ENUMERATION OF MAPS AND COVERINGS

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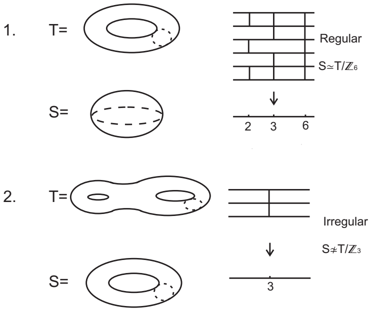
- Surface coverings

Definition. Let T and S are Riemann surfaces. A covering $p : T \to S$ is surjective map locally looking as a complex map $z \to z^n, \ z \in \mathbb{C}$, where n is an integer ≥ 1. We refer to n as a branch order at the point $z = 0$.

Examples.



Coverings

Definition. A covering $p : T \to S$ is said to be *unbranched* (or *smooth*) if all branch indices of p are equal to 1.

Two coverings $p : T \to S$ and $p' : T' \to S$ are *equivalent* if there is a homeomorphism $h : T \to T'$ such that $p = p' \circ h$.

$$\begin{array}{ccc} T & \xrightarrow[homeo]{h} & T' \\ p\downarrow & & \downarrow p' \\ S & \xrightarrow{id} & S \end{array}$$

Coverings

Let $p : T \to S$ be n-fold unbranched covering and $\Gamma = \pi_1(S)$ be the fundamental group of S. Then there is an embedding

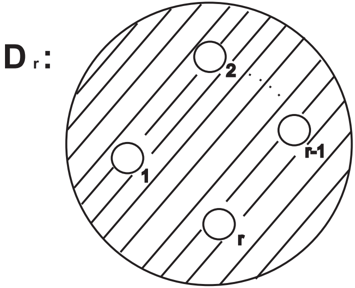
$$H = \pi_1(T) \underset{index\ n}{\subset} \Gamma = \pi_1(S).$$

Two embeddings $H = \pi_1(T) \underset{n}{\subset} \Gamma$ and $H' = \pi_1(T') \underset{n}{\subset} \Gamma$ produce *equivalent coverings* if and only if H and H' *are conjugate* in Γ.

Coverings

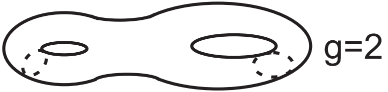
We will be mostly interesting in the following three cases.

Case 1. Let S be a bordered surface of Euler characteristic $\chi = 1 - r,\ r \geq 0$. Than $\Gamma = \pi_1(S) \cong F_r$ is a free group of rank r. A typical example of S is the disc D_r with r holes removed:



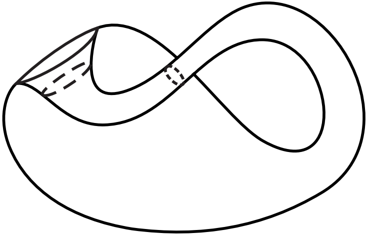
Case 2. Let S be a closed orientable surface of genus $g \geq 0$. Then

$$\pi_1(S) = \Phi_g = \langle a_1, b_1, \ldots, a_g, b_g : \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$$



Case 3. Let S be a closed non-orientable surface of genus $p \geq 1$.

$$\pi_1(S) = \Lambda_p = \langle a_1, a_2, \ldots, a_p : \prod_{i=1}^{p} a_i^2 = 1 \rangle$$



p = 2

- Two main problems

From now on we deal with the following two problems.

Problem 1. Find the number $s_\Gamma(n)$ of subgroups of index n in the group Γ.

Problem 2. Find the number $c_\Gamma(n)$ of conjugacy classes of subgroups of index n in the group Γ.

Remark. In the latter case $c_\Gamma(n)$ coincides with the number of n-fold unbranched non-equivalent coverings of surface S with

$$\pi_1(S) \equiv \Gamma.$$

- Short history: Problem 1: $s_\Gamma(n)$ Problem 2: $c_\Gamma(n)$

	Problem 1: $s_\Gamma(n)$	Problem 2: $c_\Gamma(n)$
1. $\Gamma = F_r$ $\Gamma = \pi_1(S),\ S = D_r$ bordered surface	M.Hall (1949)	V.Liskovets (1971) J.H.Kwak, J.Lee ($\geq$ 1971)
2. $\Gamma = \Phi_g$ $\Gamma = \pi_1(S),\ S = S_g$ orientable surface	A.Mednykh (1979)	A.Mednykh (1982)
3. $\Gamma = \Lambda_p$ $\Gamma = \pi_1(S),\ S = N_p$ non-orientable surface	G.Pozdnyakova, A.Mednykh (1986)	
4. $\Gamma = \pi_1(M)$ M is a Seifert 3-manifold	V.Liskovets & A.M. (2000)	A.M.& G. Chelnokov (2017-2021)

Coverings

- More deep history

A.Hurwitz	-	E.Lasker	-	G.Frobenius
1891		1900		1902

- Modern exposition

  Subgroup growth estimates and explicit asymptotic formulae were obtained in the paper by T.W.Müller, J.-Chr.Schlage-Puchta, J.Wolfart and others.

  Excellent exposition of these results is given in the book: A.Lubotzky and D.Segal "Subgroup growth", Birkhäuser, 2003.

  A. Okun'kov (Field's medalist, 2006)
  At the present, the method of non-linear differential equations is widely used to find generating functions for the number of coverings and maps. See papers by R. Pandharipande, A. Okunkov, M. Kazarian, S. Lando and others.

- Main counting principle

Theorem 1 (M., 2006)

Let Γ be an arbitrary finitely generated group. Then the number of conjugacy classes of subgroups of index n in Γ is given by the formula

$$c_\Gamma(n) = \frac{1}{n} \sum_{\substack{\ell \mid n \\ \ell m = n}} \sum_{K \underset{m}{<} \Gamma} |\mathrm{Epi}(\mathrm{K}, \mathbb{Z}_\ell)|,$$

where the second sum is taken over all subgroups K of index m in Γ and $|\mathrm{Epi}(\mathrm{K}, \mathbb{Z}_\ell)|$ is the number of epimorphism of K onto cyclic group $\mathbb{Z}_\ell$ of order ℓ.

- Proof of Theorem 1

The proof is based on two lemmas. Let $N(P,\Gamma)$ be the normalizer of P in Γ.

Lemma 1

$$c_\Gamma(n) = \frac{1}{n} \sum_{P \underset{n}{<} \Gamma} |N(P,\Gamma)/P|.$$

Lemma 2

Let P be a subgroup of index n in Γ. Then

$$|N(P,\Gamma)/P| = \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{P \underset{\mathbb{Z}_\ell}{\lhd} K \underset{m}{<} \Gamma} \varphi(\ell),$$

where $\varphi(\ell)$ is Euler function.

- Proof of Lemma 1

Fix a class E of index n subgroups in Γ. By standart arguments for given $P' \in E$ we have $|E| = |\Gamma : N(P', \Gamma)|$. Hence

$$\sum_{P \in E} |N(P, \Gamma)/P| = |E||N(P', \Gamma)/P'|$$
$$= |\Gamma : N(P', \Gamma)||N(P', \Gamma) : P'| = |\Gamma : P'| = n.$$

We obtain

$$nN(n) = \sum_{E} n = \sum_{E} \sum_{P \in E} |N(P, \Gamma)/P| = \sum_{P \underset{n}{<} \Gamma} |N(P, \Gamma)/P|.$$

- Proof of Lemma 2

Set $G = N(P,\Gamma)/P$. Given a cyclic subgroup $\mathbb{Z}_\ell < G$ there are exactly $\varphi(\ell)$ elements of G which are generators of $\mathbb{Z}_\ell$. Hence

$$|G| = \sum_{\ell | n} \varphi(\ell) \sum_{\mathbb{Z}_\ell < G} 1 = \sum_{\ell | n} \varphi(\ell) \sum_{P \underset{\mathbb{Z}_\ell}{\triangleleft} K \underset{m}{<} \Gamma} 1$$

$$= \sum_{\ell | n} \sum_{P \underset{\mathbb{Z}_\ell}{\triangleleft} K \underset{m}{<} \Gamma} \varphi(\ell).$$

Coverings

To finish the proof of the theorem we apply Lemma 1 and Lemma 2 for the case $\ell m = n$. We have

$$nN(n) = \sum_{P \underset{n}{<} \Gamma} |N(P,\Gamma)/P| = \sum_{P \underset{n}{<} \Gamma} \sum_{\ell | n} \sum_{P \underset{\mathbb{Z}_\ell}{\triangleleft} K \underset{m}{<} \Gamma} \varphi(\ell) = \sum_{\ell | n} \sum_{P \underset{n}{<} \Gamma} \sum_{P \underset{\mathbb{Z}_\ell}{\triangleleft} K \underset{m}{<} \Gamma} \varphi(\ell)$$

$$= \sum_{\ell | n} \sum_{K \underset{m}{<} \Gamma} \sum_{P \underset{\mathbb{Z}_\ell}{\triangleleft} K} \varphi(\ell) = \sum_{\ell | n} \sum_{P \underset{\mathbb{Z}_\ell}{\triangleleft} K} |\mathrm{Epi}(\mathrm{K}, \mathbb{Z}_\ell|.$$

For the last equaality we note that for given subgroup $P \underset{\mathbb{Z}_\ell}{\triangleleft} K$ there are exactly $\varphi(\ell)$ epimorphisms with $\varphi : K \to \mathbb{Z}_\ell,\ \mathcal{K}er\varphi = P$. That is

$$\sum_{P \underset{\mathbb{Z}_\ell}{\triangleleft} K} \varphi(\ell) = |\mathrm{Epi}(\mathrm{K}, \mathbb{Z}_\ell|.$$

The theorem is proved.

Coverings

- How calculate the number of epimorphisms $|\mathrm{Epi}\,(\mathrm{K}, \mathbb{Z}_\ell)|$?

Quite easy. Since the group under consideration is finite generated we have for abelizator: $K' = K/[K, K] = \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \ldots \oplus \mathbb{Z}_{m_s} \oplus \mathbb{Z}^r$.

Lemma 3

The number of homomorphisms from K into $\mathbb{Z}_d$ is given by

$$|\mathrm{Hom}\,(\mathrm{K}, \mathbb{Z}_\mathrm{d})| = (\mathrm{m}_1, \mathrm{d})(\mathrm{m}_2, \mathrm{d}) \ldots (\mathrm{m}_\mathrm{s}, \mathrm{d})\mathrm{d}^\mathrm{r}.$$

Proof. Since $\mathbb{Z}_d$ is Abelian one can change K by K'. We note $|\mathrm{Hom}\,(\mathbb{Z}_\mathrm{m}, \mathbb{Z}_\mathrm{d})| = (\mathrm{m}, \mathrm{d})$ and $|\mathrm{Hom}\,(\mathbb{Z}, \mathbb{Z}_\mathrm{d})| = \mathrm{d}$. Hence $|\mathrm{Hom}\,(\mathrm{K}, \mathbb{Z}_\mathrm{d})| = |\mathrm{Hom}\,(\mathrm{K}', \mathbb{Z}_\mathrm{d})| = (\mathrm{m}_1, \mathrm{d})(\mathrm{m}_2, \mathrm{d}) \ldots (\mathrm{m}_\mathrm{s}, \mathrm{d})\mathrm{d}^\mathrm{r}$.
Following to P.Hall (1936) we have

$$|\mathrm{Hom}\,(\Gamma, \mathbb{Z}_\ell)| = \sum_{\mathrm{d}|\ell} |\mathrm{Epi}(\Gamma, \mathbb{Z}_\mathrm{d})|.$$

By the Möbius inversion formula

$$|\mathrm{Epi}\,(\Gamma, \mathbb{Z}_\ell)| = \sum_{\mathrm{d}|\ell} \mu(\frac{\ell}{\mathrm{d}})|\mathrm{Hom}\,(\Gamma, \mathbb{Z}_\ell)|,$$

where $\mu(n)$ is the Möbius function. We obtain as result:

Lemma 4

The number of epimorphisms of group K on $\mathbb{Z}_\ell$ is given by

$$|\mathrm{Epi}(\mathrm{K}, \mathbb{Z}_\ell)| = \sum_{\mathrm{d}|\ell} \mu(\frac{\ell}{\mathrm{d}})(\mathrm{m}_1, \mathrm{d})(\mathrm{m}_2, \mathrm{d}) \ldots (\mathrm{m}_\mathrm{s}, \mathrm{d})\mathrm{d}^\mathrm{r}.$$

Corollary.

(i) $\mathrm{Epi}(\mathrm{F}_\mathrm{r}, \mathbb{Z}_\ell) = \sum\limits_{\mathrm{d}|\ell} \mu(\frac{\ell}{\mathrm{d}})\mathrm{d}^\mathrm{r}.$ Follows from $F_r' = \mathbb{Z}^r$ and Lemma 4.

(ii) $\mathrm{Epi}(\Phi_\mathrm{g}, \mathbb{Z}_\ell) = \sum\limits_{\mathrm{d}|\ell} \mu(\frac{\ell}{\mathrm{d}})\mathrm{d}^{2\mathrm{g}}.$ Since $\Phi_g' = \mathbb{Z}_{2g}$.

(iii) $\mathrm{Epi}(\Lambda_\mathrm{p}, \mathbb{Z}_\ell) = \sum\limits_{\mathrm{d}|\ell} \mu(\frac{\ell}{\mathrm{d}})(2, \mathrm{d})\mathrm{d}^{\mathrm{p}-1}.$ Since $\Lambda_p' = \mathbb{Z}_2 \oplus \mathbb{Z}^{p-1}$.

- Counting surface coverings

As an application of the above results we have the following

Theorem 2 (V. Liskovets, 1971)

Let S be a bordered surface with the fundamental group $\pi_1(S) = F_r$. Then the number of non-equivalent n-fold coverings of S is given by

$$N(n) = \frac{1}{n} \sum_{\substack{\ell|n \\ \ell m = n}} \sum_{d|\ell} \mu(\frac{\ell}{d}) d^{(r-1)m+1} M(m).$$

where $M(m)$ is the number of subgroups of index m in the group F_r.

Recall the M.Hall's recursive formula

$$M(m) = m(m!)^{r-1} - \sum_{j=1}^{m-1} (m-j)!^{r-1} M(j), \quad M(1) = 1.$$

- Proof of theorem 2

Proof. By the Schreier theorem any subgroup of index m in F_r is isomorphic to $\Gamma_m = F_{(r-1)m+1}$. By theorem 1

$$N(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} |\mathrm{Epi}(\Gamma_{\mathrm{m}}, \mathbb{Z}_\ell)| \mathrm{M}(\mathrm{m}).$$

By Corollary (i) we have

$$|\mathrm{Epi}(\Gamma_{\mathrm{m}}, \mathbb{Z}_\ell)| = \sum_{\mathrm{d}|\ell} \mu(\frac{\ell}{\mathrm{d}}) \mathrm{d}^{(\mathrm{r}-1)\mathrm{m}+1}$$

and the result follows.

Coverings

- Counting surface coverings

The next application of Theorem 1 is the following result.

Theorem 3 (M., 1982)

Let S be a closed orientable surface with the fundamental group $\pi_1(S) = \Phi_g$. Then the number of non-equivalent n-fold coverings of S is given by

$$N(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{d|\ell} \mu(\frac{\ell}{d}) d^{2(g-1)m+2} M(m),$$

where $M(m)$ is the number of subgroups of index m in the group Φ_g.

Coverings

- Proof of theorem 3

Proof. By the Riemann-Hurwitz formula any subgroup K_m of index m in the group Φ_g is isomorphic to $\Phi_{g'}$, where $2g' - 2 = m(2g - 2)$. Hence, $K_m = \Phi_{(g-1)m+1}$. By Theorem 1 we have

$$N(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} |\mathrm{Epi}(\mathrm{K_m}, \mathbb{Z}_\ell)| \mathrm{M}(\mathrm{m}),$$

where

$$|\mathrm{Epi}(\mathrm{K_m}, \mathbb{Z}_\ell)| = \sum_{\mathrm{d}|\ell} \mu(\frac{\ell}{\mathrm{d}}) \mathrm{d}^{2(\mathrm{g}-1)\mathrm{m}+2}$$

is given by Corollary (ii). The proof is complete.

Coverings

- Remark

Recall that (M., 1982) the number of subgroups $M(m)$ in the fundamental group Φ_g of closed orientable surface of genus g is given by the following recurcive formula

$$M(m) = m\beta_m - \sum_{j=1}^{m-1} \beta_{m-j} M(j), \quad M(1) = 1,$$

where

$$\beta_k = \sum_{\chi \in D_k} \left(\frac{k!}{f\chi} \right)^{2g-2},$$

D_k is the set of irreducible representations of a symmetric group S_k and f^χ is the degree of the representation χ.

One can change Φ_g by Λ_p and $2g-2$ by $p-2$ in this statement.

Some more result can be obtained in a similar way.

Theorem 4 (G. Pozdnyakova and M., 1986)

Let S be a closed non-orientable surface with the fundamental group $\pi_1(S) = \Lambda_p$. The number of non-equivalent n-fold coverings of S is given by

$$N(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right)\left(d^{m(p-2)+2} M^{+}(m) + (2, d) d^{m(p-2)+1} M^{-}(m)\right),$$

where $M^{+}(m)$ and $M^{-}(m)$ are the numbers of orientable and non-orientable subgroups of index m in the group Λ_p, respectively.

- Proof of theorem 4

Proof. Recall there are two kinds of subgroups of index m in the group Λ_p, namely $\Gamma_m^+ = \Phi_{\frac{m}{2}(p-2)+1}$ and $\Gamma_m^- = \Lambda_{m(p-2)+2}$. They represent orientable and non-orientable m-fold coverings of S, respectively. The index m is even in the first case. Again, by theorem 1 we get

$$N(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} (|\mathrm{Epi}(\Gamma_{\mathrm{m}}^+, \mathbb{Z}_\ell)|\mathrm{M}^+(\mathrm{m}) + |\mathrm{Epi}(\Gamma_{\mathrm{m}}^-, \mathbb{Z}_\ell)|\mathrm{M}^-(\mathrm{m})).$$

By Corollaries (ii) and (iii)

$$|\mathrm{Epi}(\Gamma_{\mathrm{m}}^+, \mathbb{Z}_\ell)| = \sum_{\mathrm{d}|\ell} \mu(\frac{\ell}{\mathrm{d}})\mathrm{d}^{\mathrm{m(p-2)+2}}$$

$$|\mathrm{Epi}(\Gamma_{\mathrm{m}}^-, \mathbb{Z}_\ell)| = \sum_{\mathrm{d}|\ell} \mu(\frac{\ell}{\mathrm{d}})(2, \mathrm{d})\mathrm{d}^{\mathrm{m(p-2)+1}}$$

and the result follows.

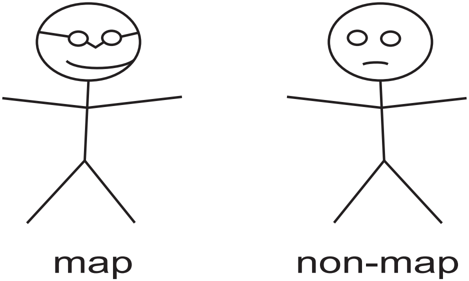
- Remark

The numbers $M^{+}(m)$ and $M^{-}(m)$ can be derived through irreducible characters of the symmetric group similar to those for number of subgroups $M(m)$ in the group Φ_g.

Maps

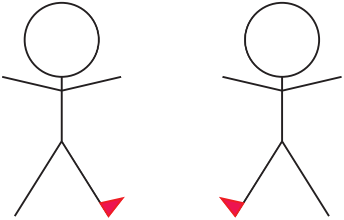
- Maps on surfaces

Map on surface is an embedding $G \subset S$ of a graph G into S such that $S \setminus G$ is a union of 2-discs.



- Rooted maps

Rooted map is a map with a distinguished semiedge ($\equiv$ dart, bit, pin, blade, brin ...).



Two different rooted maps

Maps

Two rooted maps (S, G) and (S, G') are equivalent if there exists an orientation preserving homeomorphism $h : (S, G) \to (S, G')$ sending root to root.
Two (unrooted) maps (S, G) and (S, G') are equivalent if there exists an orientation preserving homeomorphism $h : (S, G) \to (S, G')$.

Problem 1. Find the number $R_g(e)$ of non-equivalent rooted maps with e edges on a closed orientable surface of genus g.

Problem 2. Find the number $U_g(e)$ of non-equivalent maps with e edges on a closed orientable surface of genus g.

Maps

- Counting maps on orientable surface

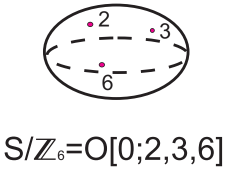
Maps	Groups
Trivial map $\circ -$	$\Gamma = T(2, \infty, \infty)$ $= \langle x, y : (xy)^2 = 1 \rangle$
Rooted maps of genus g with n edges ($= 2n$ darts)	Torsion free subgroups of genus g and of index $2n$ in Γ
Unrooted maps of genus g with n edges ($= 2n$ darts)	Conjugacy classes of torsion free subgroups of genus g and of index $2n$ in Γ

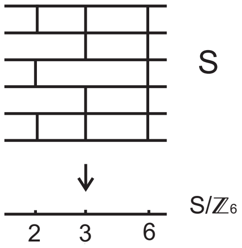
- Cyclic orbifold and its fundamental group

Let S be a closed surface of genus g and $\mathbb{Z}_\ell$ acts on S by homeomorphisms. We consider the factor space as orbifold (e.m. surface with prescribed signature).

$$S/\mathbb{Z}_\ell \equiv O[\gamma; m_1, m_2, \ldots, m_r].$$

Example. S is a torus, $\ell = 6$.





Maps

- Cyclic orbifold and its fundamental group

W.Harvey (1966) gave a complete description of signatures for cyclic orbifolds. In particular, the Riemann-Hurwitz formula holds

$$2g - 2 = \ell(2\gamma - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}))$$

and the fundamental group of orbifold O is given by

$$\pi_1^{orb}(O) = \langle a_1, b_1, \ldots, a_\gamma, b_\gamma, e_1, \ldots, e_r : \prod_{i=1}^{g}[a_i, b_i]\prod_{j=1}^{r} e_j = e_1^{m_1} = e_2^{m_2} = \ldots = e_r^{m_r} = 1\rangle.$$

Maps

One of the most important consequences of Theorem 1 is the following result.

Theorem 5 (R. Nedela and M., 2006)

Let S be a closed oriented surface of genus g. Then the number of maps having e edges and counting up to orientation preserving homeomorphism of S is given by the formula

$$U_g(e) = \frac{1}{2e} \sum_{\substack{\ell | 2e \\ \ell m = 2e}} \sum_{O = S/\mathbb{Z}_\ell} \mathrm{Epi}^\circ(\pi_1(\mathrm{O}), \mathbb{Z}_\ell) \nu_{\mathrm{O}}(\mathrm{m}),$$

where $\mathrm{Epi}^\circ(\pi_1(\mathrm{O}), \mathbb{Z}_\ell)$ is the number of order preserving epimorphisms $\pi_1(O) \to \mathbb{Z}_\ell$ and $\nu_O(m)$ is the number of rooted maps on the orbifold O having m darts.

Maps

Explicit formula for $\mathrm{Epi}^\circ(\pi_1(\mathrm{O}), \mathbb{Z}_\ell)$ is given by the following proposition.

Proposition 1

Let $O = O(\gamma; m_1, m_2, \dots, m_r)$ be an orbifold and $\Gamma = \pi_1(O)$ is the orbifold fundamental group and $m = \mathrm{l.c.m.}(m_1, m_2, \dots, m_r)$. Then

$$Epi^\circ(\Gamma, \mathbb{Z}_\ell) = \sum_{m|d|\ell} \mu(\frac{\ell}{d}) d^{2\gamma} E(m_1, m_2, \dots, m_r), \ \textit{where}$$

$$E(m_1, m_2, \dots, m_r) = \frac{1}{m} \sum_{k=1}^{m} \Phi(k, m_1) \cdot \ldots \cdot \Phi(k, m_r)$$

$$\textit{and } \Phi(k, n) = \sum_{\substack{1 \le s \le n \\ (s,n)=1}} \exp \frac{2\pi iks}{n}$$

is the von Sterneck function.

Maps

Remark. By O. Hölder

$$\Phi(k,n) = \frac{\varphi(n)}{\varphi(\frac{n}{(k,n)})}\mu(\frac{n}{(k,n)})$$

where $\varphi(n)$ and $\mu(n)$ are Euler and Möbius functions, respectively.

The number $\nu_O(m)$ of rooted maps on the orbifold O having m darts is given by the following proposition.

Proposition 2

Let $O = O[\gamma; 2^{q_2}3^{q_3}\ldots\ell^{q_\ell}]$ be an orbifold. Then

$$\nu_O(m) = \sum_{s=0}^{q_2}\binom{m}{s}\binom{\frac{m-s}{2}+2-2\gamma}{q_2-s, q_3, \ldots, q_\ell}N_g\Big(\frac{m-s}{2}\Big),$$

where $N_g(e)$ is the number of rooted maps with e edges on a closed orientable surface of genus g.

- Rooted maps

The numbers $N_g(e)$ were calculated by many people: Tutte, Arques, Giorgetti, Bender, Wormald, Walsh, Lehman, Canfield, Robinson and others. In particular,

$$N_0(e) = \frac{2(2e)!3^e}{e!(e+2)!}, \qquad \textbf{(Tutte, 1963)}$$

$$N_1(e) = \sum_{k=0}^{e-2} 2^{e-3-k}(3^{e-1} - 3^k)\binom{e+k}{k}. \qquad \textbf{(D. Arques, 1987)}$$

- Denerating function for the number of rooted maps

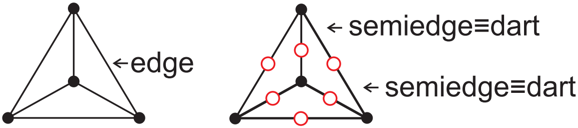
More generally, for $g \geq 1$ the ordinary generating function $Q_g(z) = \sum\limits_{n \geq 0} N_g(n) z^n$ is given by

$$Q_g(z) = \frac{m^{2g}(1-3m)^{2g-2} P_g(m)}{(1-6m)^{5g-3}(1-2m)^{5g-4}},$$

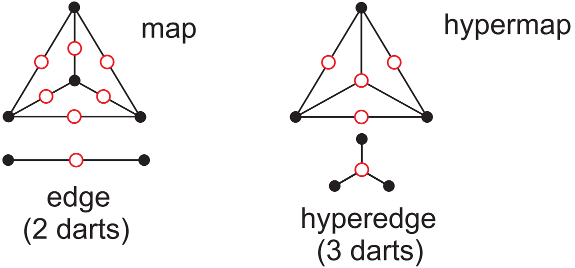
where $m = \frac{1-\sqrt{1-12z}}{6}$ and $P_g(m)$ is an integer polynomial of m of degree $6g-6$.

On the date (**2010**) only the following polynomial were known: $P_1(m),\ P_2(m),\ P_3(m),\ P_4(m),\ P_5(m),\ P_6(m)$.
Recent progress is done in paper by A. Giorgetti, G. Chapuy, P. Zograf, M. Kazaryan and others.

From maps to hypermaps

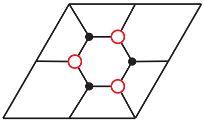


Idea: edge consists of two darts
hyperedge consists of a few darts



Hypermap H on the surface S is a 2-closed map on S.
Black verticies are *verticies* of H.
Red verticies are *hyperedges* of H.
Edges of map are *darts* of H.

Example. Hypermap on torus.



Hypermaps

- Counting unrooted hypermaps through rooted ones

Theorem 6 (R. Nedela and M., 2006)

The number of unrooted hypermaps with n darts on a closed orientable surface S_g of genus g is given by

$$H_g(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{O = S_g/\mathbb{Z}_\ell} Epi^\circ(\pi_1(O), \mathbb{Z}_\ell) \binom{m + 2 - 2\gamma}{q_2, q_3, \ldots, q_\ell} h_\gamma(m),$$

where the second sum is taken over all cyclic orbifolds $O = S_g/\mathbb{Z}_\ell$ of the signature $[\gamma; 2^{q_2} 3^{q_3} \ldots \ell^{q_\ell}]$, $\binom{p}{q_2, q_3, \ldots, q_\ell}$ is the multinomial coefficient and $h_\gamma(m)$ is the number of rooted hypermaps with m darts on S_γ.

- Rooted hypermaps

  Before it was known that

$$h_0(m) = \frac{3 \cdot 2^{m-1}}{(m+1)(m+2)} \binom{2m}{m} \qquad \textbf{T.Walsh(1975)}$$

and

$$h_1(m) = \frac{1}{3} \sum_{k=0}^{m-3} 2^k (4^{m-2-k} - 1) \binom{m+k}{k} \qquad \textbf{D.Arquès(1987)}$$

- The Liskovets problem

Let $\mathcal{M}$ be a non-orientable manifold with a finitely generated fundamental group $\Gamma = \pi_1(\mathcal{M})$.

Liskovets Problem (Dresden, 1996)

To find the number of n-fold non-equivalent orientable coverings of $\mathcal{M}$.

To solve the problem we have to use the following version of the main counting principle. Let $\mathcal{P}$ be a property of subgroups of Γ invariant under conjugation (for example: to be normal, to be torsion free, to be orientable and so on).

Orientable coverings

- Counting conjugacy classes of subgroups with prescribed property

Theorem 7 (R. Nedela and M., 2006)

Let Γ a finitely generated group. Then the number of conjugacy classes of subgroups of index n in Γ satisfying property $\mathcal{P}$ is given by

$$N^{\mathcal{P}}(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{K \underset{m}{<} \Gamma} Epi^{\mathcal{P}}(K, \mathbb{Z}_\ell),$$

where $Epi^{\mathcal{P}}(K, \mathbb{Z}_\ell)$ is the number of epimorphisms of the group K onto $\mathbb{Z}_\ell$ whose kernel has the property $\mathcal{P}$.

Orientable coverings

Fix the property $\mathcal{P} = \mathcal{P}^-$ for subgroups of Γ "to be non-orientable". Then a complete solution of the Liskovets problem is given by

Theorem 8 (J. Ho Kwak, R. Nedela and M., 2008)

Let $\mathcal{M}$ be a connected non-orientable manifold with a finitely generated fundamental group $\Gamma = \pi_1(\mathcal{M})$. Then the number non-equivalent n-fold non-orientable coverings of $\mathcal{M}$ is equal to

$$N_\Gamma^-(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{K^- \underset{m}{<} \Gamma} Epi^-(K^-, \mathbb{Z}_\ell),$$

where the second sum is taken over all non-orientable subgroups of index m in Γ and $Epi^-(K^-, \mathbb{Z}_\ell)$ is the number of epimorphisms of the group K^- onto $\mathbb{Z}_\ell$ with non-orientable kernel.

Orientable coverings

We note that $N_\Gamma(n) = N_\Gamma^-(n) + N_\Gamma^+(n)$ and

$$Epi(K, \mathbb{Z}_\ell) = Epi^-(K, \mathbb{Z}_\ell) + \mathrm{Epi}^+(\mathrm{K}, \mathbb{Z}_\ell).$$

By G. Jones the function $\ell \to \mathrm{Epi}^+(\mathrm{K}, \mathbb{Z}_\ell)$ is multiplicative. This gives

Theorem 9 (J. Ho Kwak, R. Nedela and M., 2008)

Let $K = \pi_1(\mathcal{M})$ be finitely generated and $H_1(\mathcal{M}) = \mathbb{Z}_{s_1}^{(-1)^{p_1}} \oplus \mathbb{Z}_{s_2}^{(-1)^{p_2}} \oplus \ldots \oplus \mathbb{Z}_{s_n}^{(-1)^{p_n}}$ is the orient homology group of $\mathcal{M}$. Then $\mathrm{Epi}^+(\mathrm{K}, \mathbb{Z}_\ell) = 0$, if ℓ is odd and

$$\mathrm{Epi}^+(\mathrm{K}, \mathbb{Z}_{2\ell}) = \prod_{j=1}^{n} \frac{1 + (-1)^{\frac{s_j p_j}{(s_j, \ell)}}}{2} \sum_{\substack{m \\ \frac{\ell}{m} - \mathrm{odd}}} \mu\left(\frac{\ell}{m}\right)(s_1, m)(s_2, m) \ldots (s_n, m).$$

Note. The function $\ell \to \mathrm{Epi}^-(\mathrm{K}, \mathbb{Z}_\ell)$ is not multiplicative.

Orientable coverings

- Reflexible coverings

Let $\mathcal{M}$ be a non-orientable manifold or orbifold. An orientable covering $p : U^+ \to \mathcal{M}$ is called to be *reflexible* if there exists an orientation reversing homeomorphism $h : U^+ \to U^+$ such that $p \circ h = p$. In particular, any regular covering p is reflexible.

Теорема 10 (J. Ho Kwak, R. Nedela and M., 2008)

Let $\mathcal{M}$ be a connected non-orientable manifold with $\pi_1(\mathcal{M}) = \Gamma$. Then the number of $2n$-fold reflexible coverings of $\mathcal{M}$ is equal to

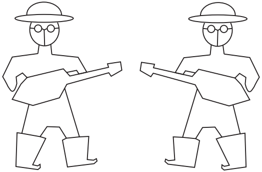
$$A_\Gamma(n) = \frac{1}{2n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{K^- \underset{m}{<} \Gamma} Epi^+(K^-, \mathbb{Z}_{2\ell}),$$

where the second sum is taken over all non-orientable subgroups of index m in Γ and $Epi^+(K^-, \mathbb{Z}_{2\ell})$ is the number of epimorphisms of the group K^- onto $\mathbb{Z}_{2\ell}$ with orientable kernel.

- Chiral pairs and twins

Two maps on a closed orientable surface are *chiral* (or *twins*) if they are homeomorphic under orientation reversing homeomorphism but are not homeomorphic under orientation preserving one.

Problem. Find the number of twins on closed orientable surface with a given number of edges.



A.Breda, R.Nedela and A.Mednykh applied the above theorem on reflexible coverings to find the number of twins with prescribed number of edges. (Discrete Mathematics, Vol. 310, No. 6–7, P. 1184–1203, 2010).

Disconnected coverings

Let $p : U \to \mathcal{M}$ be (possibly *disconnected*) covering over connected manifold $\mathcal{M}$. Taking $x_0 \in \mathcal{M}$ and $\gamma \in \pi_1(\mathcal{M}, x_0)$ define a $1-1$ map $L_\gamma : \widetilde{x_0} \in F \to \widetilde{\gamma}(\widetilde{x_0}) \in F$, where $F = p^{-1}(x_0)$ and $\widetilde{\gamma}$ is a lifting of γ on U at the point $\widetilde{x_0}$.

Thus, we get a correspondence between coverings and homomorphisms $\theta : \pi_1(\mathcal{M}, x_0) \to S_F$. By the monodromy theorem it is well defined.

Two coverings $p : U \to \mathcal{M}$ and $p' : U \to \mathcal{M}$ are *equivalent* if and only if corresponding homomorphisms $\theta : \pi_1(\mathcal{M}, x_0) \to S_F$ and $\theta' : \pi_1(\mathcal{M}, x_0) \to S_F$ are conjugate by a suitable element $h \in S_F : \theta' = h \circ \theta \circ h^{-1}$.

In such away n-fold coverings (*connected* or not) are classified by $Hom(\pi_1(\mathcal{M}), S_n)/S_n$.

Disconnected coverings

Теорема 11 (V. A. Liskovets and M., 2009)

Let $\mathcal{M}$ be a connected manifold with finitely generated fundamental group $\Gamma = \pi_1(\mathcal{M})$. Denote by b_n the number of non-equivalent (connected or not) n-fold coverings over $\mathcal{M}$ and set $b(x) = 1 + b_1 x + b_2 x^2 + \ldots$. Then

$$b(x) = \exp\Big(\sum_{n=1}^{\infty}\Big(\sum_{\substack{\ell | n \\ \ell m = n}} \sum_{K \underset{m}{<} \Gamma} Hom(K, \mathbb{Z}_\ell)\Big)\frac{x^n}{n}\Big),$$

where $Hom(K, \mathbb{Z}_\ell)$ is the number of homomorphisms of the group K into a cyclic group $\mathbb{Z}_\ell$ of order ℓ.

We note that the number of connected n-fold coverings $c_\Gamma(n)$ is related with $b(x)$ by the following Euler transform $b(x) = \prod_{n=1}^{\infty}(1 - x^n)^{-c_\Gamma(n)}$.

Disconnected coverings

- Examples

1°. Let $\mathcal{M} = S^1$ be the unite circle. Then $b_n = p(n)$ is the Hardy-Ramanujan partion function.

2°. Let G be a finite graph with Betty number $r = \beta(G)$. Then $\Gamma = \pi_1(G) = F_r$ is a free group of rank r

$$b_n = \sum_{c_1+2c_2+\ldots+nc_n=n} \prod_{i=1}^{n} (i^{c_i} c_i!)^{r-1}.$$

This is the result by J.H. Kwak and Y. Lee (1996).

3°. Let $\mathcal{M} = S_g$ be a closed orientable surface of genus g. Then $b_1 = 1,\ b_2 = 4 \cdot 2^\nu,\ b_3 = 2 \cdot 6^\nu + 4 \cdot 3^\nu + 2 \cdot 2^\nu$, $b_4 = 2 \cdot 24^\nu + 12^\nu + 6 \cdot 8^\nu + 9 \cdot 4^\nu + 3 \cdot 3^\nu$, where $\nu = 2g - 2$.

In particular, for $g = 1$ this is the sequence **A 061256** from "On-Line Encyclopedia of Integer Sequences"

1, **4**, **8**, **21**, **39**, **92**, **170**, **360**, **667**, **1316**, $\ldots$.

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