# Isospectral Riemann surfaces and graphs 

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## Isospectral surfaces and graphs

Since the classical paper by Mark Kac "Can one hear the shape of a drum?"(1966), the question of what geometric properties are determined by the Laplace operator has inspired many intriguing results.

Wolpert (1979) showed that a generic Riemann surface is determined by its Laplace spectrum. Nevertheless, pairs of isospectral non-isometric Riemann surfaces in every genus $\geq 4$ are known. See papers by Buser (1986), Brooks and Tse (1987), and others. There are also examples of isospectral non-isometric surfaces of genus two and three with variable curvature Barden and Hyunsuk Kang (2012). In the same time, isospectral genus one Riemann surfaces (flat tori) are isometric (Brooks, 1988). Similar result for Klein bottle was obtained by R. Isangulov (2000). Similar results are also known for graphs (see survey by E.R.van Dam and W.H.Haemers (2003)).

## Isospectral surfaces and graphs

Peter Buser (1992) posed an interesting problem: are two isospectral Riemann surfaces of genus two isometric? Up to our knowledge the problem is still open but, quite probably, can be solved positively. The aim of this lecture is to give a positive solution of this problem for theta graphs. Because of the intrinsic link between Riemann surfaces and graphs we hope that our result will be helpful to make a progress in solution of the Buser problem. Recall that two graphs $G$ and $G^{\prime}$ are isospectral if an only if they share the same Laplacian polynomial. That is

$$
\mu(G, x)=\mu\left(G^{\prime}, x\right)
$$

## Laplacian polynomial

A. K. Kel'mans (1967) gave a combinatorial interpretation to all the coefficients of $\mu(X, x)$ in terms of the numbers of certain subforests of the graph. We present the result in the following form.

## Theorem

If $\mu(X, x)=x^{n}-c_{n-1} x^{n-1}+\ldots+(-1)^{n-i} c_{i} x^{i}+\ldots+(-1)^{n-1} c_{1} x$ then

$$
c_{i}=\sum_{S \subset V,|S|=i} T\left(X_{S}\right)
$$

where $T(H)$ is the number of spanning trees of $H$, and $X_{S}$ is obtained from $X$ by identifying all points of $S$ to a single point.

## Theta graphs



Fig.1. Theta graph $\Theta(k, l, m)$.

## Theta graphs

Let $u$ and $v$ are two (not necessary distinct) vertices. Denote by $\Theta(k, I, m)$ the graph consisting of three internally disjoint paths joining $u$ to $v$ with lengths $k, I, m \geq 0$ (see Fig. 1). We set
$\sigma_{1}=\sigma_{1}(k, I, m)=k+I+m, \sigma_{2}=\sigma_{2}(k, I, m)=k I+I m+k m$, and $\sigma_{3}=\sigma_{3}(k, l, m)=k / m$.
It is easy to see that two graphs $\Theta(k, l, m)$ and $\Theta\left(k^{\prime}, l^{\prime}, m^{\prime}\right)$ are isomorphic if and only if the unordered triples $\{k, I, m\}$ and $\left\{k^{\prime}, I^{\prime}, m^{\prime}\right\}$ coincide.
Equivalently, $\sigma_{1}=\sigma_{1}^{\prime}, \sigma_{2}=\sigma_{2}^{\prime}$ and $\sigma_{3}=\sigma_{3}^{\prime}$, where $\sigma_{1}^{\prime}=\sigma_{1}\left(k^{\prime}, I^{\prime}, m^{\prime}\right), \sigma_{2}^{\prime}=\sigma_{2}\left(k^{\prime}, l^{\prime}, m^{\prime}\right)$, and $\sigma_{3}^{\prime}=\sigma_{1}\left(k^{\prime}, l^{\prime}, m^{\prime}\right)$.

## The main result

The first result is the following theorem.

## Theorem

Two theta graphs are Laplacian isospectral if and only if they are isomorphic.

The proof of the theorem is based on the following three lemmas.

## Lemma 2.

## Lemma 2

Let $G=\Theta(k, l, m)$ be a theta graph and

$$
\mu(X, x)=x^{n}-c_{n-1} x^{n-1}+\ldots+(-1)^{n-i} c_{i} x^{i}+\ldots+(-1)^{n-1} c_{1} x
$$

be its Laplacian polynomial. Then $n=k+I+m-1, c_{n-1}=2(k+I+m)$ and $c_{1}=(k I+I m+k m)(k+I+m-1)$.

Proof. The number of vertices, edges and spanning trees of graph $G$ are given by

$$
V(G)=k+I+m-1, E(G)=k+I+m, T(G)=k I+I m+k m
$$

Then by the Kel'mans theorem we have
$n=V(G)=k+I+m-1, c_{n-1}=2 E(G)=2(k+I+m)$ and $c_{1}=V(G) \cdot T(G)=(k I+I m+k m)(k+I+m-1)$.

## Lemma 3.

## Lemma 3

Let $G=\Theta(k, l, m)$ be a theta graph and
$\mu(X, x)=x^{n}-c_{n-1} x^{n-1}+\ldots+(-1)^{n-i} c_{i} x^{i}+\ldots+(-1)^{n-1} c_{1} x$ be its Laplacian polynomial. Then

$$
c_{2}=A\left(\sigma_{1}, \sigma_{2}\right)+B\left(\sigma_{1}, \sigma_{2}\right) \sigma_{3}
$$

where $A(s, t)=\left(4 t-3 s t-2 s^{2} t+s^{3} t+4 t^{2}-s t^{2}\right) / 12, B(s, t)=$ $\left(3-4 s+s^{2}-3 t\right) / 12$,
$\sigma_{1}=k+I+m, \sigma_{2}=k I+I m+k m$, and $\sigma_{3}=k I m$.
Proof. By the Kelmans theorem $c_{2}=\sum_{S \subset V,|S|=2} T\left(X_{S}\right)$, where $X_{S}$ runs through all graphs obtained from $G=\Theta(k, l, m)$ by gluing two vertices. There are exactly four types of such graphs shown on the Fig.2. We will enumerate the spanning trees of each type separately.

## Lemma 3.



Fig. 2. The graphs obtained from $\Theta(k, I, m)$ by gluing two vertices

## Lemma 3.

Type $G_{1}$. Glue two 3-valent vertices of graph $G$. As a result we obtain the graph $G_{1}$ shown on Fig. 2. The number of spanning trees of this graph is $T_{1}=T\left(C_{k}\right) \cdot T\left(C_{l}\right) \cdot T\left(C_{m}\right)=k / m$.
Type $G_{2}$. Glue one 3 -valent and one 2 -valent vertices of graph $G$. For given $i, 1 \leq i \leq k-1$ the number of spanning trees for graph $G_{2}$ is equal to $T\left(C_{i}\right) \cdot T(\Theta(k-i, I, m))=i \sigma_{2}(k-i, l, m)$. We set
$F(k, I, m)=\sum_{i=1}^{k-1} i \sigma_{2}(k-i, I, m)$. Then the total number of spanning trees for graphs of type $G_{2}$ is

$$
T_{2}=2(F(k, I, m)+F(I, m, k)+F(m, k, I))
$$

The multiple 2 is needed since the graph $\Theta(k, l, m)$ has two 3-valent vertices.

## Lemma 3.

In a similar way we calculate the numbers $T_{3}$ and $T_{4}$. Finally, we have

$$
c_{2}=T_{1}+T_{2}+T_{3}+T_{4}=A\left(\sigma_{1}, \sigma_{2}\right)+B\left(\sigma_{1}, \sigma_{2}\right) \sigma_{3}
$$

## Lemma 4.

## Lemma 4

Let $G=\Theta(k, l, m)$ be a theta graph and $\mu(X, x)=x^{n}-c_{n-1} x^{n-1}+\ldots+(-1)^{n-i} c_{i} x^{i}+\ldots+(-1)^{n-1} c_{1} x$ be its Laplacian polynomial. Then

$$
c_{3}=C\left(\sigma_{1}, \sigma_{2}\right)+D\left(\sigma_{1}, \sigma_{2}\right) \sigma_{3}+E\left(\sigma_{1}, \sigma_{2}\right) \sigma_{3}^{2}
$$

where

$$
\begin{aligned}
C(s, t) & =\left(-34 t+21 s t+25 s^{2} t-10 s^{3} t-3 s^{4} t+s^{5} t-50 t^{2}+10 s t^{2}\right. \\
& \left.+12 s^{2} t^{2}-2 s^{3} t^{2}-16 t^{3}+s t^{3}\right) / 360 \\
D(s, t) & =\left(-45+50 s+5 s^{2}-12 s^{3}+2 s^{4}+24 s t-9 s^{2} t+15 t^{2}\right) / 360 \\
E(s, t) & =-3(-8+3 s) / 360
\end{aligned}
$$

## Lemma 4.



Fig. 3. The graphs obtained from $\Theta(k, l, m)$ by gluing three vertices

## Lemma 4.

The proof of Lemma 4 is based on similar arguments given in the proof of Lemma 3. One can consider the six cases shown on the Fig. 3.

## Proof of the Main Theorem

Let $G$ and $G^{\prime}$ be two bridgeless graphs of genus two. Then by Lemma 1 for suitable $\{k, l, m\}$ and $\left\{k^{\prime}, l^{\prime}, m^{\prime}\right\}$ we have

$$
G=\Theta(k, l, m) \text { and } G^{\prime}=\Theta\left(k^{\prime}, l^{\prime}, m^{\prime}\right)
$$

Denote by
$\mu(X, x)=x^{n}-c_{n-1} x^{n-1}+\ldots+(-1)^{n-i} c_{i} x^{i}+\ldots+(-1)^{n-1} c_{1} x$ and $\mu\left(G^{\prime}, x\right)=\mu(X, x)=$
$x^{n^{\prime}}-c_{n^{\prime}-1} x^{n^{\prime}-1}+\ldots+(-1)^{n^{\prime}-i} c_{i} x^{i}+\ldots+(-1)^{n^{\prime}-1} c_{1} x$ their Laplacian polynomials.

## Proof of the Main Theorem

Suppose that the graphs $G$ and $G^{\prime}$ are isospectral. Then $n^{\prime}=n, c_{1}^{\prime}=c_{1}, \ldots, c_{n-1}^{\prime}=c_{n-1}$. By Lemma 2 we obtain

$$
2 \sigma_{1}=2 \sigma_{1}^{\prime} \text { and } \sigma_{2}\left(\sigma_{1}-1\right)=\sigma_{2}^{\prime}\left(\sigma_{1}^{\prime}-1\right)
$$

Since both graphs are of genus 2 we have $\sigma_{1}>1$ and $\sigma_{1}^{\prime}>1$. Then the obtained system of equations gives $\sigma_{1}=\sigma_{1}^{\prime}$ and $\sigma_{2}=\sigma_{2}^{\prime}$. The theorem will be proved if we show that $\sigma_{3}=\sigma_{3}^{\prime}$. We will do this in two steps. First of all, we note that isospectral graphs with $n \leq 5$ vertices are isomorphic. So, we can assume that $n=k+I+m-1>5$, that is $\sigma_{1}=k+I+m>6$.

## Proof of the Main Theorem

By Lemma 3

$$
c_{2}=A\left(\sigma_{1}, \sigma_{2}\right)+B\left(\sigma_{1}, \sigma_{2}\right) \sigma_{3}
$$

where $A(s, t)=\left(4 t-3 s t-2 s^{2} t+s^{3} t+4 t^{2}-s t^{2}\right) / 12$ and $B(s, t)=\left(3-4 s+s^{2}-3 t\right) / 12$.

Step 1. $B\left(\sigma_{1}, \sigma_{2}\right) \neq 0$. Since $c_{2}^{\prime}=c_{2}, \sigma_{1}=\sigma_{1}^{\prime}$ and $\sigma_{2}=\sigma_{2}^{\prime}$ we obtain

$$
B\left(\sigma_{1}, \sigma_{2}\right) \sigma_{3}^{\prime}=B\left(\sigma_{1}, \sigma_{2}\right) \sigma_{3}
$$

Hence $\sigma_{3}=\sigma_{3}^{\prime}$ and the theorem is proved.

## Proof of the Main Theorem

Step 2. $B\left(\sigma_{1}, \sigma_{2}\right)=0$. Then $c_{3}^{\prime}=c_{3}$ gives

$$
D\left(\sigma_{1}, \sigma_{2}\right) \sigma_{3}^{\prime}+E\left(\sigma_{1}, \sigma_{2}\right){\sigma_{3}^{\prime}}^{2}=D\left(\sigma_{1}, \sigma_{2}\right) \sigma_{3}+E\left(\sigma_{1}, \sigma_{2}\right) \sigma_{3}^{2}
$$

where $C(s, t), D(s, t)$ and $E(s, t)$ are as in Lemma 4.
We note that $E\left(\sigma_{1}, \sigma_{2}\right)=-3\left(-8+3 \sigma_{1}\right) / 360 \neq 0$ for any integer $\sigma_{1}$.
Then the above equation has two solution with respect to $\sigma_{3}^{\prime}$.
The first solution is $\sigma_{3}^{\prime}=\sigma_{3}$ and the second one

$$
\sigma_{3}^{\prime}=-\frac{D\left(\sigma_{1}, \sigma_{2}\right)}{E\left(\sigma_{1}, \sigma_{2}\right)}-\sigma_{3}
$$

## Proof of the Main Theorem

In the first case the theorem is proved. So we assume the second case. Since $B\left(\sigma_{1}, \sigma_{2}\right)=0$ we have $\sigma_{2}=\left(3-4 \sigma_{1}+\sigma_{1}^{2}\right) / 3$. Hence

$$
\begin{equation*}
\sigma_{3}^{\prime}=\frac{1}{729}\left(2\left(425-357 \sigma_{1}-144 \sigma_{1}^{2}+27 \sigma_{1}^{3}\right)-\frac{490}{-8+3 \sigma_{1}}\right)-\sigma_{3} \tag{*}
\end{equation*}
$$

## Proof of the Main Theorem

Since $\sigma_{3}$ and $\sigma_{3}^{\prime}$ are integers we obtain
(i) $N=2\left(425-357 \sigma_{1}-144 \sigma_{1}^{2}+27 \sigma_{1}^{3}\right)-\frac{490}{-8+3 \sigma_{1}}$ is divisible by 729 ;
(ii) $-8+3 \sigma_{1}$ is a divisor of 490 ;
(iii) $\sigma_{2}=\left(3-4 \sigma_{1}+\sigma_{1}^{2}\right) / 3$ is a positive integer.

There only finite number possibilities to satisfy these three conditions $\sigma_{1} \in\{6,19,166\}$.
The case $\sigma_{1}=6$ can be excluded since we suggested that $\sigma_{1}>6$.

## Proof of the Main Theorem

Two cases remained $\sigma_{1}=19$ and $\sigma_{1}=166$. Here by $(*)$ we have $\sigma_{3}^{\prime}=348-\sigma_{3}$ and $\sigma_{3}^{\prime}=327789-\sigma_{3}$ respectively. The respective values of $\sigma_{2}$ are 96 and 8965.
Let $\sigma_{1}=19$. We have

$$
k+I+m=19, k I+I m+m k=96, k I m=\sigma_{3} .
$$

This system has only one solution $\{k, I, m\}=\{3,4,12\}, \sigma_{3}=144$.
Now we are able to find parameters $k^{\prime}, l^{\prime}, m^{\prime}, \sigma_{3}^{\prime}$ of the graph $G^{\prime}=\Theta\left(k^{\prime}, I^{\prime}, m^{\prime}\right)$. First of all, $\sigma_{3}^{\prime}=348-\sigma_{3}=204$. Then we have

$$
k^{\prime}+l^{\prime}+m^{\prime}=19, k^{\prime} l^{\prime}+l^{\prime} m^{\prime}+m^{\prime} k=96, k^{\prime} I^{\prime} m^{\prime}=204 .
$$

The latter system has no integer solutions. So the case $\sigma_{1}=19$ is impossible.

## Proof of the Main Theorem

Let $\sigma_{1}=166$. We have

$$
k+I+m=166, k I+I m+m k=8965, k I m=\sigma_{3} .
$$

This system has only one solution $\{k, I, m\}=\{39,59,68\}, \sigma_{3}=39 \cdot 59 \cdot 68$. Find parameters $k^{\prime}, l^{\prime}, m^{\prime}, \sigma_{3}^{\prime}$ of the graph $G^{\prime}=\Theta\left(k^{\prime}, l^{\prime}, m^{\prime}\right)$. Now, $\sigma_{3}^{\prime}=327789-\sigma_{3}=171321$. Then we have

$$
k^{\prime}+l^{\prime}+m^{\prime}=166, \quad k^{\prime} I^{\prime}+I^{\prime} m^{\prime}+m^{\prime} k^{\prime}=8965, \quad k^{\prime} I^{\prime} m^{\prime}=171321 .
$$

The system has no integer solutions. The case $\sigma_{1}=166$ is also impossible. The proof of the theorem is finished.

## Final remarks

$1^{\circ}$. The Theorem is not valid for genus two graph with bridges.


Fig. 4. Genus 2 graphs with the same spectrum. The second has a bridge.
The graphs share the following Laplacian polynomial:

$$
-72 x+192 x^{2}-176 x^{3}+73 x^{4}-14 x^{5}+x^{6}
$$

$2^{\circ}$. There are isospectral bridgeless graphs of genus three which are not isomorphic.


Fig. 5. Two nonisomorphic isospectral graphs of genus 3 .
These two graph share the following Laplacian polynomial:

$$
-384 x+1520 x^{2}-2288 x^{3}+1715 x^{4}-708 x^{5}+164 x^{6}-20 x^{7}+x^{8} .
$$

$3^{\circ}$. Any bridgeless graph of genus one is isomorphic to a cyclic graph $C_{n}$ for some $n \geq 1$. If two cyclic graphs $C_{m}$ and $C_{n}$ are isospectral then their Laplace polynomials are of the same degree $m=n$. Hence, the graphs are isomorphic.
In the same time, there are isospectral genus one graphs with bridges that are non-isomorphic.


Fig. 6. Isospectral genus 1 graphs with bridges.
$4^{\circ}$. One can hear genus of a graph. That is genus of a graph $G$ is completely determined by its Laplace spectrum. Indeed, $g(G)=1-V(G)+E(G)$. Let

$$
\mu(G, x)=x^{n}-c_{n-1} x^{n-1}+\ldots+(-1)^{n-1} c_{1} x
$$

be the Laplacian polynomial of $G$. By the arguments from the proof of Lemma 3.2 we have $n=V(G)$ and $c_{n-1}=2 E(G)$. Thus $V(G)$ and $E(G)$, as well as the genus, are uniquely determined by the Laplacian polynomial. $5^{\circ}$. One cannot hear a bridge of a graph. Indeed, the two graphs on Fig. 4 are isospectral. We are not able to recognise the existence of a bridge of the second graph by its spectrum.

