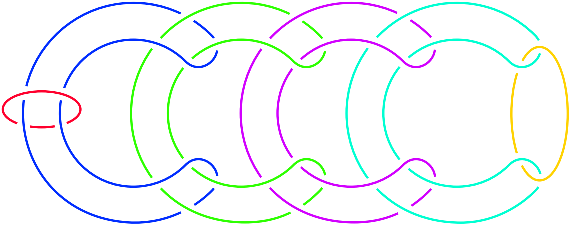

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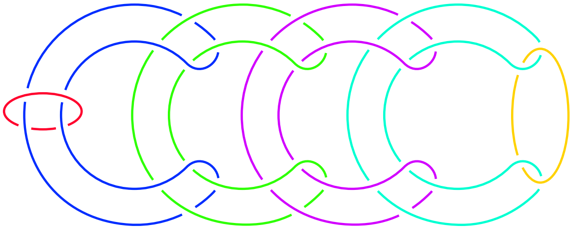

In fact, M_n is *Brunnian*, that is, if any component is omitted, it becomes link homotopic to the unlink.

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Let $f = f_+ \sqcup f_- \colon S^2 \sqcup S^2 \to S^4$ be a generic link map.

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Theorem (Schneiderman–Teichner, 2019, Ann. of Math.) σ is injective.

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$$\begin{array}{ccc} FDLM^4_{2,2} & \xrightarrow{\ \Sigma\ } & \mathbb{Z}[t^{\pm 1}] \oplus \mathbb{Z}[t^{\pm 1}] \\ \big\downarrow \text{closure} & & \big\downarrow t^n \mapsto t^{|n|} \\ LM^4_{2,2} & \xrightarrow{\ \sigma\ } & \mathbb{Z}[t] \oplus \mathbb{Z}[t] \end{array}$$

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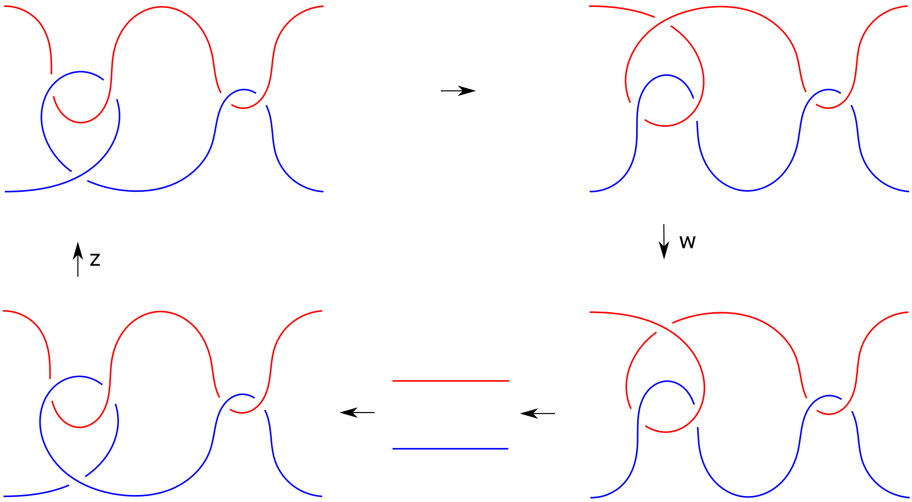

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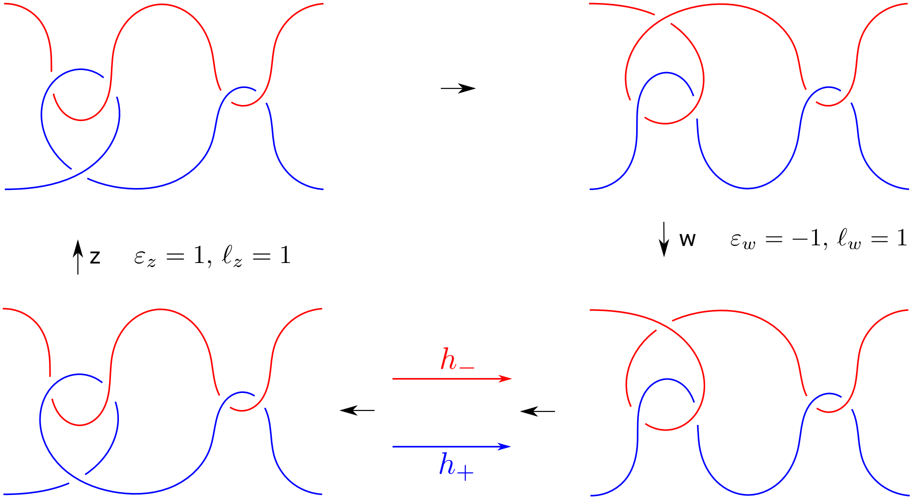

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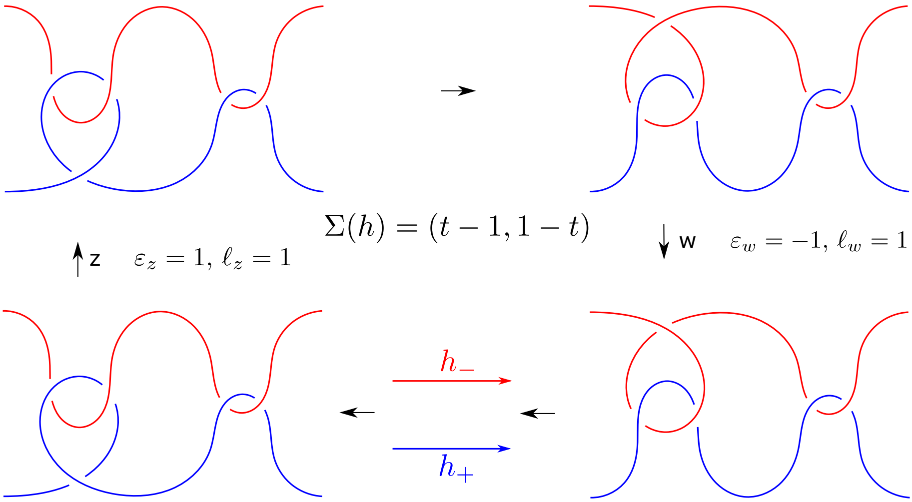

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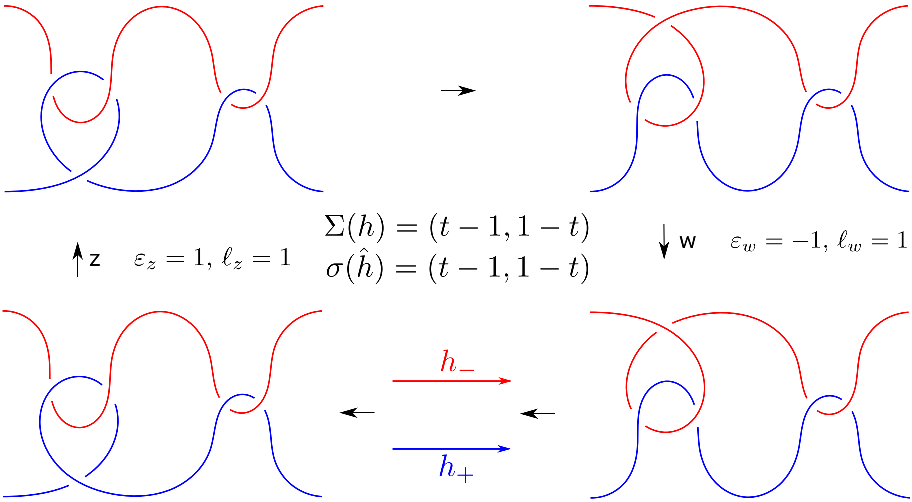

$$\Sigma(h) = (t-1, 1-t)$$
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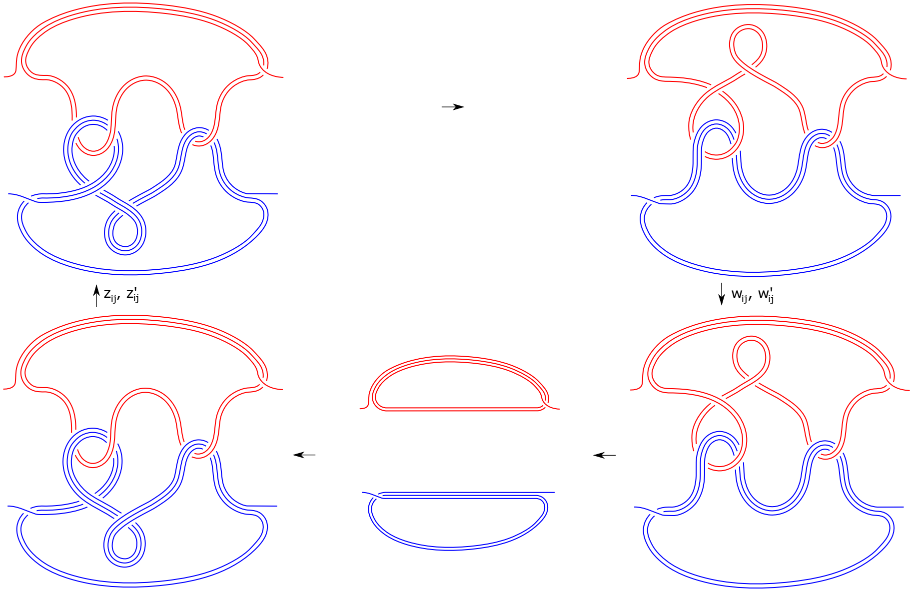
Cabled Fenn–Rolfsen link map ($m = 3$, $n = -2$)



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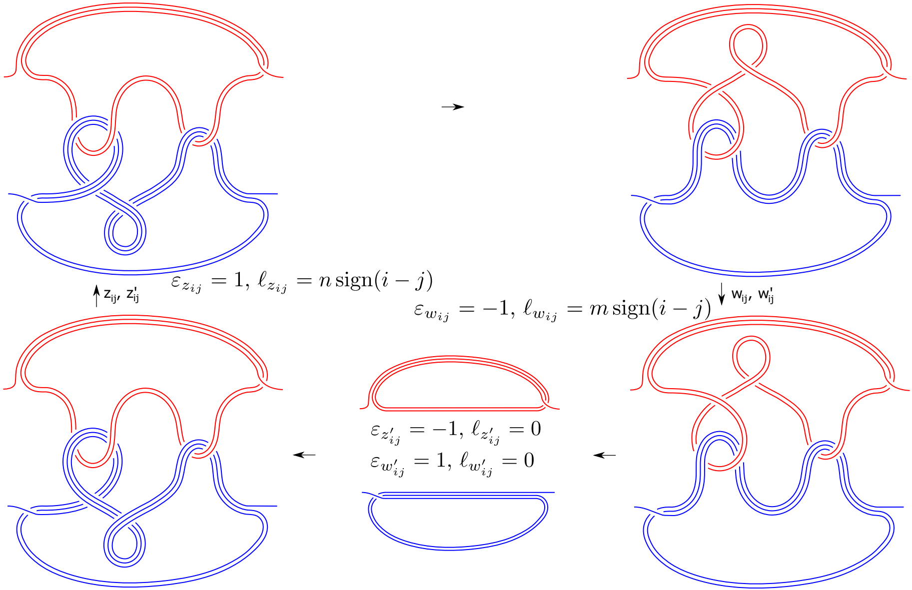

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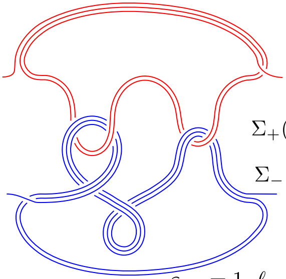

$$\Sigma_+(\Lambda_{mn}) = \tfrac{m^2+m}{2}(t^n - 1) + \tfrac{m^2-m}{2}(t^{-n} - 1)$$

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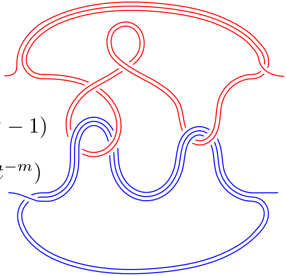

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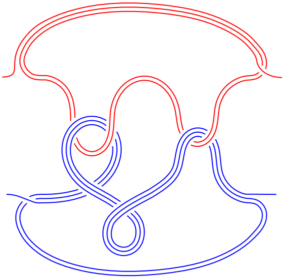

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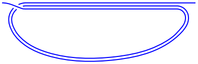

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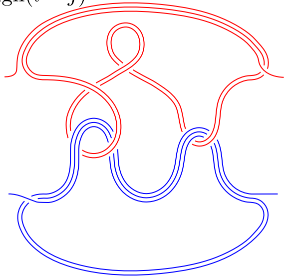

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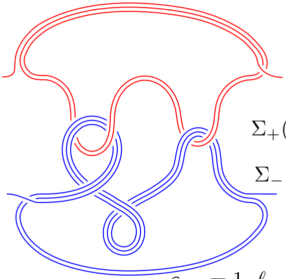

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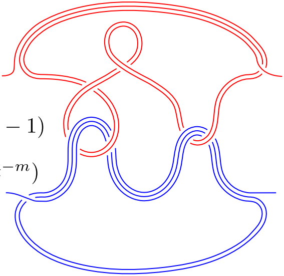

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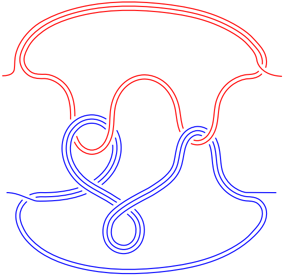

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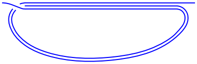

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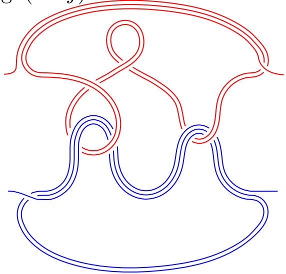

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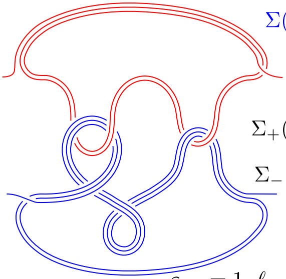

$$\Sigma(\Lambda_{n1}\#\rho\Lambda_{n1}) = \big(n(t - t^{-1}),\, t^{-n} - t^{n}\big)$$

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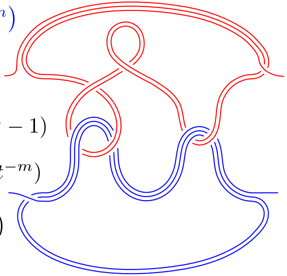

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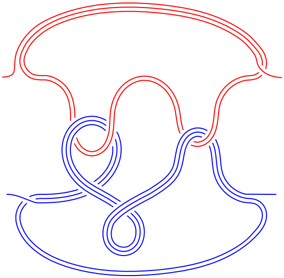

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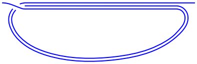

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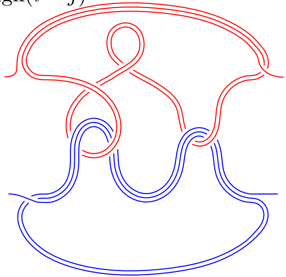

Images of σ and Σ

Theorem (Kirk, 1988). $\operatorname{im}\sigma = \ker\delta$, where $\delta\colon \mathbb{Z}[t]\oplus\mathbb{Z}[t]\to\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ is given by $(f,g)\mapsto\big(f|_{t=1},\, g|_{t=1},\, f'+g'+f''+g''|_{t=1}\big)$.

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$\supset$: because $\sigma(\hat\Lambda_{m,n})$ generate $\ker\delta$ and $\Sigma(\Lambda_{m,n})$ generate $\ker\Delta$.

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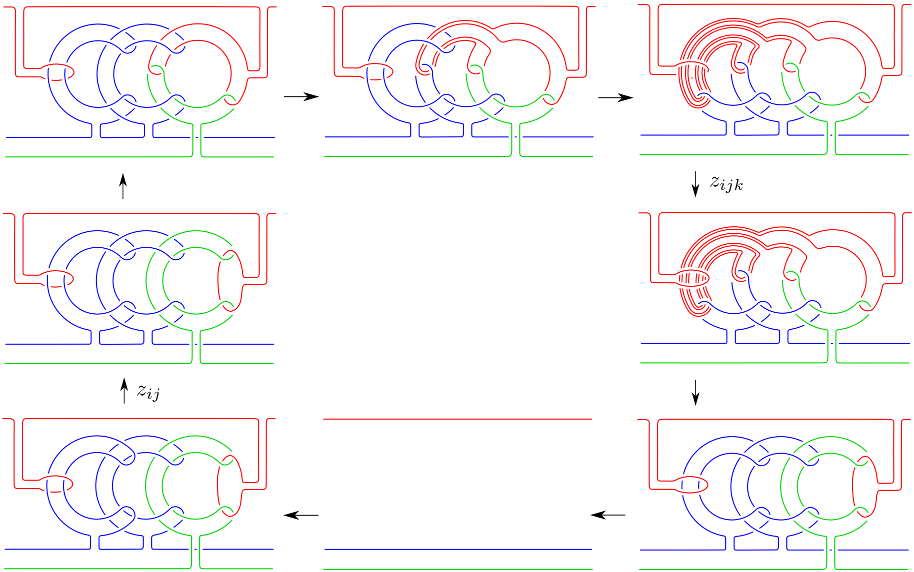
$\Rightarrow$ **Theorem 2:** A computation of $\sigma^{\mathrm{ab}}(LM^4_{2,2,2})$.

Brunnian example ($m = 3$)



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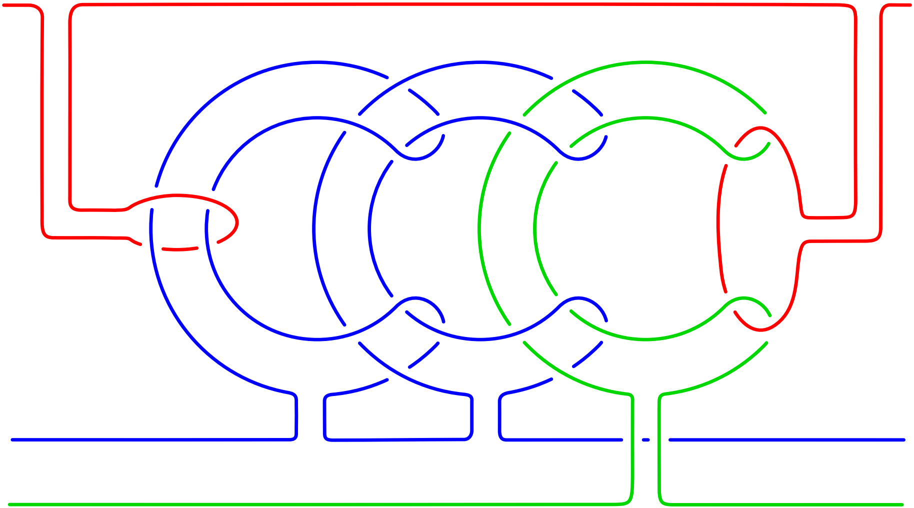

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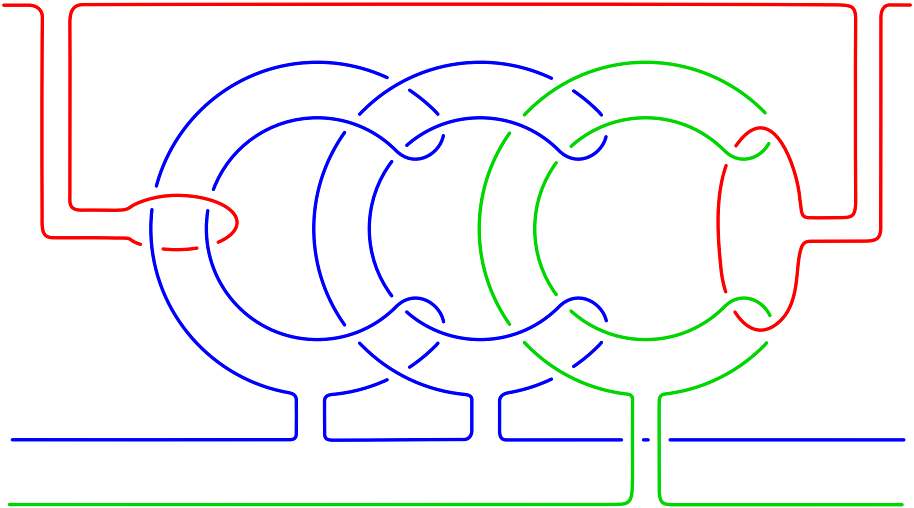

"A C_k-move (Goussarov–Habiro move)"

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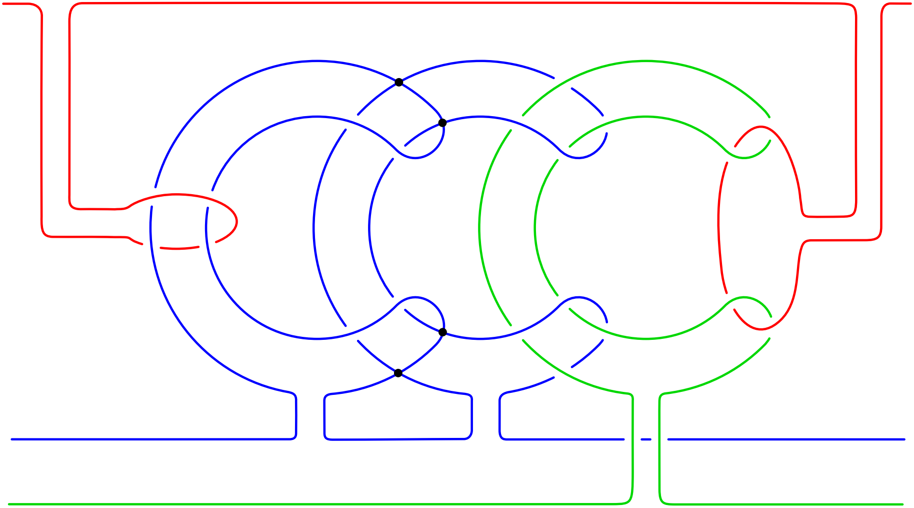

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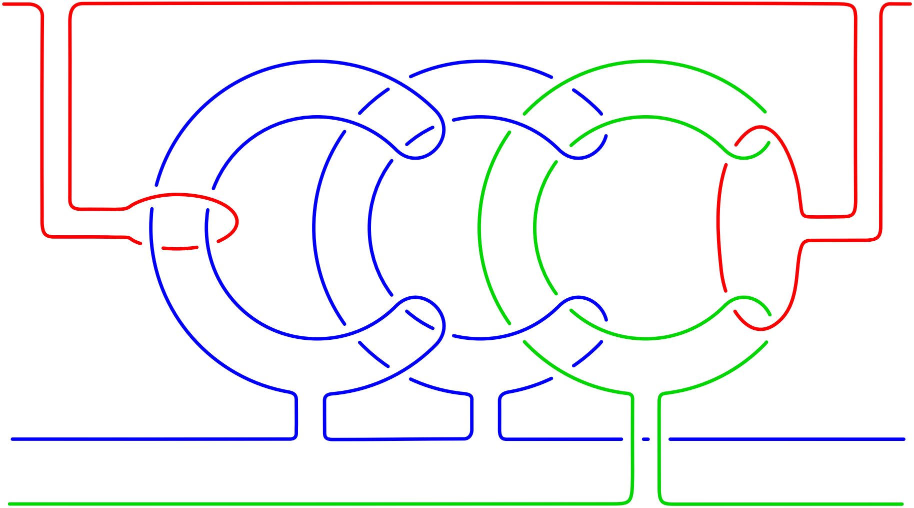

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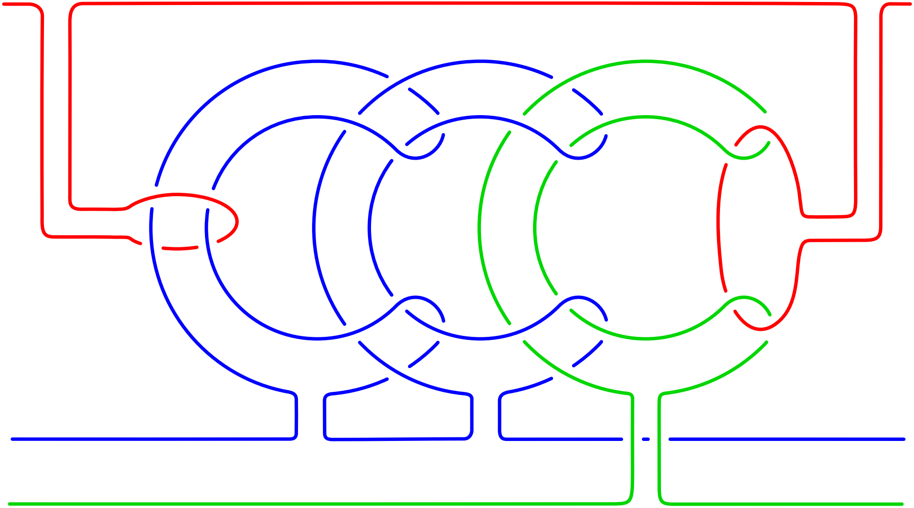

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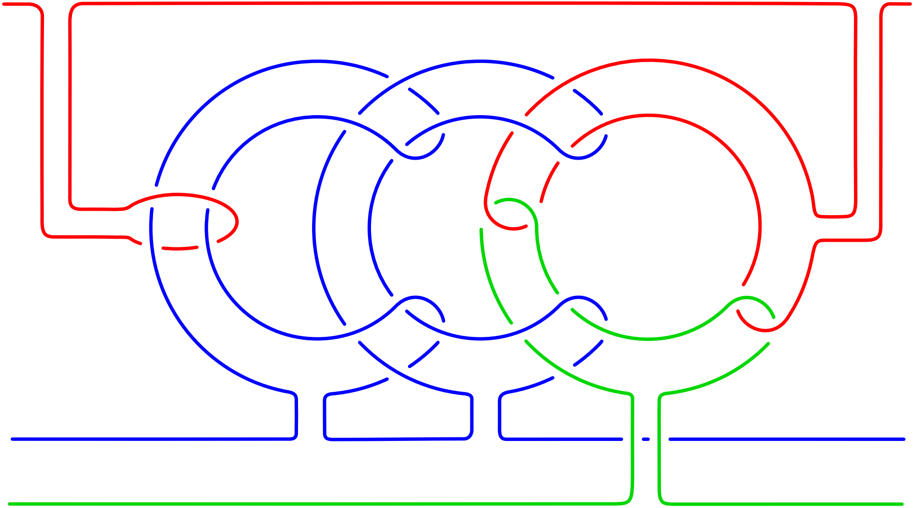

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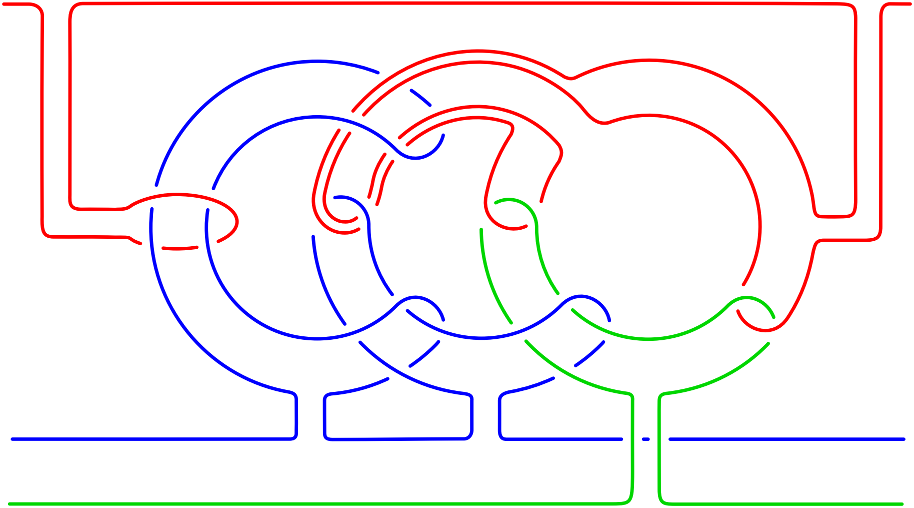

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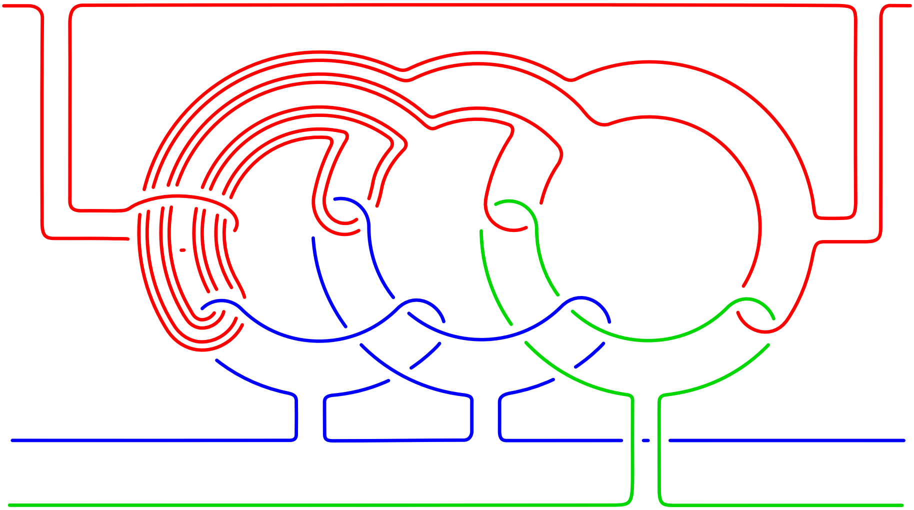

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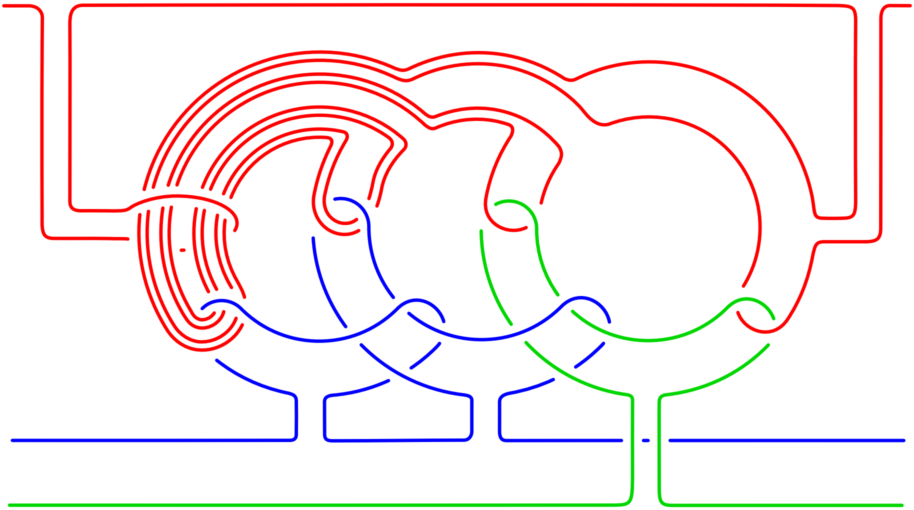

"A minimal solution of the Chinese Rings puzzle."

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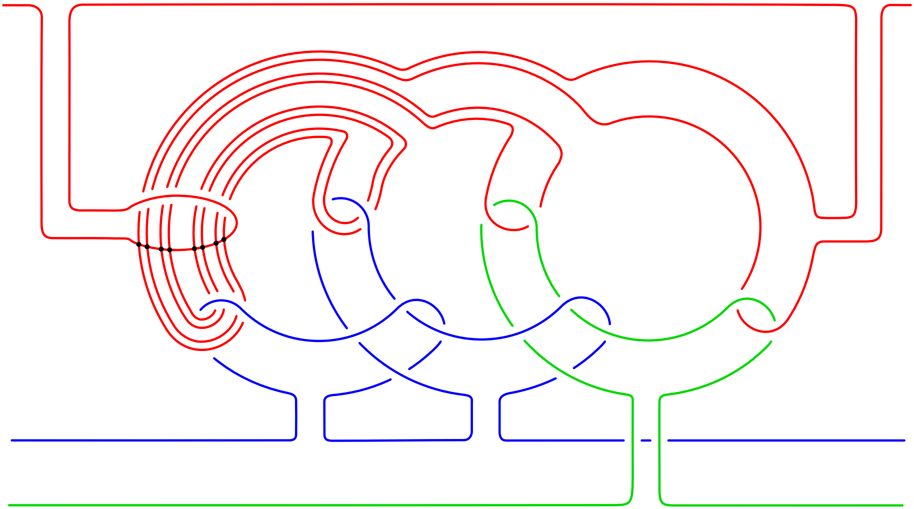

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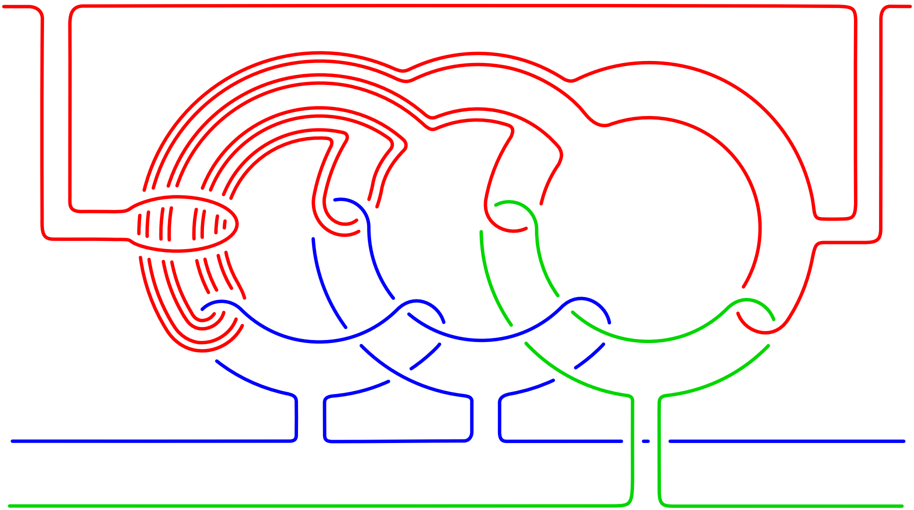

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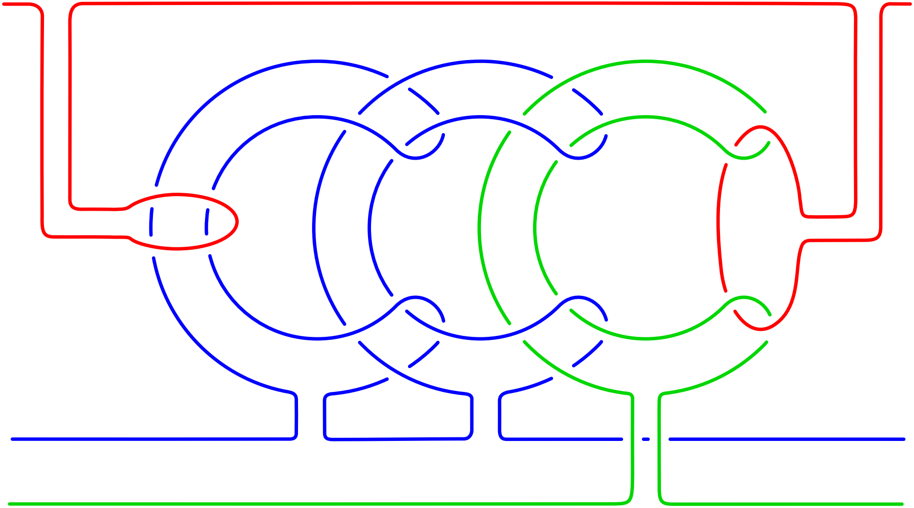

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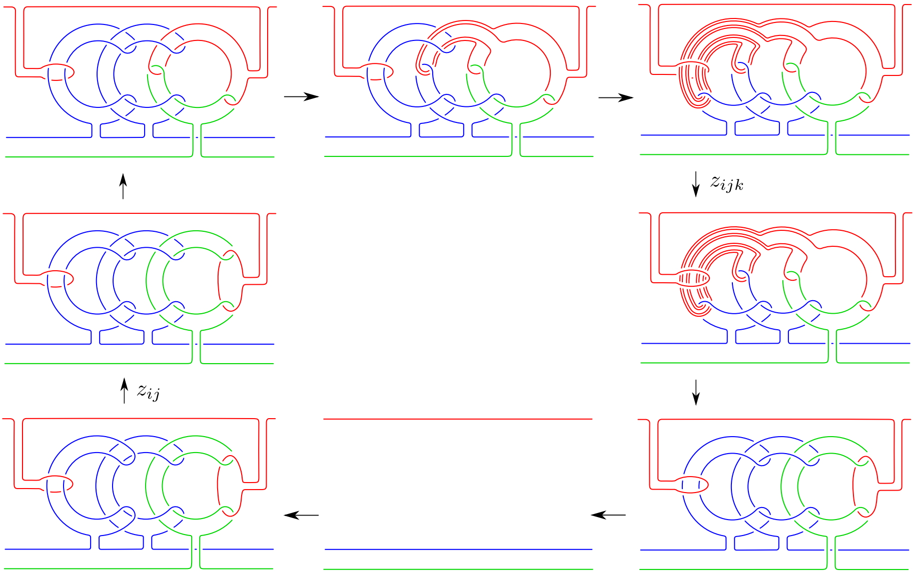

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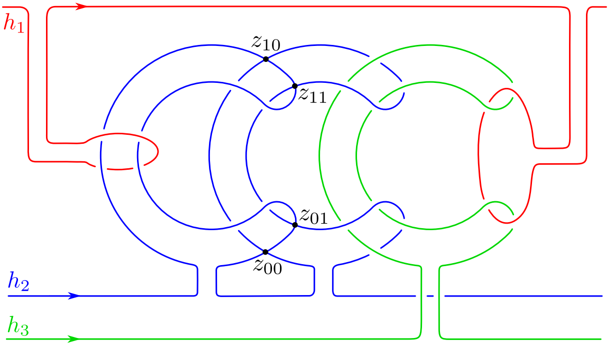

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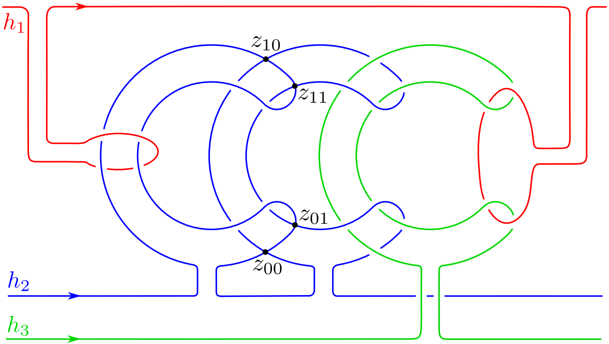

$\ell_{z_{ij}3} = 0$

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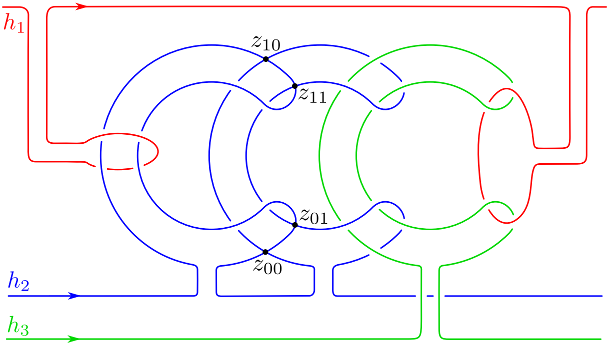

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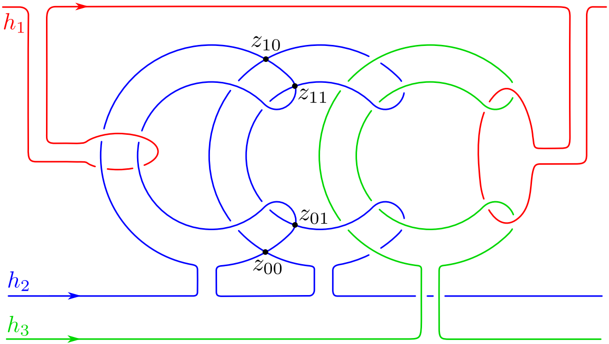

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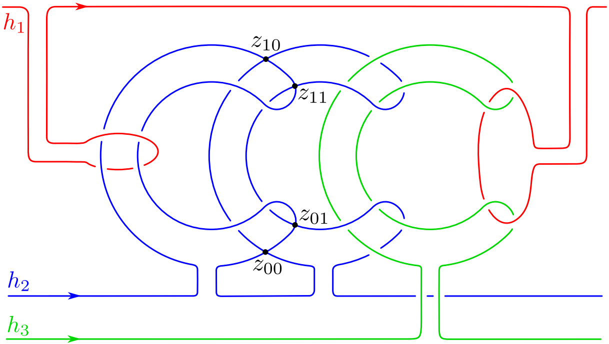

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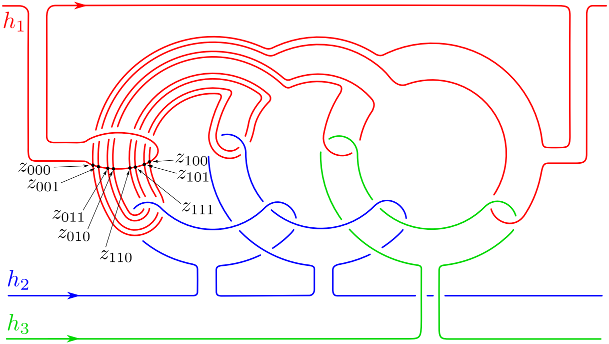
$\Sigma_2^{\mathrm{ab}}(h) = (t_1 - 1) - (t_1 - 1) = 0$

Brunnian example ($m = 3$)



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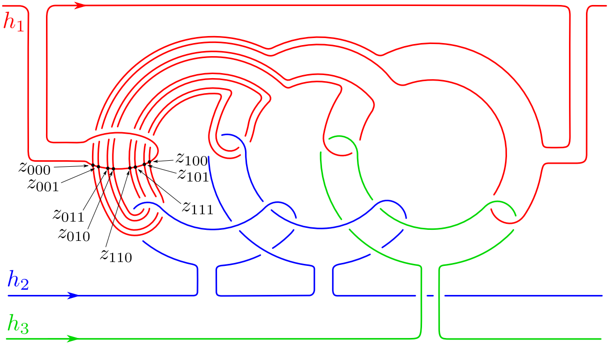

$\ell_{z_{ijk}3} = k$

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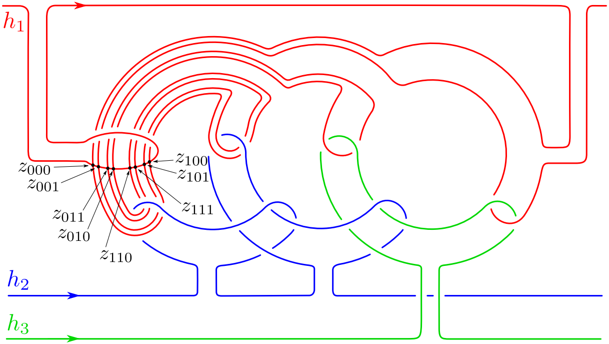

$$\ell_{z_{ijk}3} = k$$

$$\ell_{z_{ijk}2_a} = i, \quad \ell_{z_{ijk}2_b} = j$$

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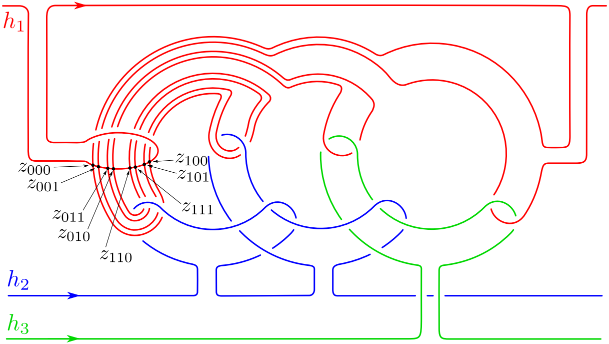

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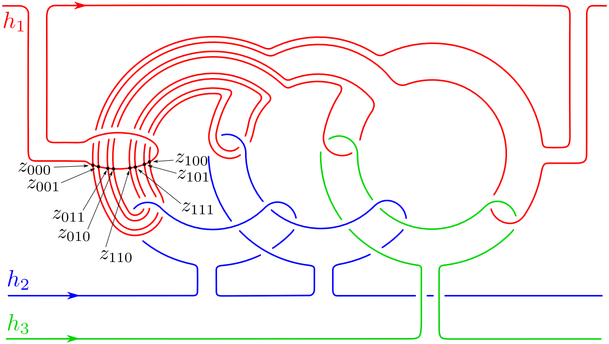

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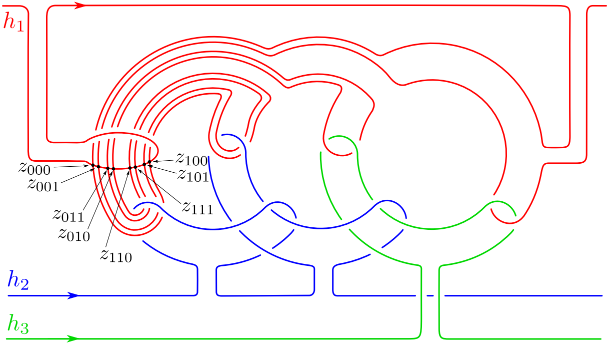

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$$\Sigma_1^{\mathrm{ab}}(h) = \sum_{i,j,k\in\{0,1\}} (-1)^{i+j+k}(t_2^{i+j} t_3^k - 1)$$

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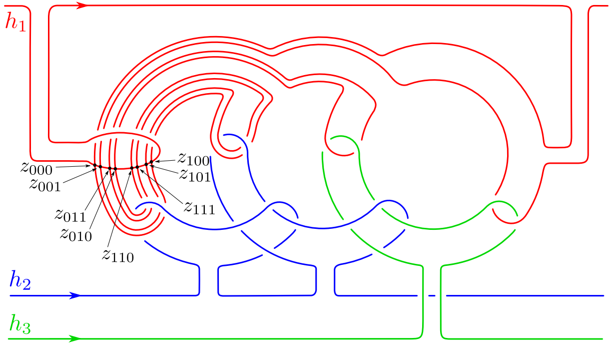

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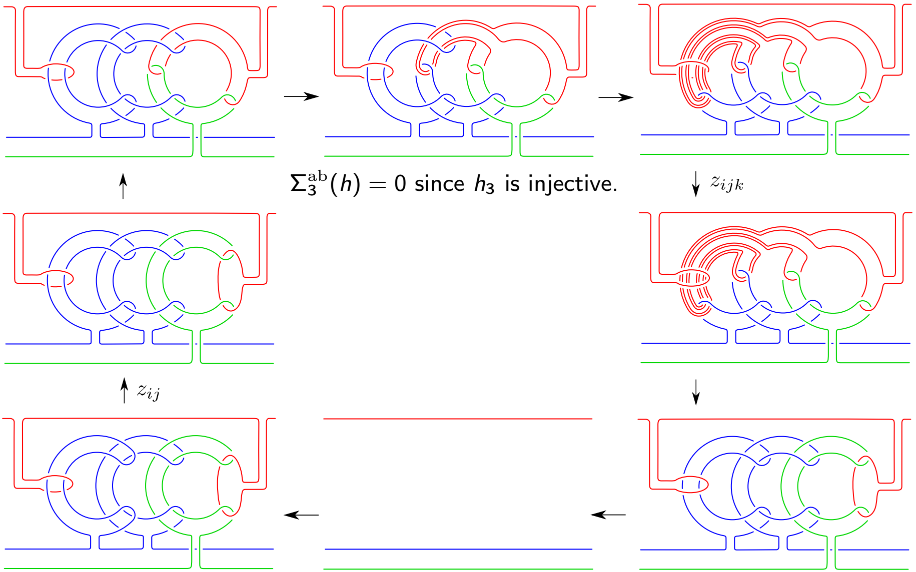

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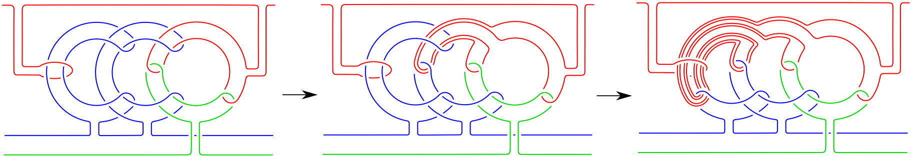
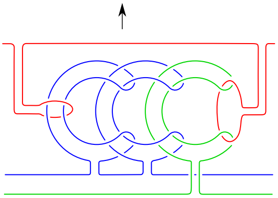
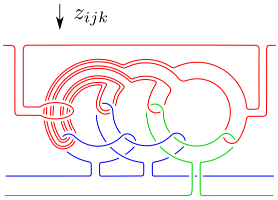


$\Sigma_3^{\mathrm{ab}}(h) = 0$ since h_3 is injective.

$\Sigma^{\mathrm{ab}}(h) = \left((1 - t_2)^2(1 - t_3), 0, 0\right)$





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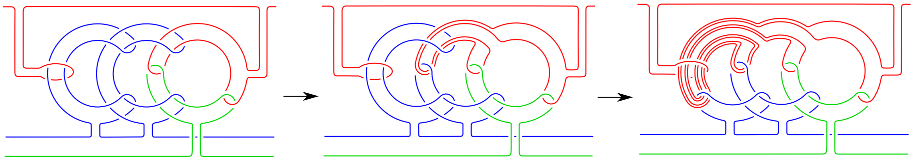

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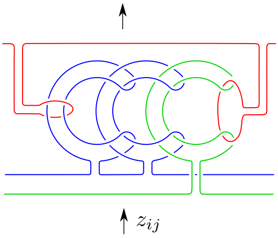

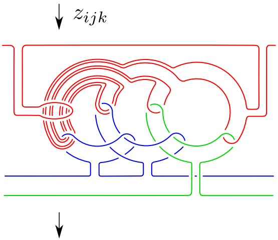

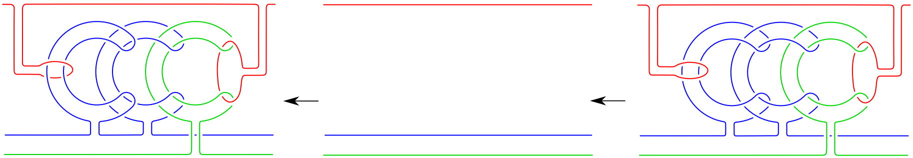

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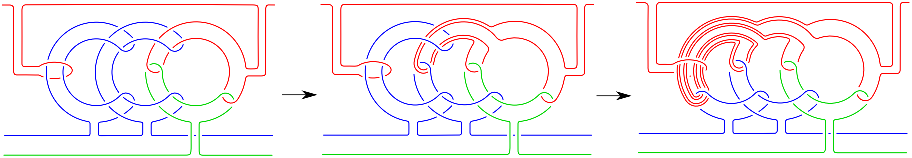

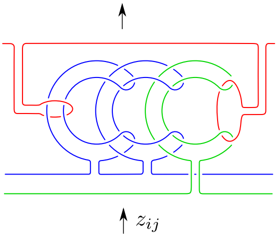

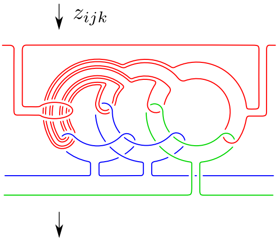

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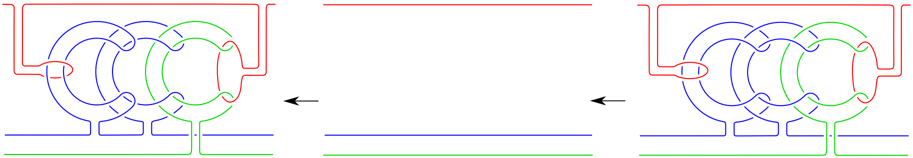
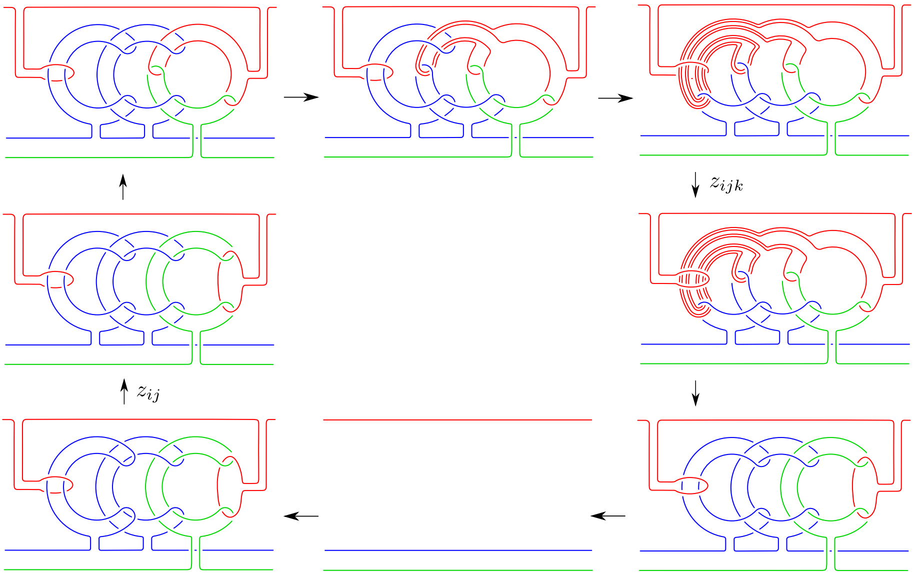
Every $f\colon S^2 \sqcup S^2 \to S^4$ with injective f_+ or f_- is trivial (Schneiderman–Teichner '19)



Brunnian example (generalization)



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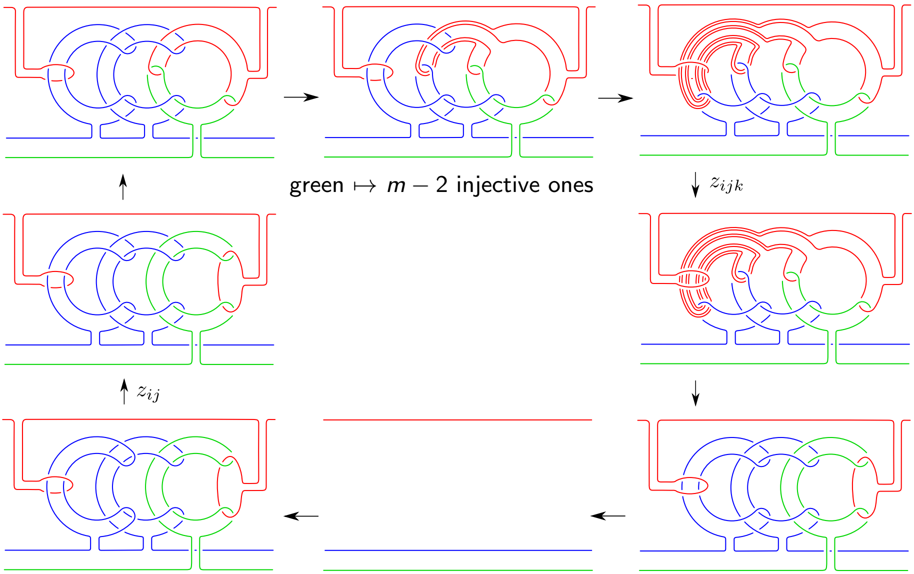

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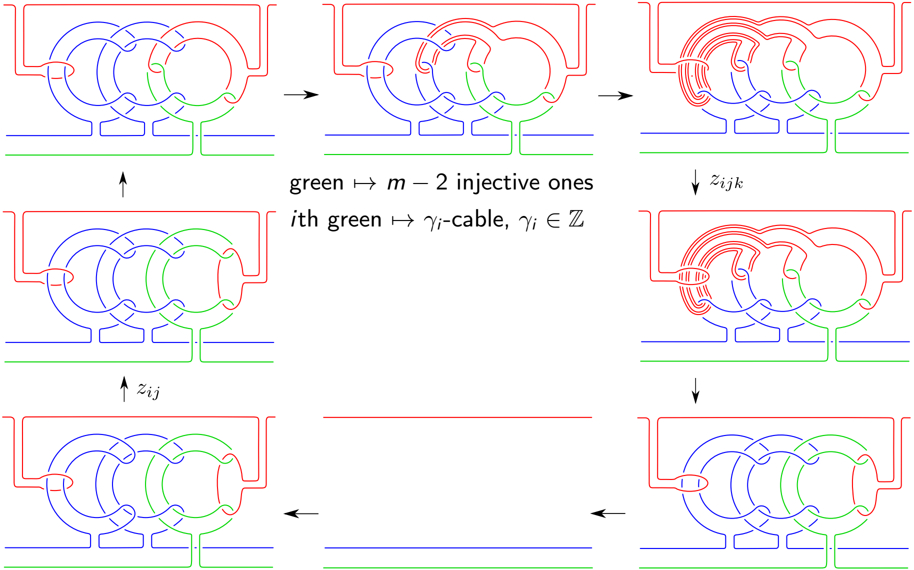

green $\mapsto$ $m-2$ injective ones

ith green $\mapsto$ γ_i-cable, $\gamma_i \in \mathbb{Z}$

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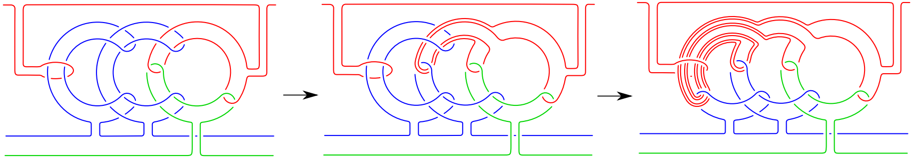

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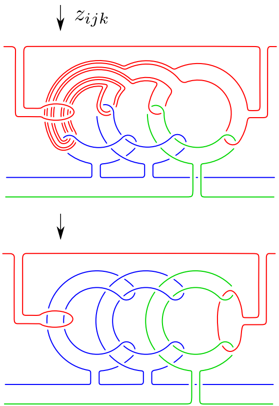

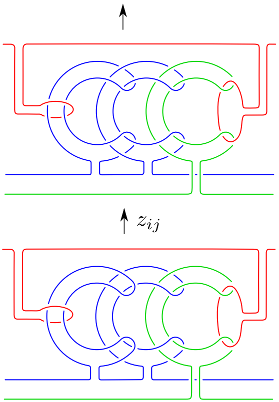

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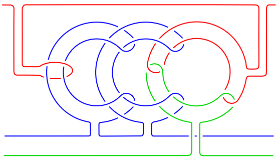
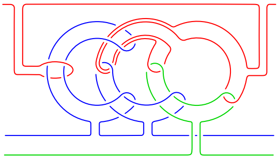

→

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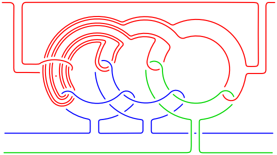

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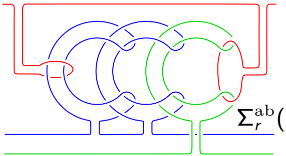
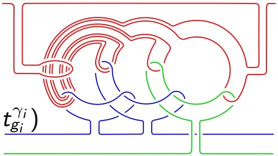
↓ z_{ijk}





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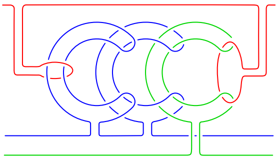
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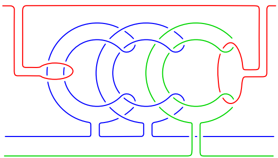

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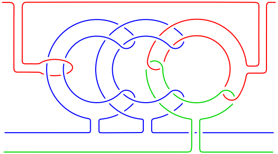

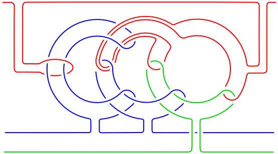

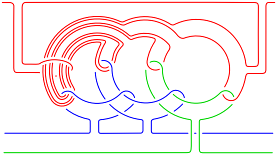

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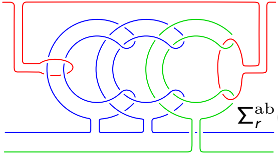

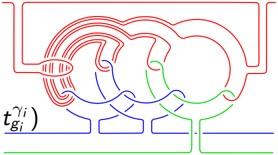

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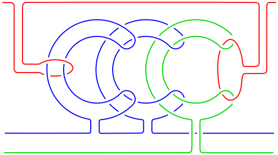

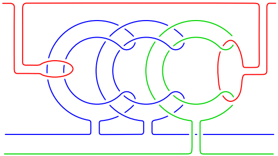

Images of σ^{ab} and Σ^{ab}

Theorem 2. $\Sigma^{\mathrm{ab}}(BFDLM^4_{2,2,2}) = \bigoplus_{i=1}^3 \operatorname{im}\Sigma_i^{\mathrm{ab}}$ and each $\operatorname{im}\Sigma_i^{\mathrm{ab}} = \ker\Delta_i^3$, where

$$\Delta_i^m\colon \mathbb{Z}[t_1^{\pm1},\dots,\hat t_a^{\pm1},\dots,t_m^{\pm1}] \to \mathbb{Z}\oplus\bigoplus_{c\neq a}\mathbb{Z}[t_1^{\pm1},\dots,\hat t_a^{\pm1},\dots,\hat t_c^{\pm1},\dots,t_m^{\pm1}]$$

is given by $f\mapsto\left(\frac{\partial^m f}{\partial t_1\cdots\partial t_m}\Big|_{t_1=\cdots=t_m=1};\ f|_{t_1=1},\dots,f|_{t_m=1}\right)$.

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Addendum. The inclusion $\operatorname{im}\Sigma_i^{\mathrm{ab}}\supset\ker\Delta_i^m$ also holds for $m>3$.

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Weak Δ-link-homotopy: C_2^{xxx}-moves (=Δ-link homotopy) + $C_3^{xx,yz}$-moves.

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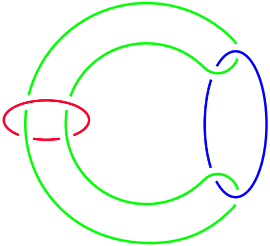

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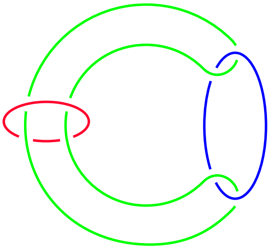

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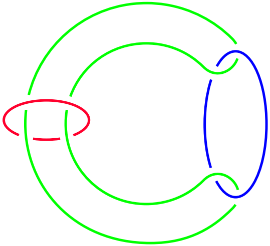

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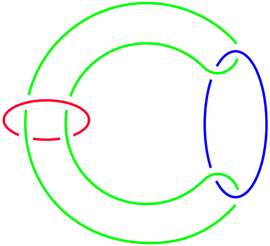

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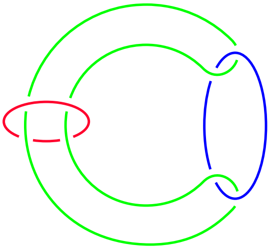

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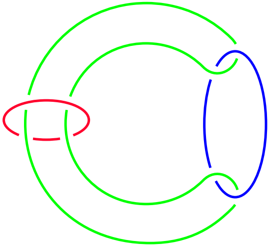

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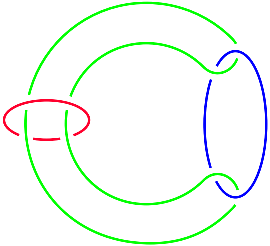

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Addendum. If $H\colon (S^2 \sqcup \cdots \sqcup S^2) \times I \to S^4 \times I$ is a generic link homotopy between link maps $F_0, F_1\colon S^2 \sqcup \cdots \sqcup S^2 \to S^4$, then the inclusion induced maps $\mathcal{G}(F_i) \to \mathcal{G}(H)$ are isomorphisms.

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Example. Let $H \subset \mathbb{Z}[RF_2]$ be the copy of $\mathbb{Z}[\mathbb{Z}]$ generated by the powers of $[y, x]$. Then $T \cap H = \langle [y, x]^m - [y, x]^{-m} \rangle$ and there is an epimorphism $\mathbb{Z}[RF_2]/T \to H/T \cap H \simeq \mathbb{Z}[t]$.

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Let $\sigma_i(f) = \sum_{z \in \Delta(f_i)} \varepsilon_z(\lambda_z - 1) \in \mathbb{Z}[RF(x_1, \dots, \hat{x}_i, \dots, x_m)]/T$.

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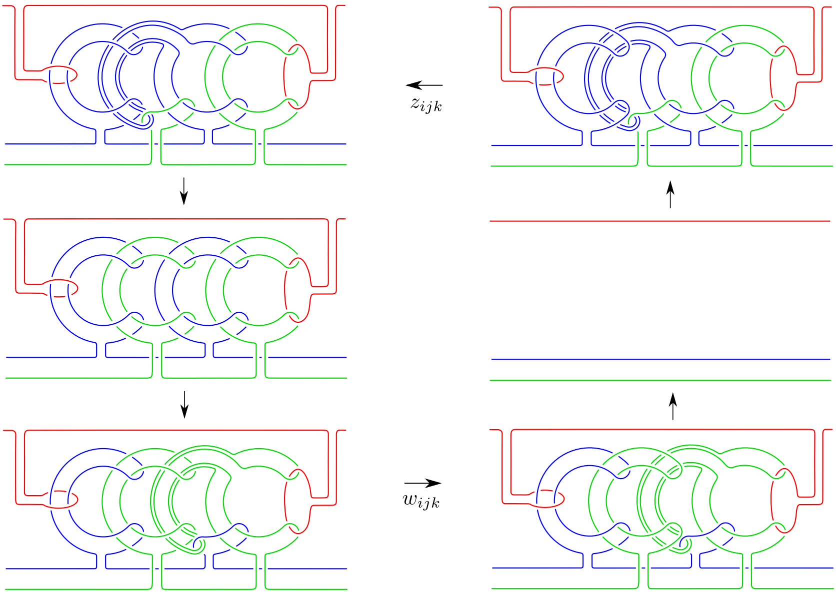
Lemma. $\sigma(f)$ is a link homotopy invariant.

Nontrivial link map $\hat{h}$ with vanishing $\sigma^{ab}(\hat{h})$



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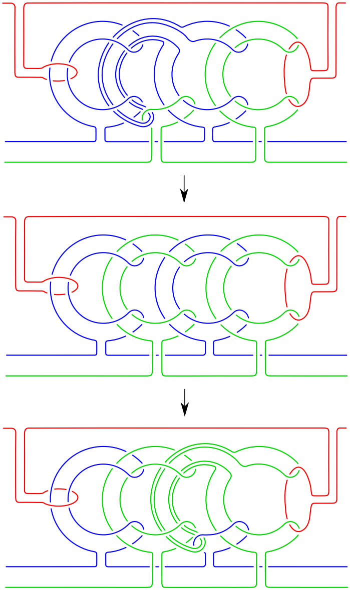
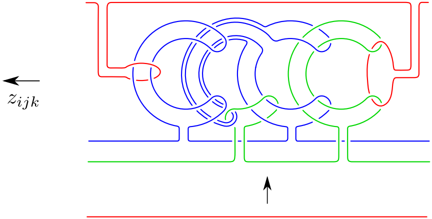
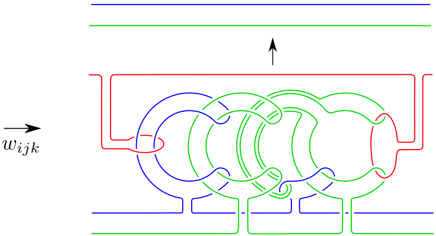




$\Sigma_1(h) = 0$ since h_1 is injective.



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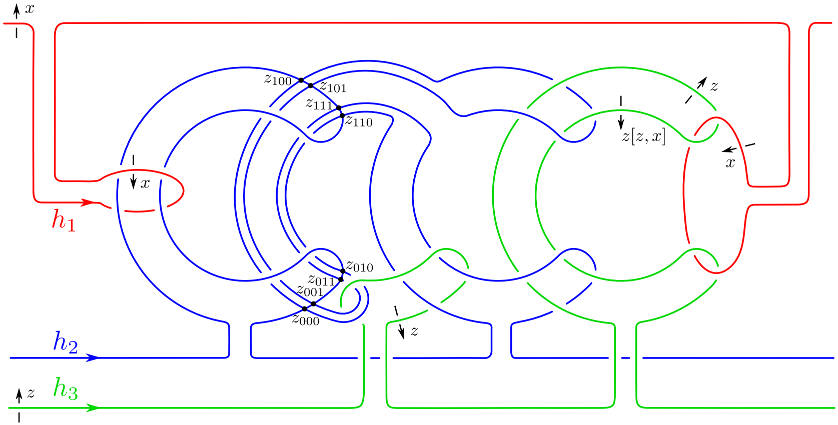

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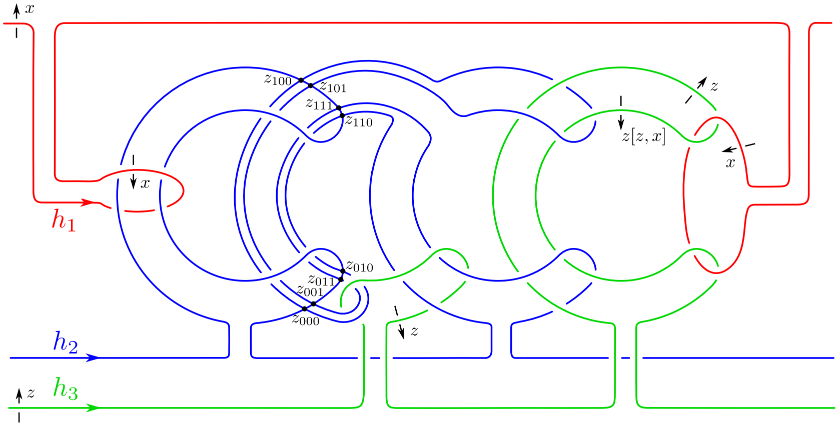

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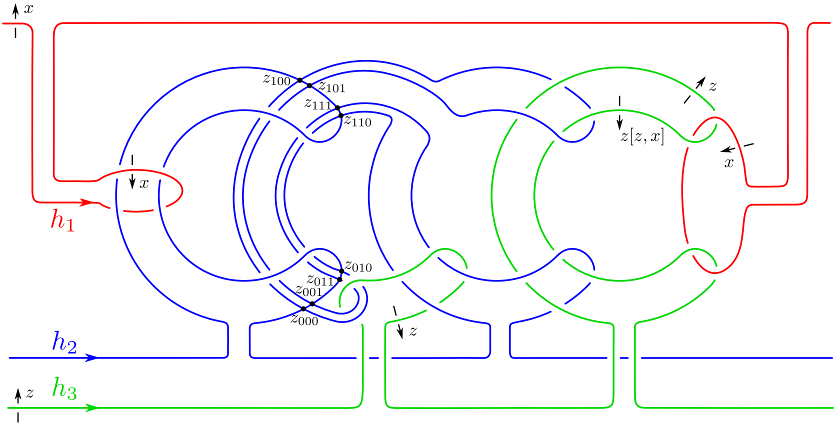

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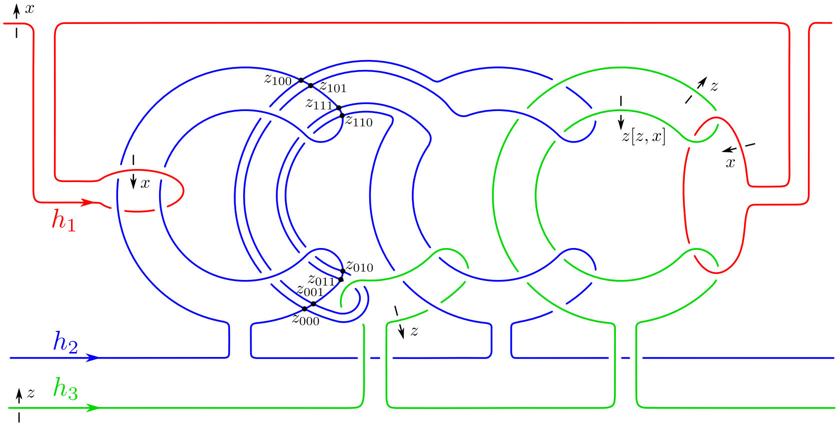
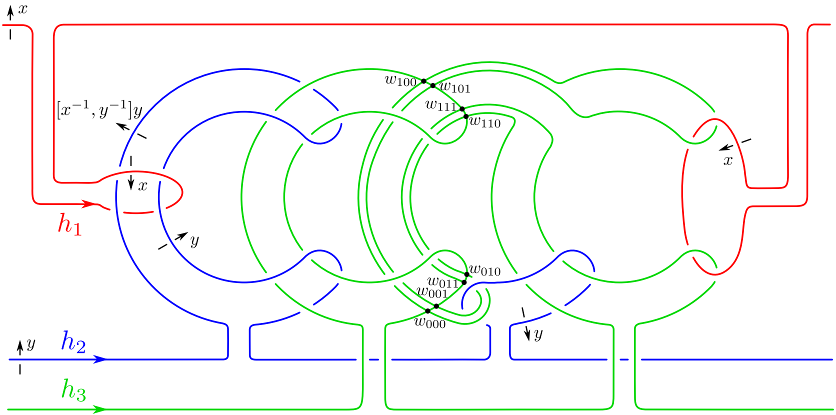


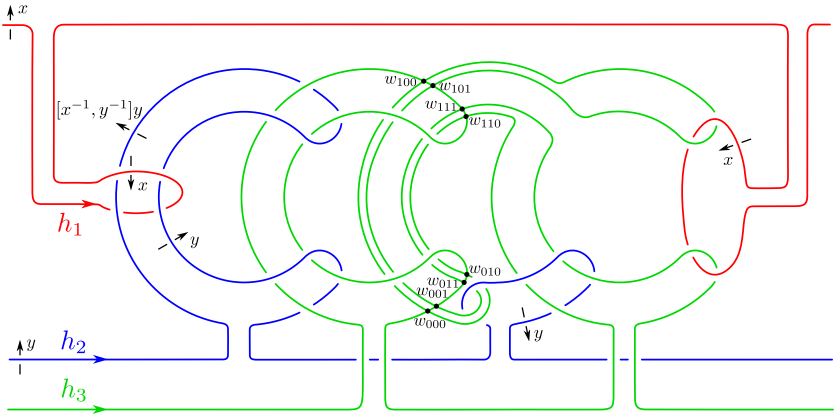
$$\Sigma_2(h) = (z-1)(x-1)([z,x]-1) \qquad \Rightarrow \qquad \Sigma_2^{\mathrm{ab}}(h) = 0$$

$$\sigma_2(\hat{h}) = (z-1)(x-1)([z,x]-1) \underset{T}{=} [z,x] - 1 \underset{T}{\neq} 0.$$

Nontrivial link map $\hat{h}$ with vanishing $\sigma^{ab}(\hat{h})$

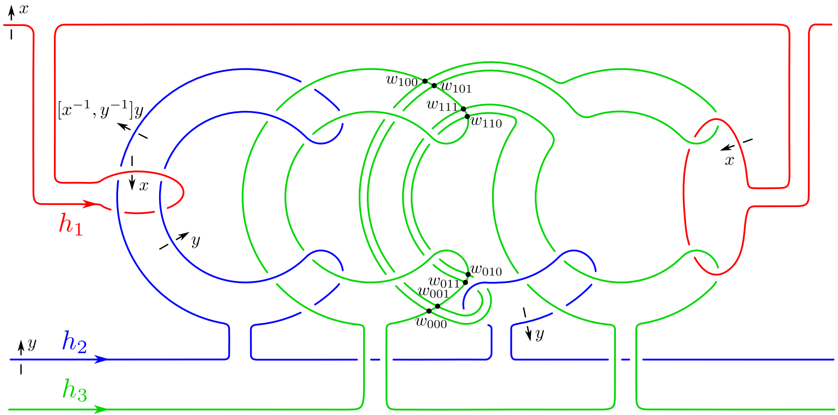


Nontrivial link map $\hat{h}$ with vanishing $\sigma^{ab}(\hat{h})$



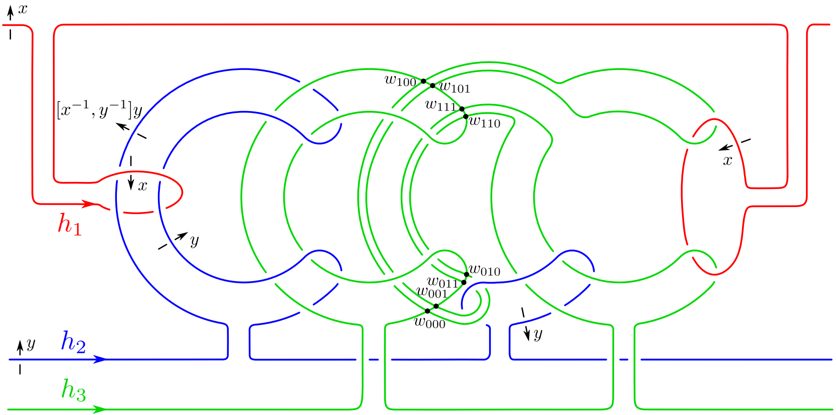
$$\Sigma_3(h) = (1 - x)(1 - y)(1 - [x, y])$$

Nontrivial link map $\hat{h}$ with vanishing $\sigma^{ab}(\hat{h})$



$$\Sigma_3(h) = (1 - x)(1 - y)(1 - [x, y]) \quad \Rightarrow \quad \Sigma_3^{\text{ab}}(h) = 0.$$

Nontrivial link map $\hat{h}$ with vanishing $\sigma^{ab}(\hat{h})$



$$\Sigma_3(h) = (1-x)(1-y)(1-[x,y]) \qquad \Rightarrow \qquad \Sigma_3^{\mathrm{ab}}(h) = 0.$$

$$\sigma_3(\hat{h}) = (1-x)(1-y)(1-[x,y]) \underset{T}{=} 1-[x,y] \underset{T}{\neq} 0.$$