Lecture 1. Count of square-tiled surfaces

Anton Zorich

(after a joint work with V. Delecroix, E. Goujard and P. Zograf;
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see `arXiv:2011.05306` and also `arXiv:2007.04740`.)

School "Moduli Spaces, Combinatorics and Integrable Systems"
St. Petersburg, November 24, 2021

Count of metric ribbon graphs and of square-tiled surfaces

Epigraph: “We are counting cards”

If you are annoyed while waiting for the beginning of the lecture, please type “Very sparkly” in YouTube search and watch the three-minutes extract from the movie “Rainman” (pay attention starting from the conversation with Iris):

https://www.youtube.com/watch?v=Wjc58nT4hUA

This is the best epigraph which I can imagine for my lectures. I fully identify myself with Rainman, except that he and his brother were counting cards for a day and, together with my collaborators, we were obsessively counting square-tiled surfaces for many years... We have not finished yet.

Intersection numbers (Witten–Kontsevich correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \ldots, P_n \in C$ is a complex orbifold of complex dimension $3g - 3 + n$.
Choose index i in $\{1, \ldots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$.
The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g - 3 + n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} .$$

The famous Witten's conjecture claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in 2-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; alternative proofs belong to A. Okounkov and R. Pandharipande, to M. Mirzakhani, to M. Kazarian and S. Lando (and there are more).

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Recursive relations

Initial data: $\langle\tau_0^3\rangle = 1, \qquad \langle\tau_1\rangle = \frac{1}{24}\,.$

String equation:

$$\langle\tau_0\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n+1} = \langle\tau_{d_1-1}\ldots\tau_{d_n}\rangle_{g,n} + \cdots + \langle\tau_{d_1}\ldots\tau_{d_n-1}\rangle_{g,n}\,.$$

Dilaton equation:

$$\langle\tau_1\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n+1} = (2g-2+n)\langle\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n}\,.$$

Virasoro constraints (in Dijkgraaf–Verlinde–Verlinde form; $k\geq 1$):

$$\langle\tau_{k+1}\tau_{d_1}\cdots\tau_{d_n}\rangle_g = \frac{1}{(2k+3)!!}\Bigg[\sum_{j=1}^{n}\frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle\tau_{d_1}\cdots\tau_{d_j+k}\cdots\tau_{d_n}\rangle_g$$

$$+\frac{1}{2}\sum_{\substack{r+s=k-1\\ r,s\geq 0}}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1}$$

$$+\frac{1}{2}\sum_{\substack{r+s=k-1\\ r,s\geq 0}}(2r+1)!!(2s+1)!!\sum_{\{1,\ldots,n\}=I\coprod J}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\Bigg]\,.$$

Uniform large genus asymptotics

We stated in August 2019 a conjecture which was proved by Amol Aggarwal already in April 2020.

Theorem (Aggarwal). *The following* ***uniform*** *asymptotic formula is valid:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} =$$
$$= \frac{1}{24^g} \cdot \frac{(6g-5+2n)!}{g!\,(3g-3+n)!} \cdot \frac{d_1! \dots d_n!}{(2d_1+1)! \cdots (2d_n+1)!} \cdot \big(1+\varepsilon(\boldsymbol{d})\big)\,,$$

where $\varepsilon(\boldsymbol{d}) = O\left(1 + \frac{(n+\log g)^2}{g}\right)$ ***uniformly*** *for all* $n = o(\sqrt{g})$ *and all partitions* $\boldsymbol{d}$*,* $d_1 + \dots + d_n = 3g-3+n$*, as* $g \to +\infty$*.*

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1,\ldots,d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1+\cdots+d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1,\ldots,d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1}\cdot\cdots\cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1,\ldots,b_n)$ of degree $6g-6+2n$ in variables $b_1,\ldots,b_n$:

$$N_{g,n}(b_1,\ldots,b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}}\,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!}\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\ldots\psi_n^{d_n}$$

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Up to a numerical factor, the polynomial $N_{g,n}(b_1, \ldots, b_n)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $V_{g,n}(b_1, \ldots, b_n)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces:

$$V_{g,n}^{top}(b) = 2^{2g-3+n} \cdot N_{g,n}(b) \,.$$

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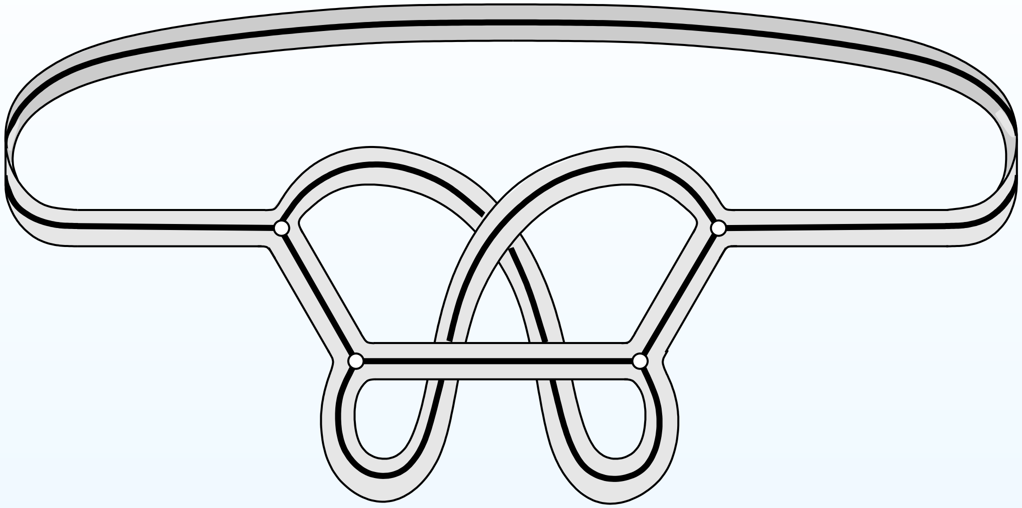
Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \ : \quad \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) ,$$

and extend it to symmetric polynomials in b_i by linearity.

Trivalent ribbon graphs



This trivalent ribbon graph defines an orientable surface of genus $g = 1$ with $n = 2$ boundary components. If we assigned lengths to all edges of the core graph, each boundary component gets induced length, namely, the sum of the lengths of the edges which it follow.

Note, however, that in general, fixing a genus g, a number n of boundary components and integer lengths $b_1, \ldots, b_n$ of boundary components, we get plenty of trivalent integral metric ribbon graphs associated to such data. The Theorem of Kontsevich counts them.

Kontsevich's count of metric ribbon graphs

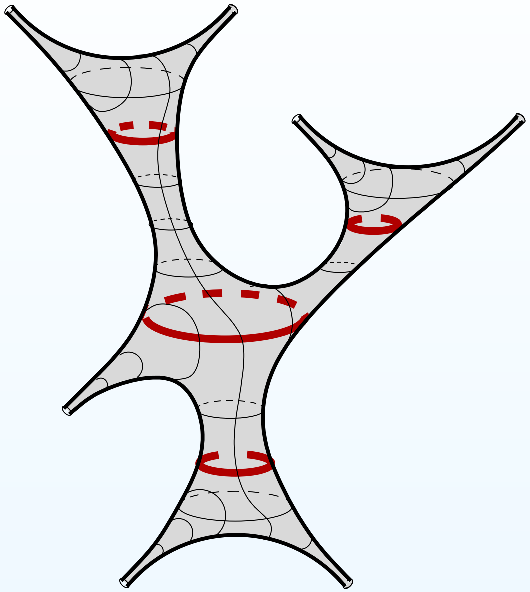
Theorem (M. Kontsevich; in this form — P. Norbury). *Consider a collection of positive integers* $b_1, \dots, b_n$ *such that* $\sum_{i=1}^n b_i$ *is even. The weighted count of genus* g *connected trivalent metric ribbon graphs* Γ *with integer edges and with* n *labeled boundary components of lengths* $b_1, \dots, b_n$ *is equal to* $N_{g,n}(b_1, \dots, b_n)$ *up to the lower order terms:*

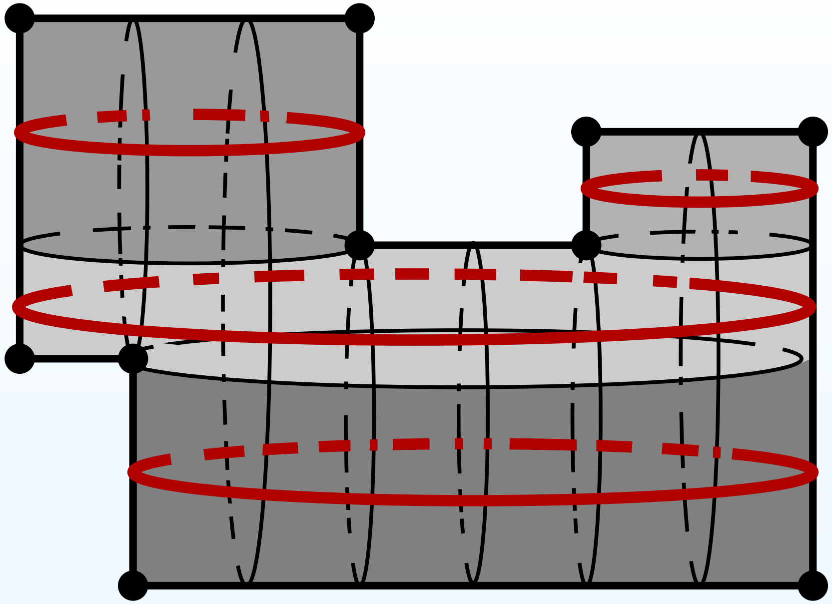
$$\sum_{\Gamma \in \mathcal{R}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} N_\Gamma(b_1, \dots, b_n) = N_{g,n}(b_1, \dots, b_n) + \textit{lower order terms}\,,$$

where $\mathcal{R}_{g,n}$ *denote the set of (nonisomorphic) trivalent ribbon graphs* Γ *of genus* g *and with* n *boundary components.*

This Theorem is an important part of Kontsevich's proof of Witten's conjecture.

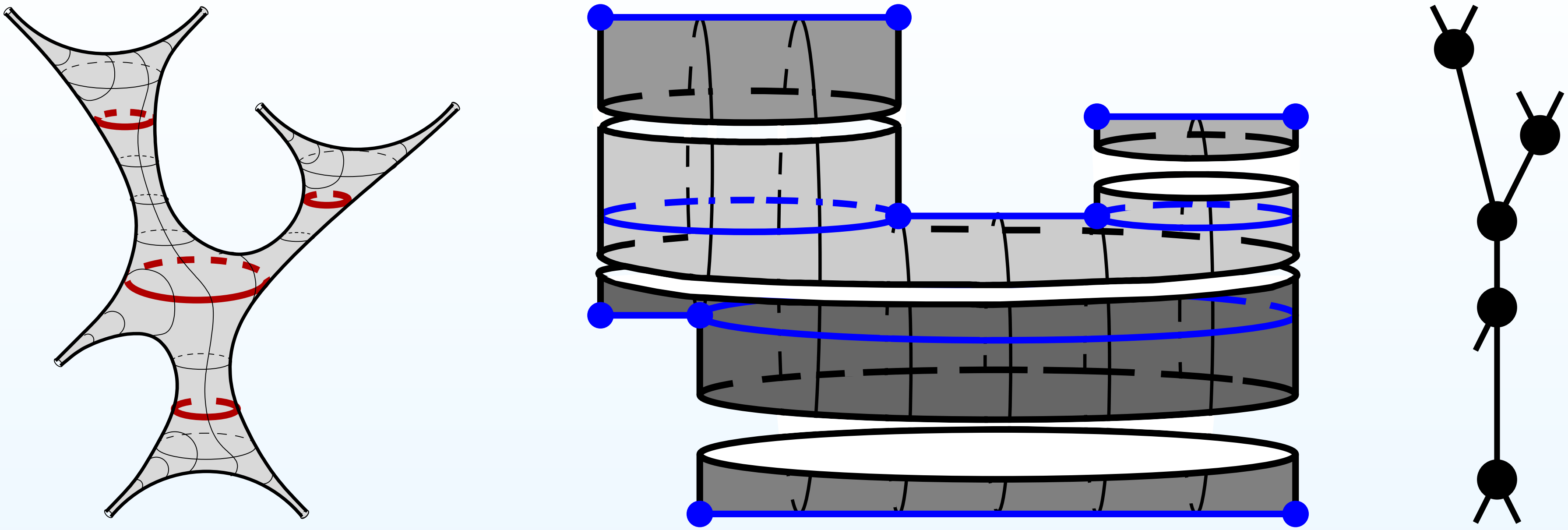
Stable graph associated to a square-tiled surface





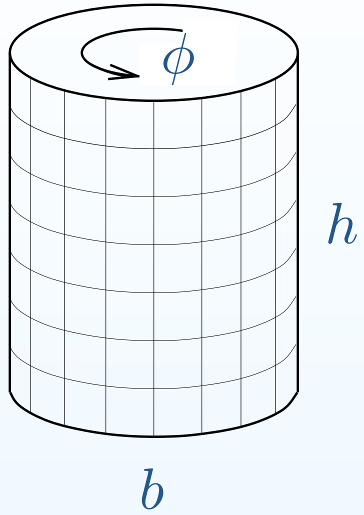
Having a square-tiled surface we associate to it a topological surface S on which we mark all "corners" with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points. We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders.

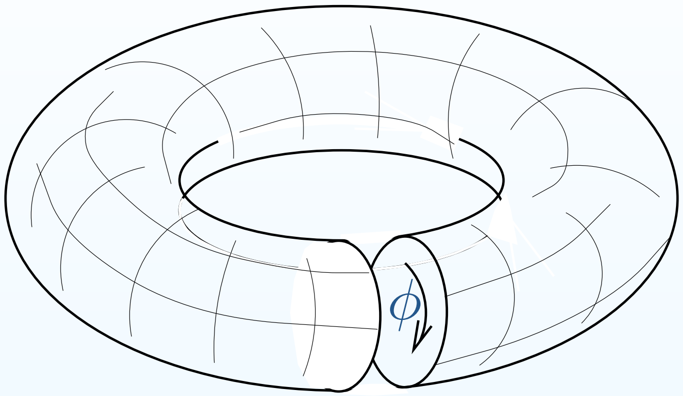
Stable graph associated to a square-tiled surface



Having a square-tiled surface we associate to it a topological surface S on which we mark all "corners" with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points. We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders. The associated *stable graph* Γ is the dual graph to the multicurve. The vertices of Γ are in the natural bijection with metric ribbon graphs given by components of $S \setminus \gamma$. The edges are in the bijection with the waist curves γ_i of the cylinders. The marked points are encoded by "legs" — half-edges of the dual graph.

Number of square-tiled tori

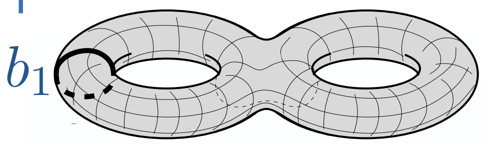




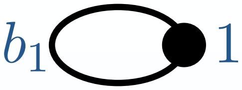
The number of square-tiled tori tiled with at most N squares has asymptotics

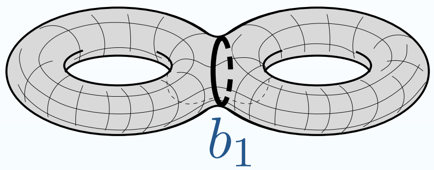
$$\sum_{\substack{b,h\in\mathbb{N}\\ b\cdot h\le N}} b = \sum_{\substack{b,h\in\mathbb{N}\\ b\le \frac{N}{h}}} b \sim \sum_{h\in\mathbb{N}} \frac{1}{2}\cdot\left(\frac{N}{h}\right)^2 = \frac{N^2}{2}\sum_{h\in\mathbb{N}}\frac{1}{h^2} = \frac{N^2}{2}\cdot\frac{\pi^2}{6} = \frac{N^2}{2}\zeta(2) =$$

$$= \frac{N^2}{2}\mathcal{Z}(b)\,, \quad \text{where} \quad \mathcal{Z}\ :\ \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n}\big(m_i!\cdot\zeta(m_i+1)\big)\,.$$

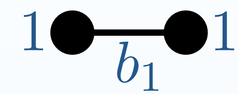


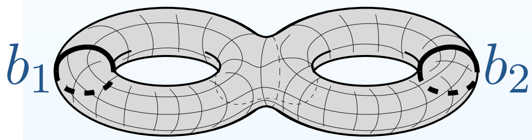
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1)$$



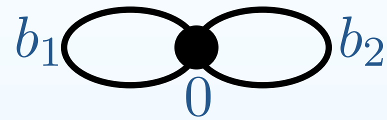


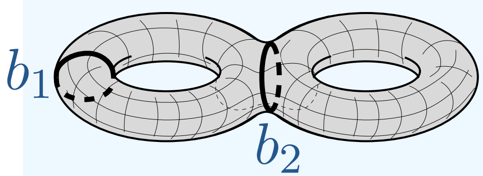
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1)$$



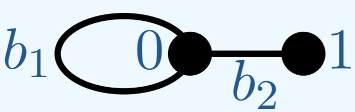


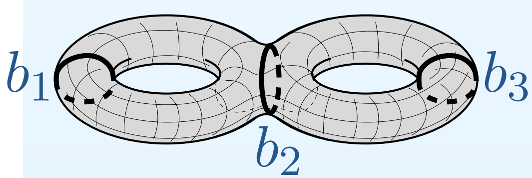
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2)$$



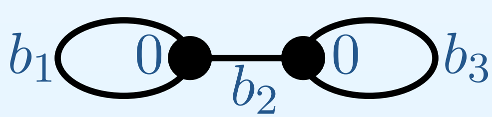


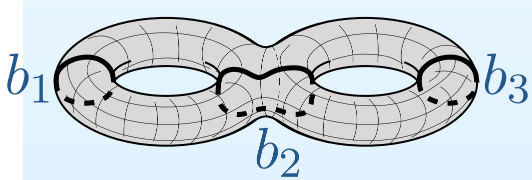
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2)$$



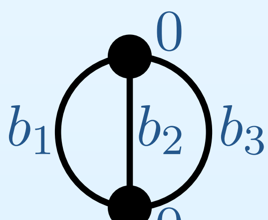


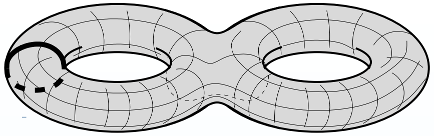
$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3)$$



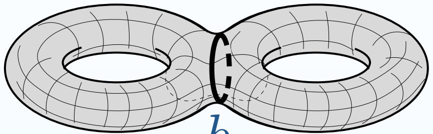


$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3)$$

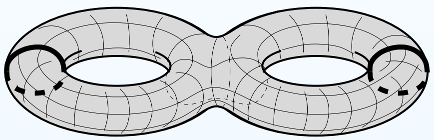




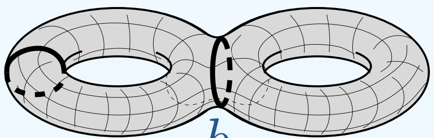
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1) \qquad = \frac{1}{2} \cdot b_1 \left(\frac{1}{384} (2b_1^2)(2b_1^2) \right)$$



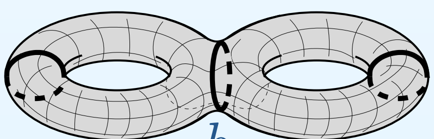
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1) \qquad = \frac{1}{4} \cdot b_1 \left(\frac{1}{48} b_1^2 \right) \left(\frac{1}{48} b_1^2 \right)$$



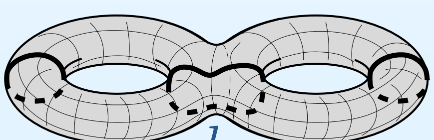
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2) \qquad = \frac{1}{8} \cdot b_1 b_2 \cdot \left(\frac{1}{4} (2b_1^2 + 2b_2^2) \right)$$



$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2) \qquad = \frac{1}{4} \cdot b_1 b_2 \cdot (1) \cdot \left(\frac{1}{48} b_2^2 \right)$$

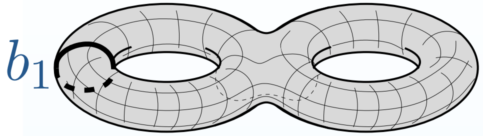


$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3) \qquad = \frac{1}{16} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

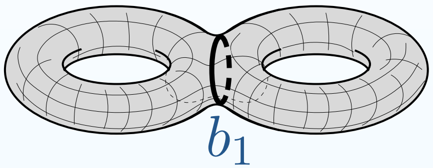


$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3) \qquad = \frac{1}{24} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

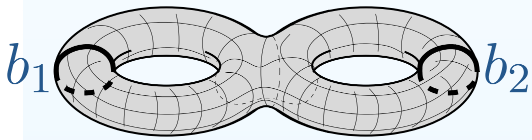
Volume of $\mathcal{Q}_2$



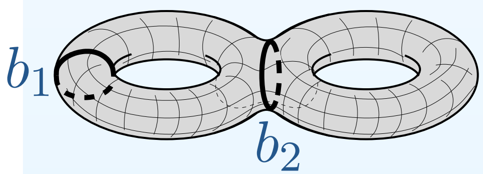
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



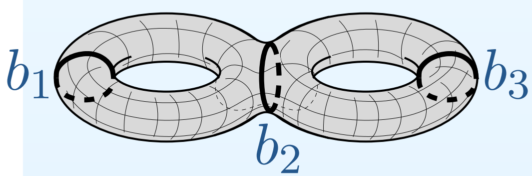
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



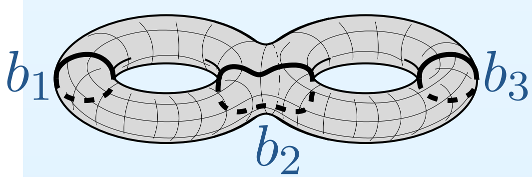
$$\frac{1}{16}(b_1^3 b_2 + {} + b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16}b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$



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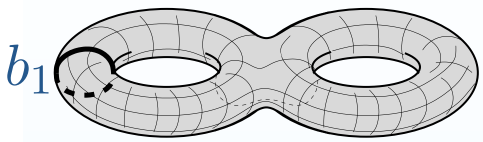
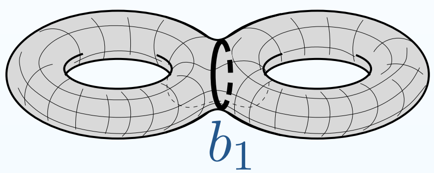
$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\left(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\right)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

Volume of $\mathcal{Q}_2$



$$\frac{1}{192}\cdot b_1^5 \quad \overset{\mathcal{Z}}{\longmapsto} \quad \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) \qquad = \frac{1}{1512}\cdot\pi^6$$



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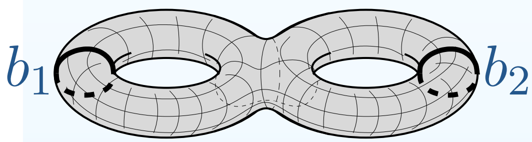
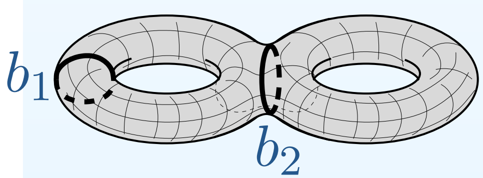


$$\frac{1}{16}\big(b_1^3 b_2 + {} + b_1 b_2^3\big) \quad \overset{\mathcal{Z}}{\longmapsto} \quad \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) \qquad = \frac{1}{720}\cdot\pi^6$$



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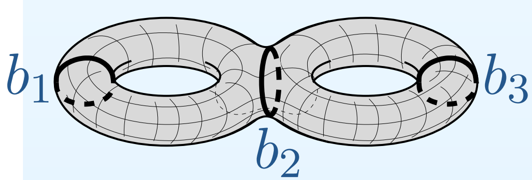

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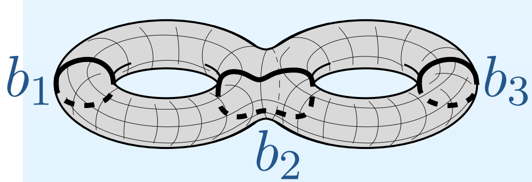

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Volume of $\mathcal{Q}_{g,n}$

Theorem (Delecroix, Goujard, Zograf, Zorich). *The Masur–Veech volume $\operatorname{Vol}\mathcal{Q}_{g,n}$ of the moduli space of meromorphic quadratic differentials with n simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot \sum_{\substack{\text{Weighted graphs }\Gamma\\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of }\Gamma-1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of }\Gamma} b_e \cdot \prod_{\text{Vertices of }\Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right)\,,$$

The partial sum for fixed number k of edges gives the contribution of k-cylinder square-tiled surfaces.

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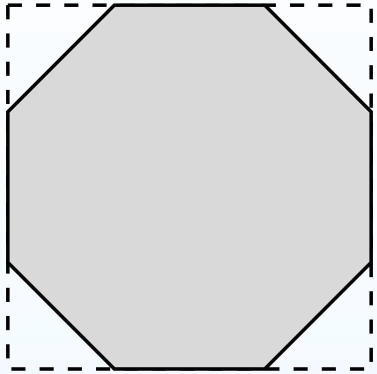
Remark. The Weil–Petersson volume of $\mathcal{M}_{g,n}$ corresponds to the *constant term* of the volume polynomial $N_{g,n}(L)$ when the lengths of all boundary components are contracted to zero. To compute the Masur–Veech volume we use the *top homogeneous parts* of volume polynomials; i.e. we use them in the opposite regime when the lengths of all boundary components tend to infinity.

Masur–Veech volumes of the moduli spaces of Abelian and quadratic differentials. Square-tiled surfaces

Very flat surface of genus two



Identifying the opposite sides of a regular octagon we get a flat surface of genus two. All the vertices of the octagon are identified into a single conical singularity. We always consider such a flat surface endowed with a distinguished (say, vertical) direction. By construction, the holonomy of the flat metric is trivial. Thus, the vertical direction at a single point globally defines vertical and horizontal foliations.

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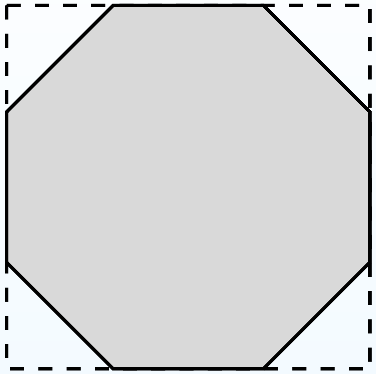

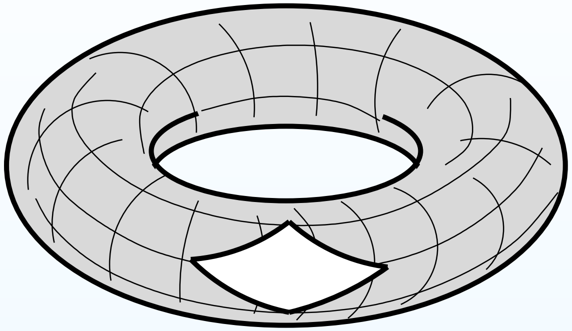

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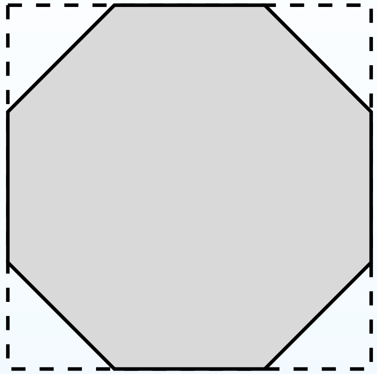

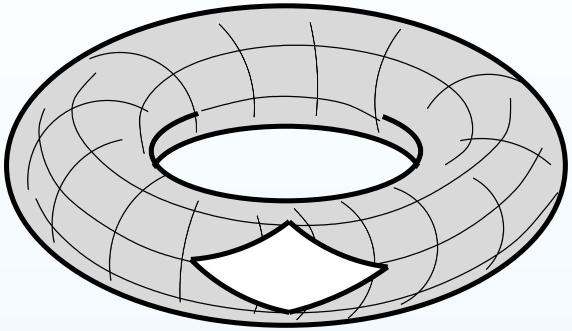

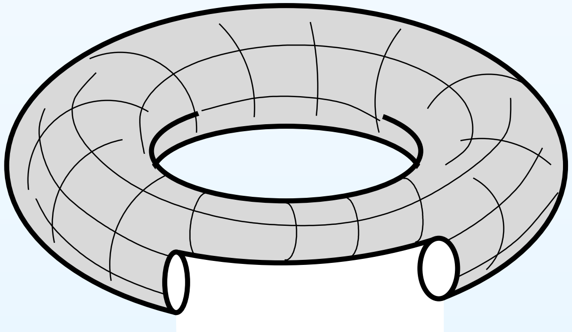

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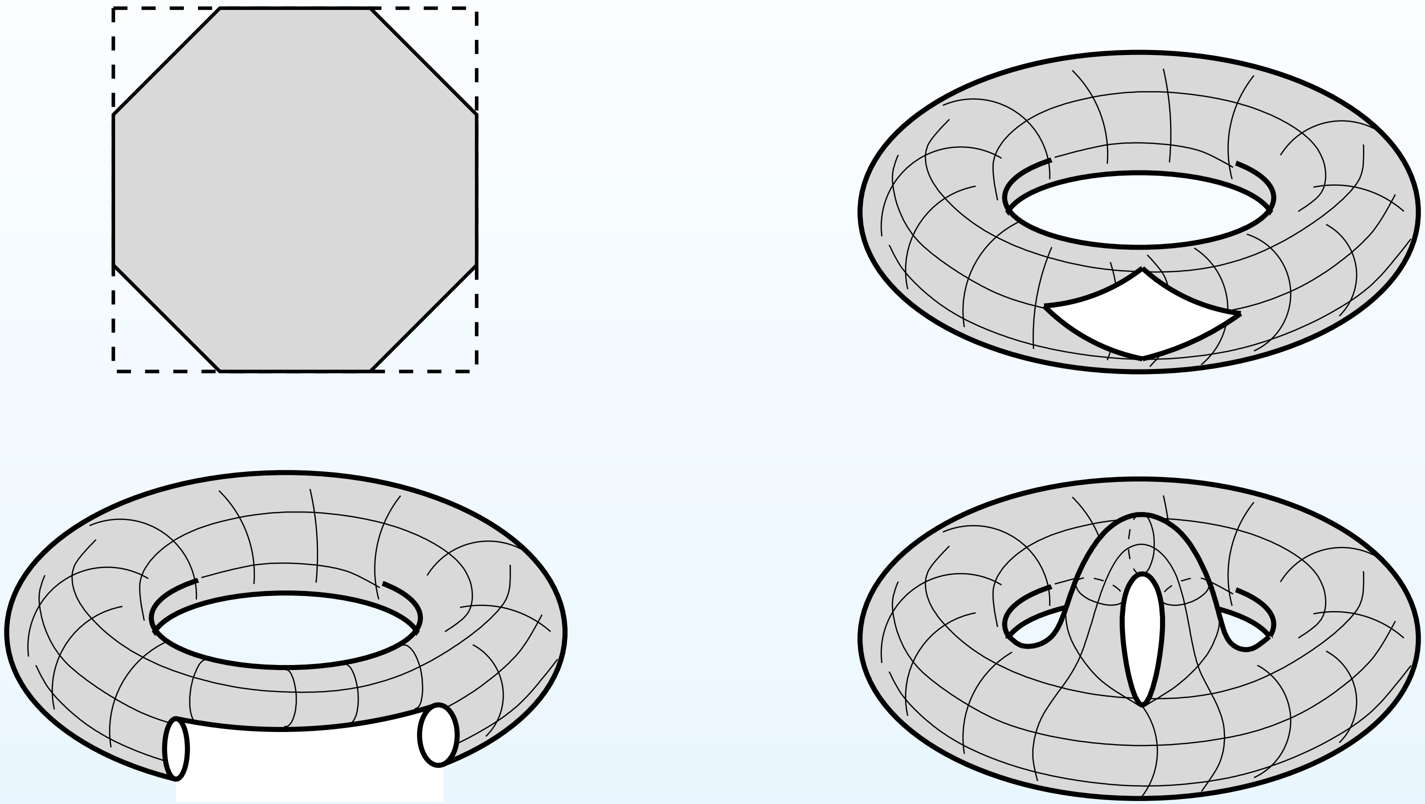

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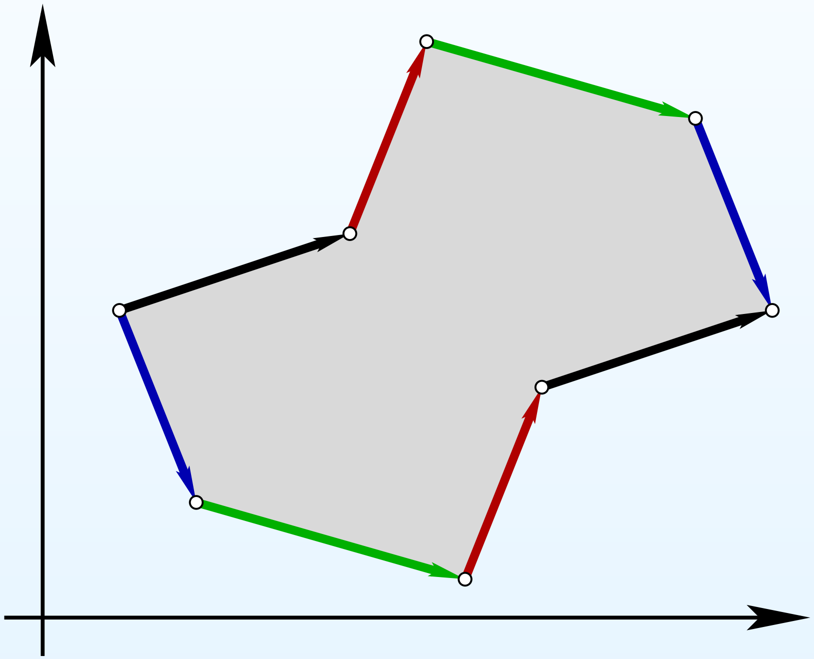
Period coordinates and Masur–Veech measure

Vectors defining the sides of the polygonal pattern serve as coordinates in the space of flat surfaces endowed with the distinguished vertical direction. The Lebesgue measure in these coordinates is called the *Masur–Veech measure*.



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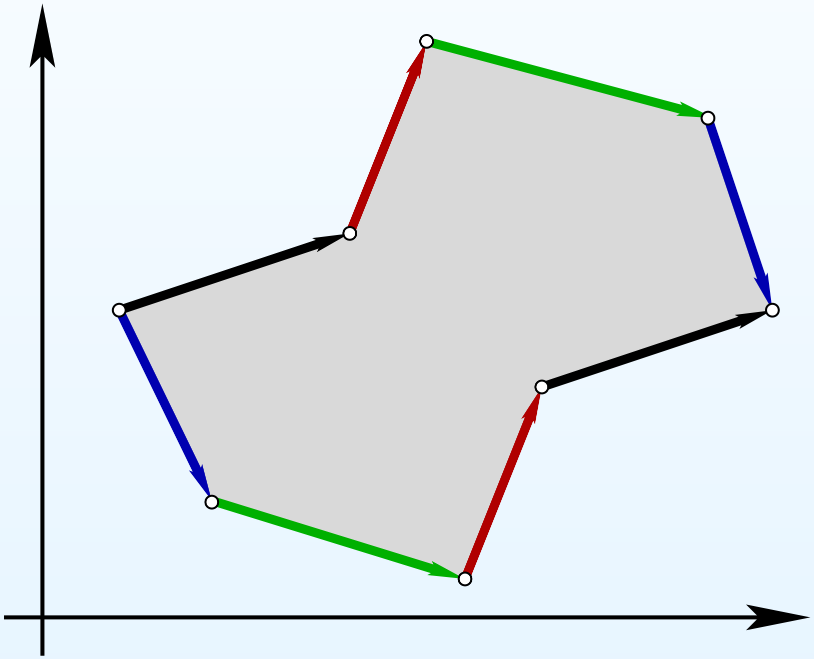

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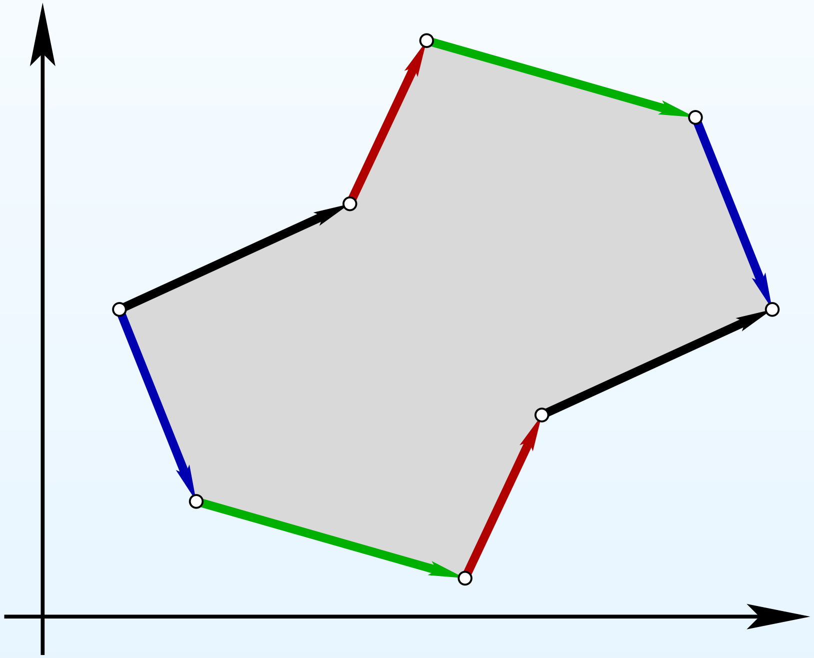

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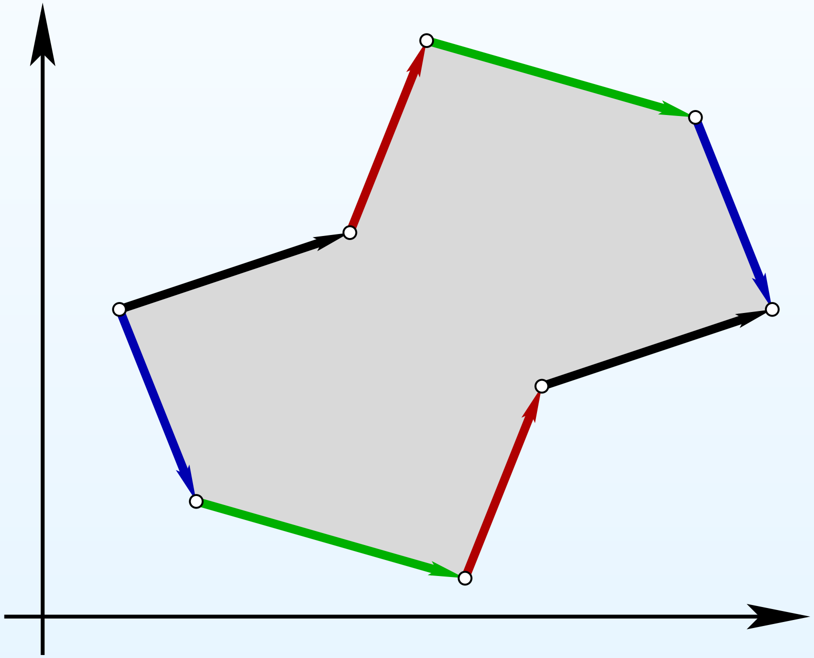
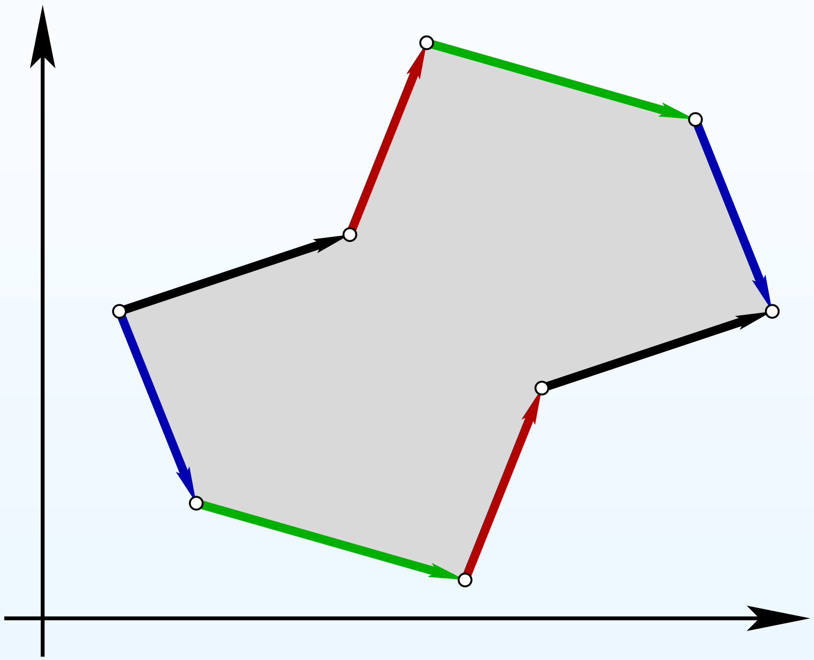


Considered as complex numbers, they represent integrals of the holomorphic form $\omega = dz$ along paths joining zeroes of the form ω. (In polygonal representation the zeroes of ω are represented by vertices of the polygon.)

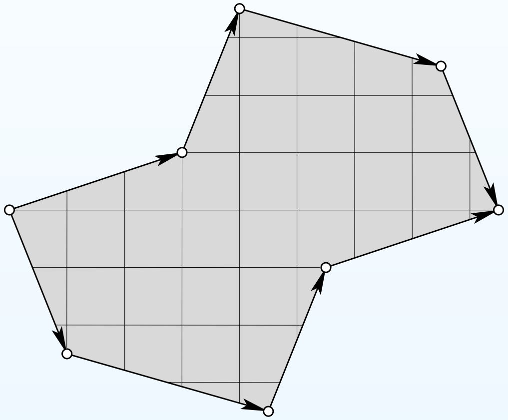
Period coordinates and Masur–Veech measure



In other words, the moduli space $\mathcal{H}(m_1, \dots, m_n)$ of pairs (C, ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1, \dots, m_n$, where $\sum m_i = 2g - 2$, is modeled on the vector space $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$. The latter vector space contains a natural lattice $H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat area of the surface as a positive homogeneous function

We have a natural action of $\mathbb{R}^+$ on any moduli space $\mathcal{H}(m_1, \ldots, m_n)$: given a positive integer $r > 0$ we can rescale a flat surface by factor r. The flat area of the surface gets rescaled by the factor r^2.



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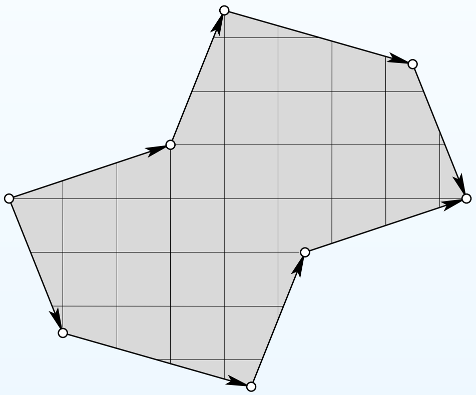

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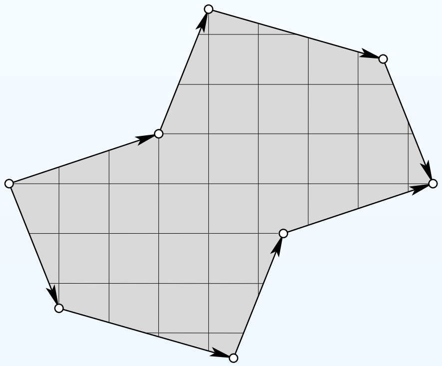

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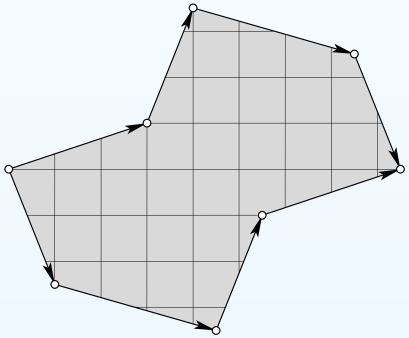

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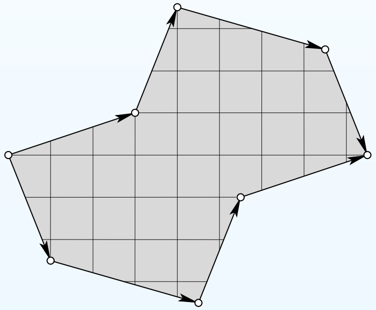

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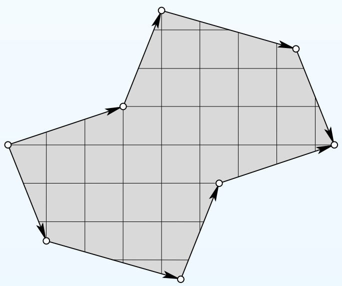


Flat surfaces of area 1 form a real hypersurface $\mathcal{H}_1=\mathcal{H}_1(m_1,\dots,m_n)$ defined in period coordinates by equation

$$1=\text{area}(S)=\frac{i}{2}\int_C \omega\wedge\bar\omega=\sum_{i=1}^{g}(A_i\bar B_i-\bar A_i B_i)\,.$$

Any flat surface S can be uniquely represented as $S=(C,r\cdot\omega)$, where $r>0$ and $(C,\omega)\in\mathcal{H}_1(m_1,\dots,m_n)$. In these “polar coordinates” the volume element disintegrates as $d\nu=r^{2d-1}dr\,d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

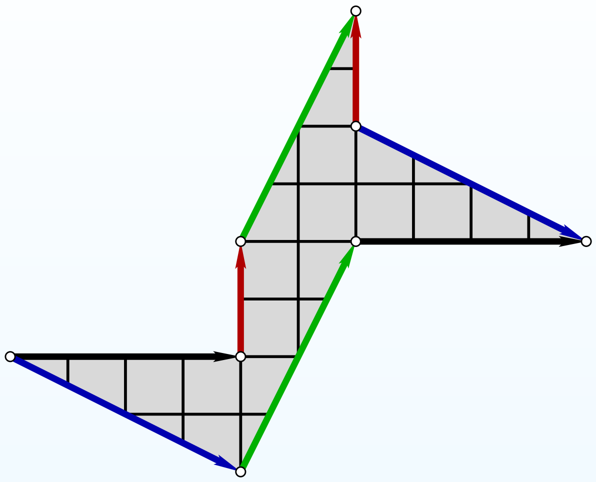
Masur–Veech volume

Summary. Every stratum of Abelian differentials admits

- A local structure of a vector space $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$;
- An integer lattice $H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$ which allows to normalize the associated Lebesgue measure;
- A positive homogeneous function which allows to define an analog of a unit sphere (or rather of a unit hyperboloid).

Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum $\mathcal{H}_1(m_1,\dots,m_n)$ or $\mathcal{Q}_1(m_1,\dots,m_n)$ of Abelian differentials or of meromorphic quadratic differentials with at most simple poles is finite.*

Integer points as square-tiled surfaces



Integer points in period coordinates are represented by *square-tiled surfaces*.

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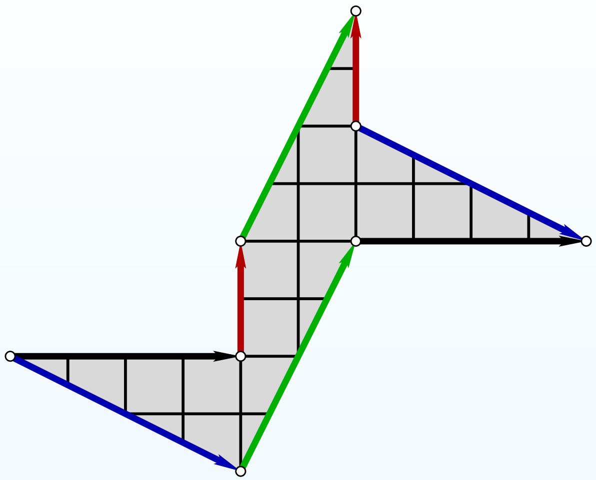

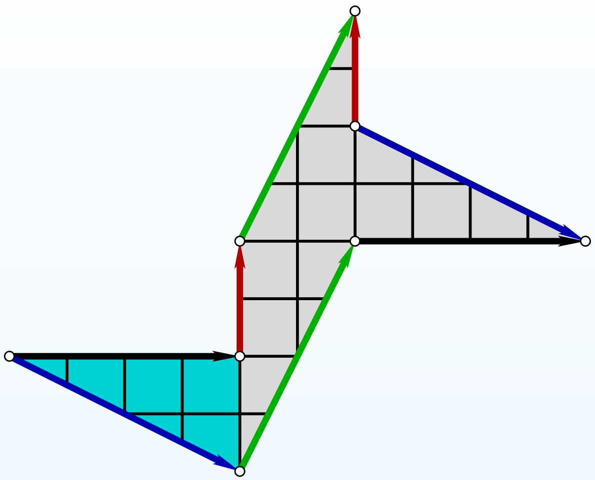

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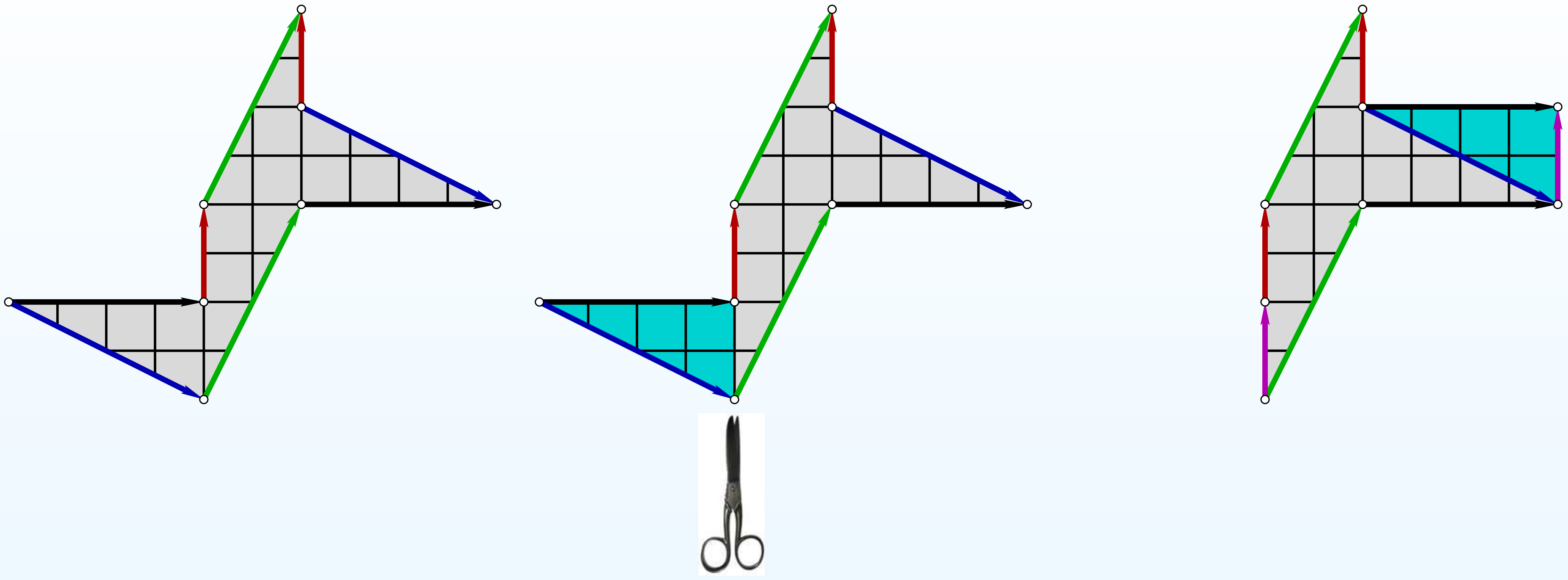

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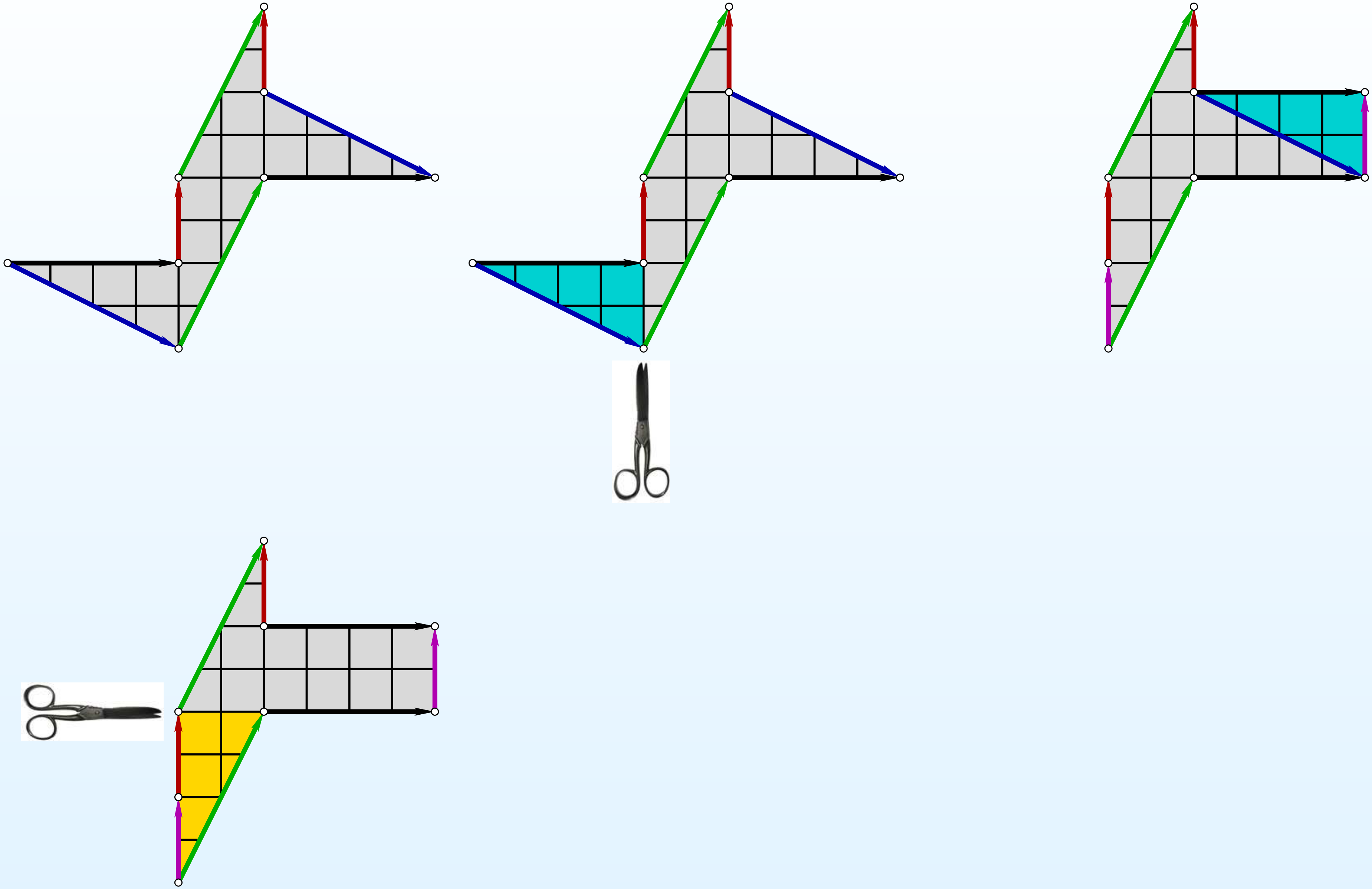

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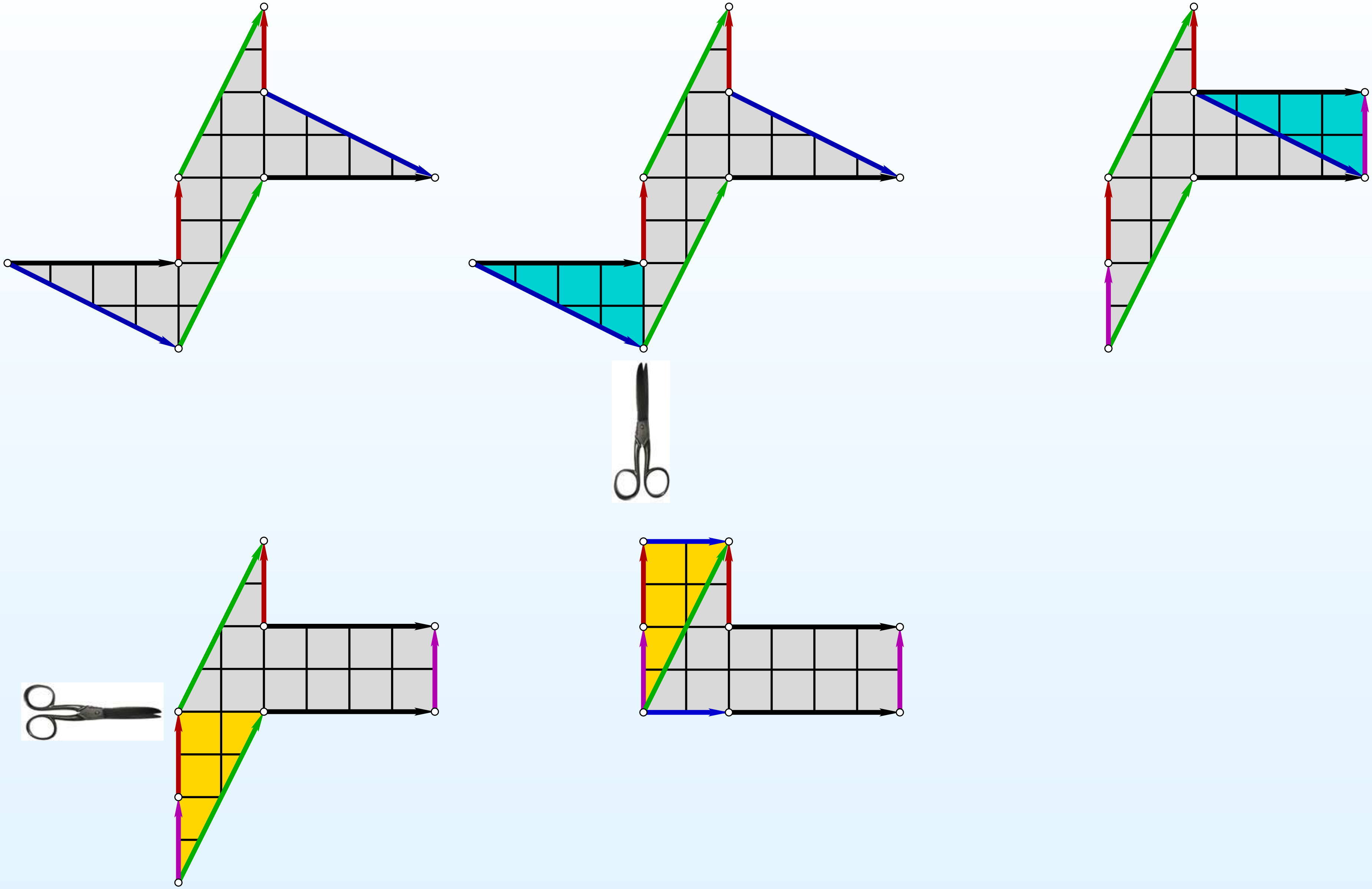

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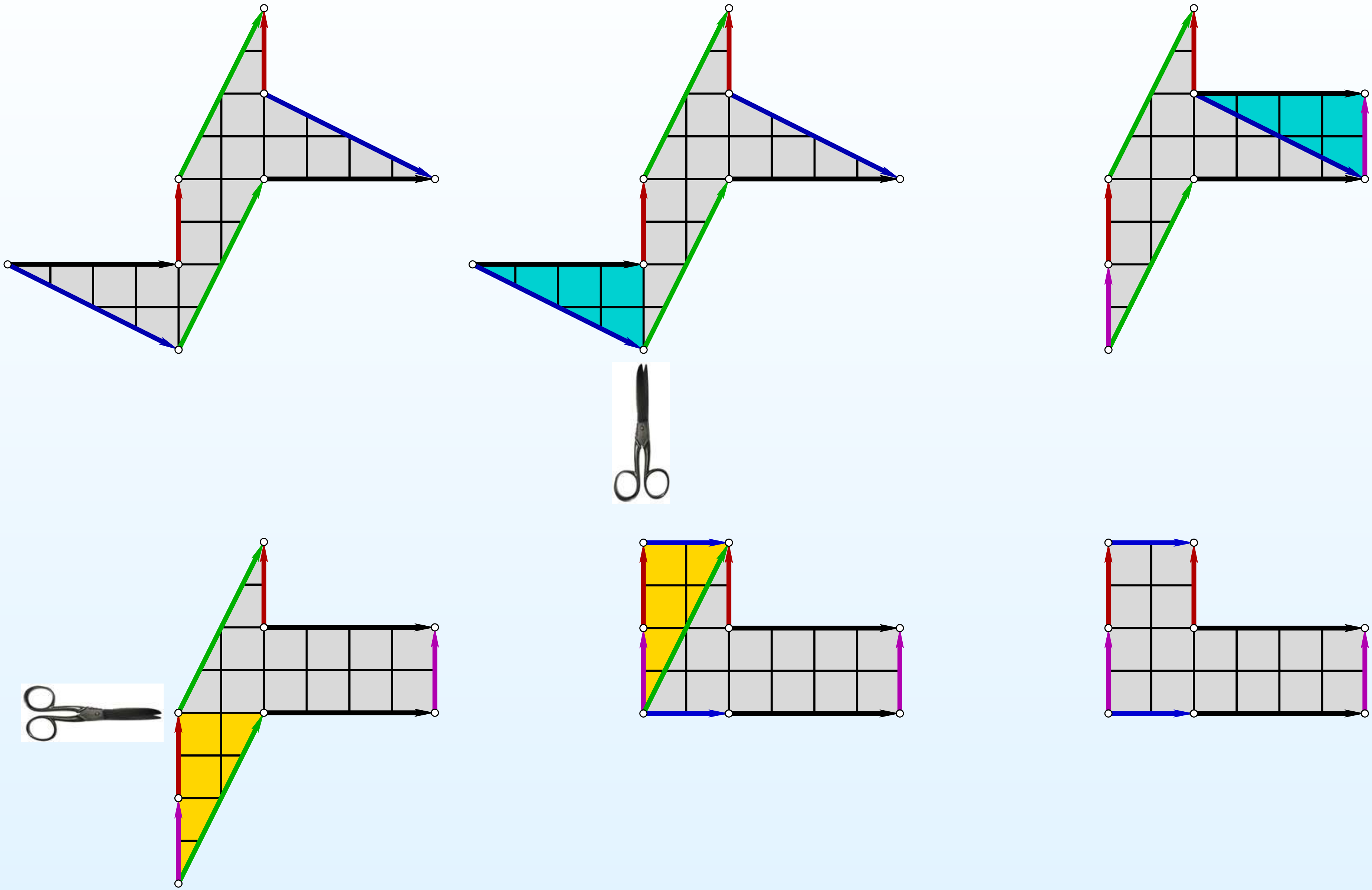

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Integer points in period coordinates are represented by *square-tiled surfaces*. Indeed, if a flat surface S is defined by a holomorphic 1-form ω such that $[\omega] \in H^1(S, \{P_1, \ldots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover p over the standard torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified over a single point:

$$S \ni P \mapsto \left(\int_{P_1}^{P} \omega \mod \mathbb{Z} \oplus i\mathbb{Z}\right) \in \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) = \mathbb{T}, \text{ where } P_1 \text{ is a zero of } \omega \,.$$

The ramification points of the cover are exactly the zeroes of ω.

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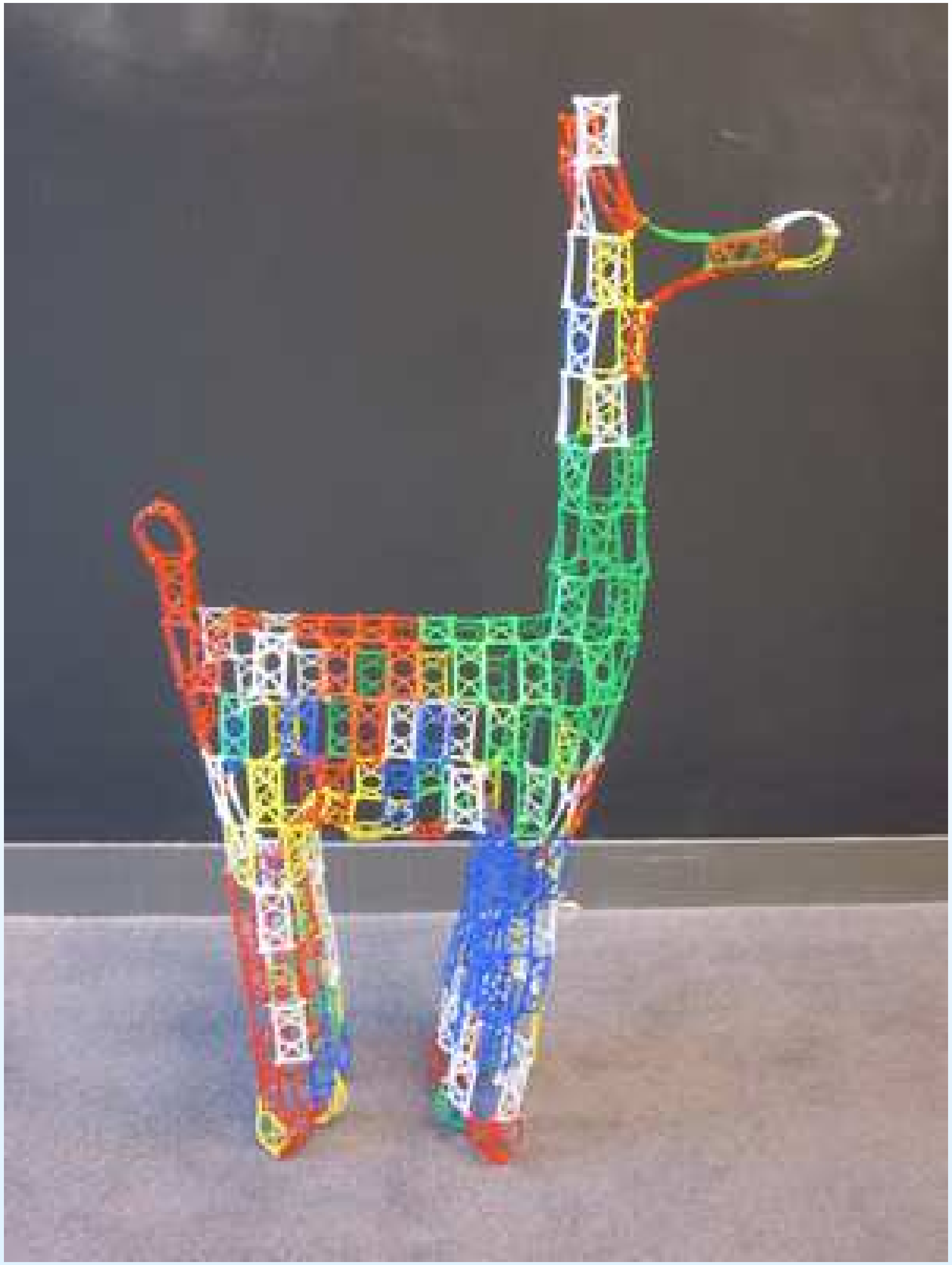
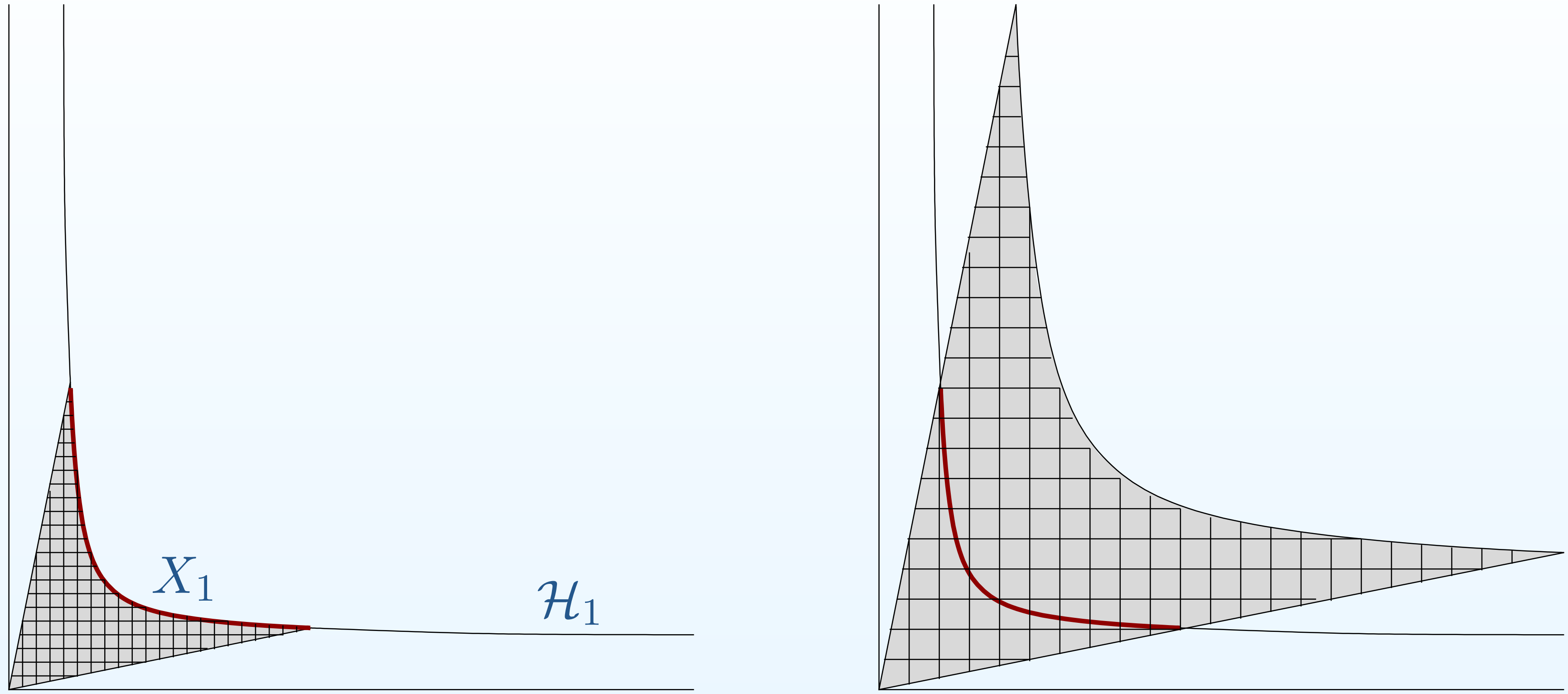
Integer points in the strata $\mathcal{Q}(d_1, \dots, d_n)$ of quadratic differentials are represented by analogous "pillowcase covers" over $\mathbb{CP}^1$ branched at four points. Thus, counting volumes of the strata is similar to counting analogs of Hurwitz numbers.

An example of a square-tiled surface



Counting volume by counting integer points in a large cone



To count volume of the cone $C(X_1)$ one can take a small grid and count the number of lattice points inside it. Counting points of the $\frac{1}{N}$-grid in the cone $C(X_1) = \{r \cdot S | S \in X_1, r \leq 1\}$ is the same as counting integer points in the larger proportionally rescaled cone $C_N(X_1) = \{r \cdot S | S \in X_1, r \leq N\}$.

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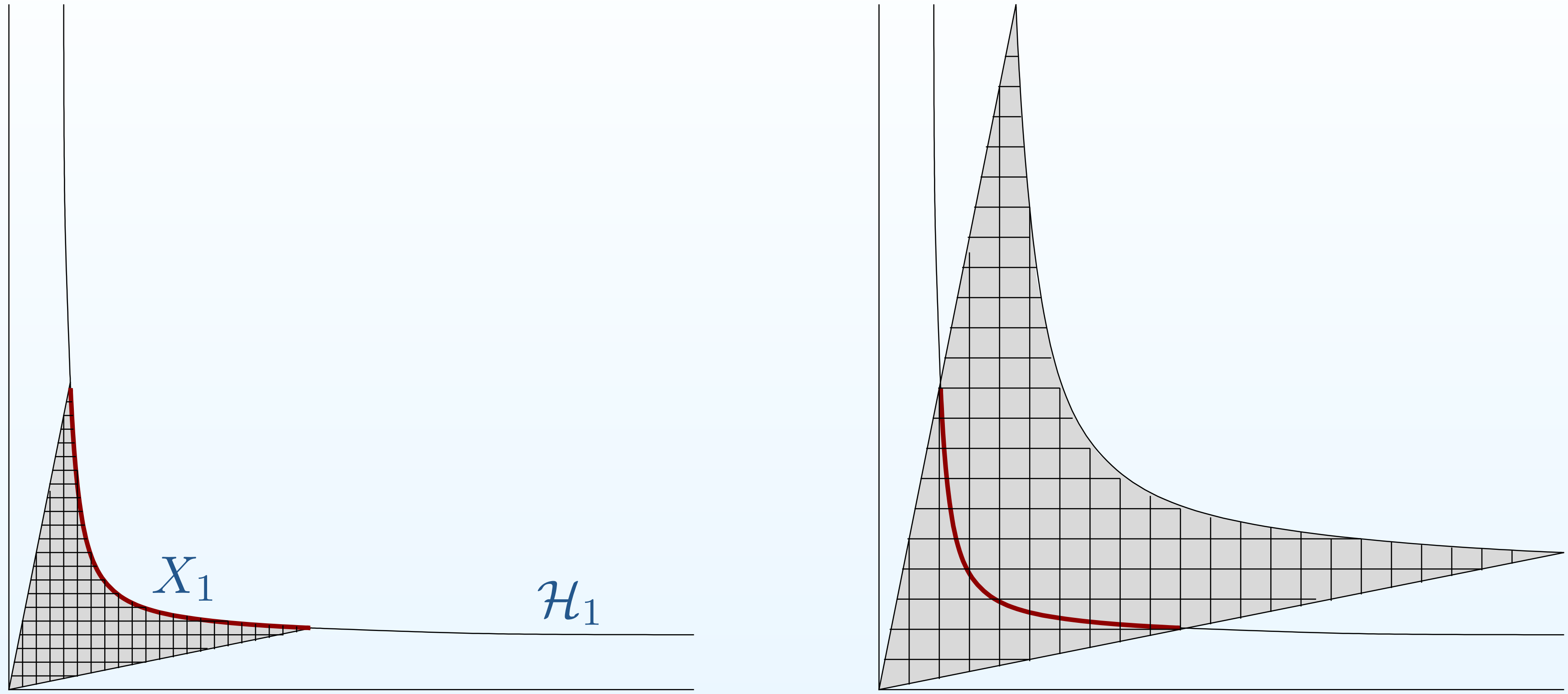
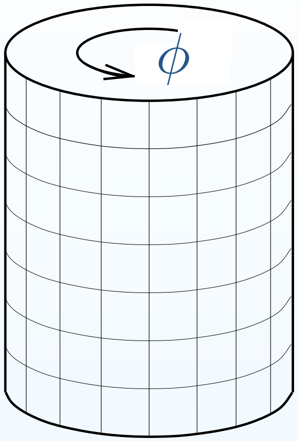
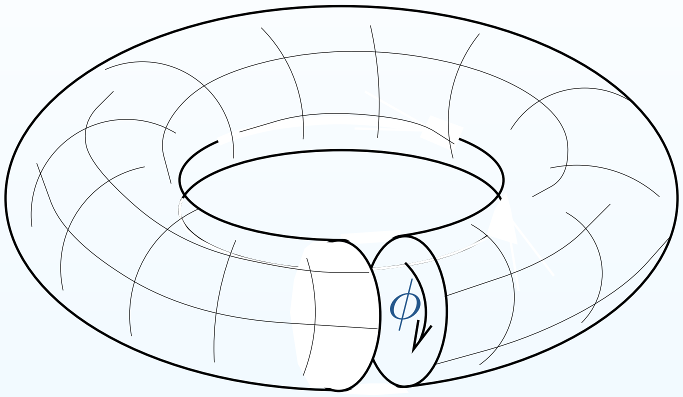


Let $\mathcal{H} = \mathcal{H}(m_1, \ldots, m_n)$; let $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \ldots, m_n) = 2g + n - 1$. We get:

$$\operatorname{Vol} \mathcal{H}_1 = 2d \cdot \lim_{N \to +\infty} \frac{\left(\begin{array}{c}\text{number of square-tiled surfaces in } \mathcal{H} \\ \text{tiled with at most } N \text{ identical squares}\end{array}\right)}{N^d} .$$

Volume of the space of flat tori





The number of square-tiled tori tiled with at most N squares has asymptotics

$$\sum_{\substack{b,h\in\mathbb{N}\\ b\cdot h\le N}} b = \sum_{\substack{b,h\in\mathbb{N}\\ b\le\frac{N}{h}}} b \sim \sum_{h\in\mathbb{N}} \frac{1}{2}\cdot\left(\frac{N}{h}\right)^2 = \frac{N^2}{2}\sum_{h\in\mathbb{N}}\frac{1}{h^2} = \frac{N^2}{2}\cdot\zeta(2) = \frac{N^2}{2}\cdot\frac{\pi^2}{6}.$$

$$\operatorname{Vol}\mathcal{H}_1(0) = 2\cdot 2\cdot \lim_{N\to+\infty}\frac{\begin{pmatrix}\text{number of square-tiled surfaces in } \mathcal{H}\\ \text{tiled with at most } N \text{ identical squares}\end{pmatrix}}{N^2} = \frac{\pi^2}{3}.$$

Methods of evaluation of Masur–Veech volumes

- M. Kontsevich–A. Zorich (1998). Straightforward calculation of square-tiled surfaces.
- (A. Eskin–A. Okounkov–R. Pandharipande; D. Chen–M. Möller–D. Zagier; E. Goujard) A. Eskin and A. Okounkov observed in 2000 that the generating function for the count of square-tiled surfaces is a quasimodular form.
- D. Chen–M. Möller–A. Sauvaget; M. Kazarian; Di Yang–D. Zagier–Y. Zhang (2018–) Using recent BCGGM smooth compactification of the moduli space, one can work with the volume element as with the cohomology class. Intersection theory.
- V. Delecroix–E. Goujard–P. Zograf–A. Zorich (2018) (F. Arana–Herrera): volume of the principal stratum of quadratic differentials through Kontsevich's count of metric ribbon graphs in terms of Witten–Kontsevich correlators.
- D. Chen–M. Möller–A. Sauvaget–D. Zagier; A. Aggarwal (2018–) Large genus asymptotics for any stratum of Abelian differentials (proving conjectures of Eskin–Zorich and of Delecroix–Goujard–Zograf–Zorich).
- Andersen–Borot–Charbonnier–Delecroix–Giacchetto–Lewanski–Wheeler, 2020 (inspired by the formula of Delecroix-Goujard–Zograf–Zorich): topological recursion.

Lecture 2. Count of simple closed geodesics on Riemann surfaces (after Maryam Mirzakhani)

Anton Zorich

(Reference: M. Mirzakhani, *“Growth of the number of simple closed geodesics on hyperbolic surfaces”*, Annals of Math. (2) **168** (2008), no. 1, 97–125.)

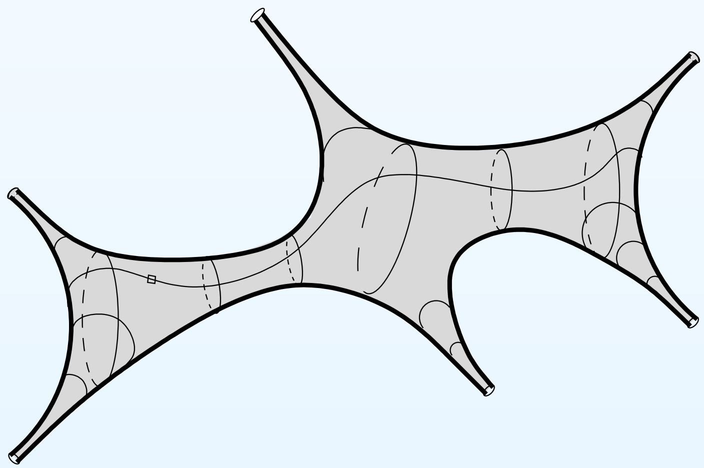
School “Moduli Spaces, Combinatorics and Integrable Systems”

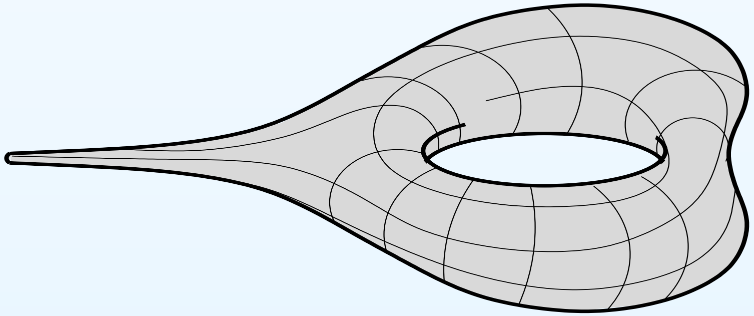
St. Petersburg, November 25, 2021

Hyperbolic geometry of surfaces

Hyperbolic surfaces

Any smooth orientable surface of genus $g \geq 2$ admits a metric of constant negative curvature (usually chosen to be -1), called *hyperbolic* metric.
Allowing to metric to have several singularities (cusps), one can construct a hyperbolic metric also on a sphere and on a torus.





Simple closed curves and simple closed geodesics

A smooth closed curve on a surface is called *simple* if it does not have self-intersections.

Suppose that we have a simple closed curve γ on a *hyperbolic surface* (possibly with cusps). Suppose that the curve is *essential*, that is not contractible to a small curve encircling some disc or some cusp.

Interpreting our curve as an elastic loop, let it slide along the surface to contract to the shortest shape in our hyperbolic metric. We get a closed geodesic, which remains to be smooth non self-intersecting curve.

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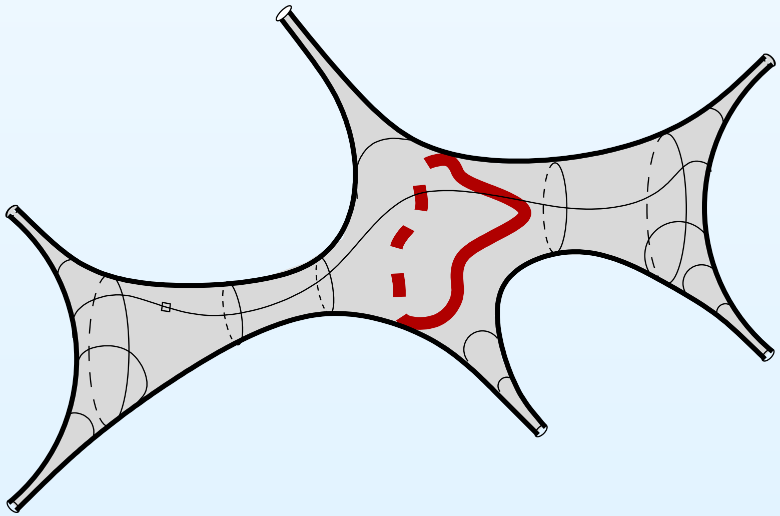

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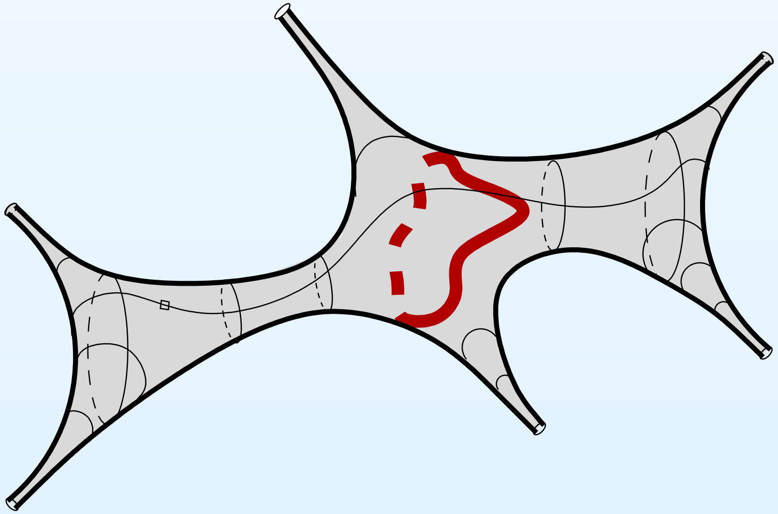

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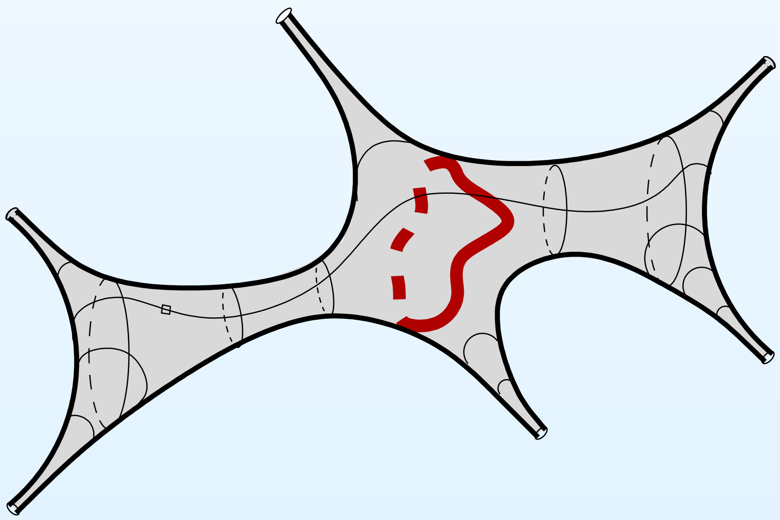

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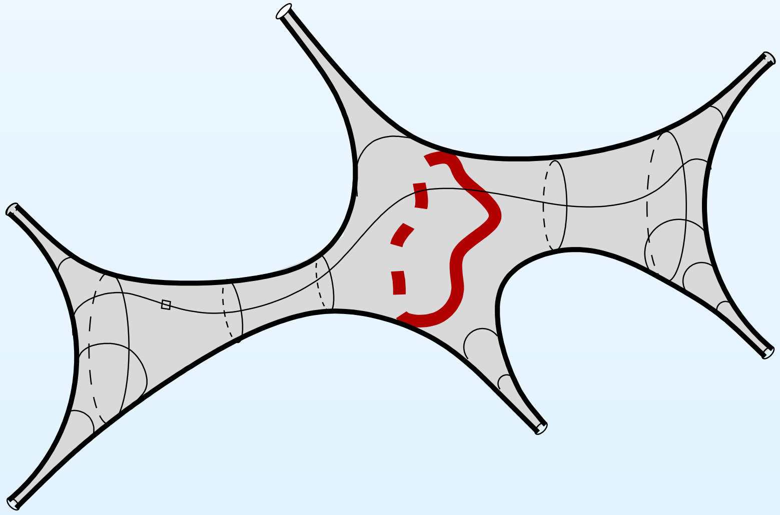

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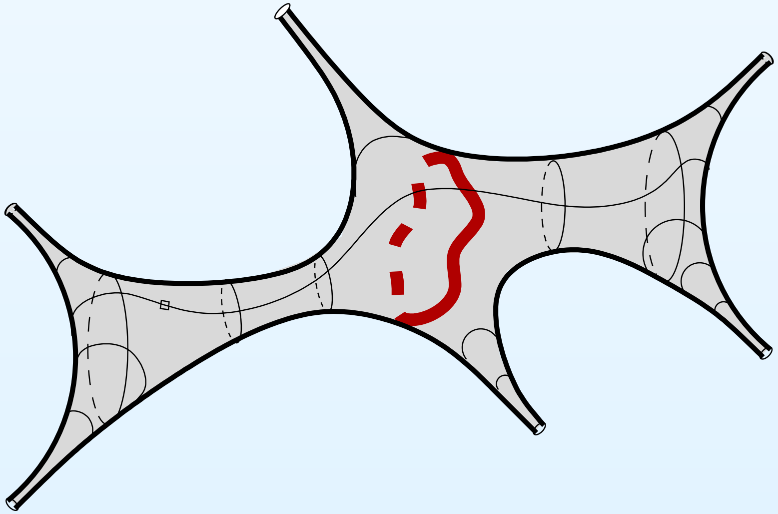

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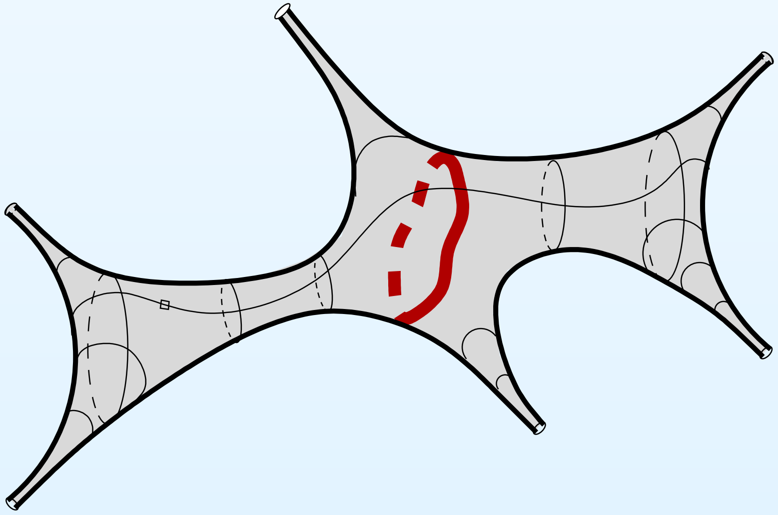

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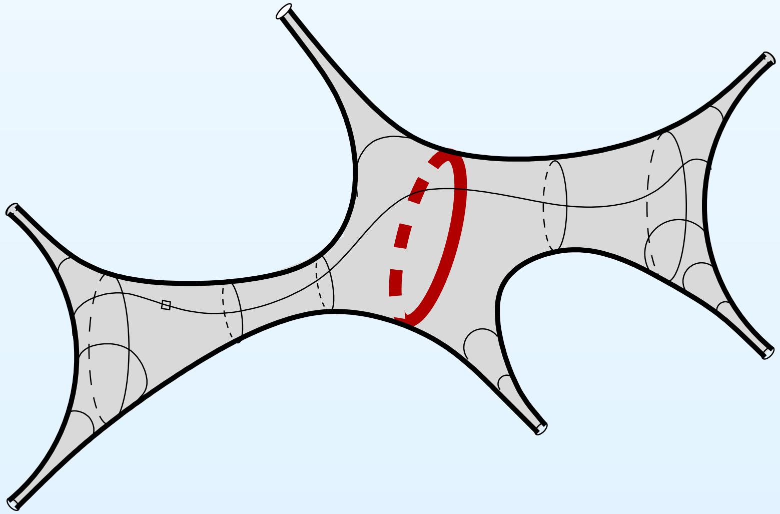

Families of hyperbolic surfaces

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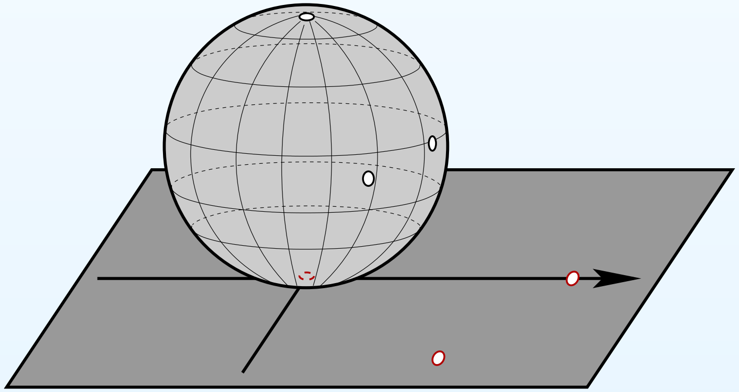

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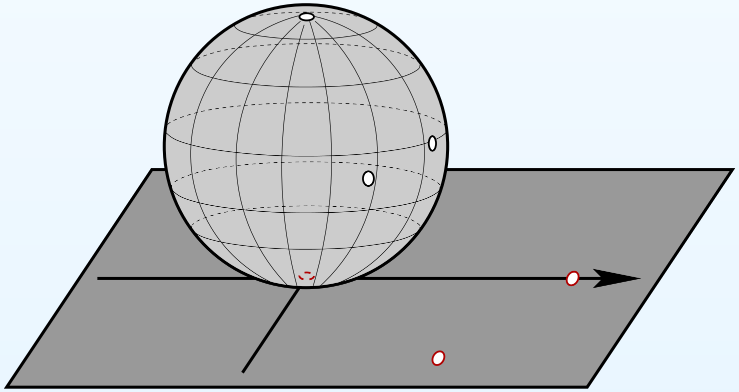

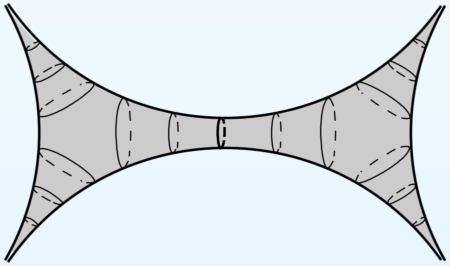

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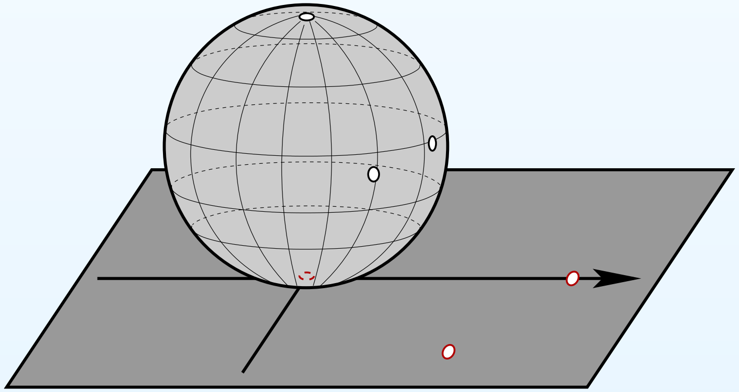

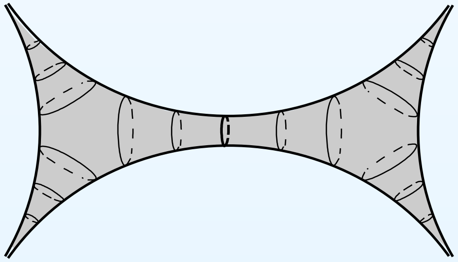

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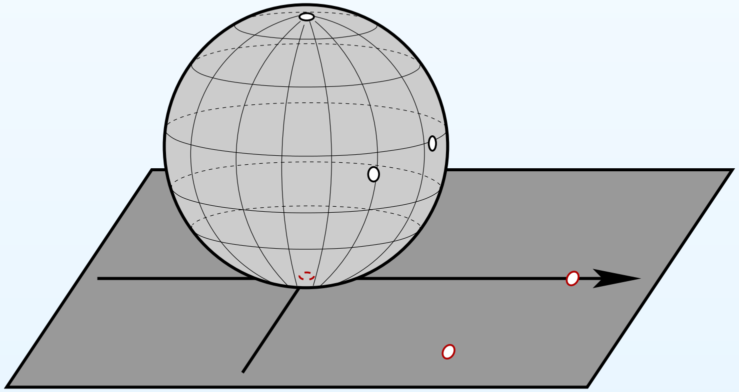

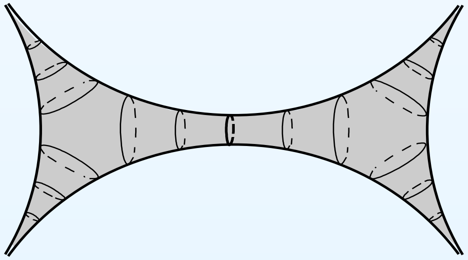

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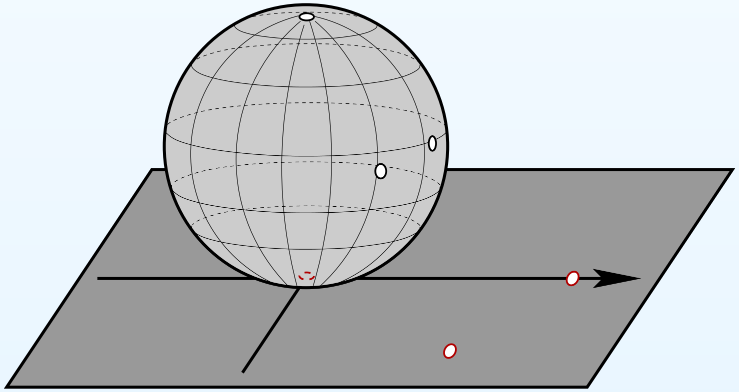

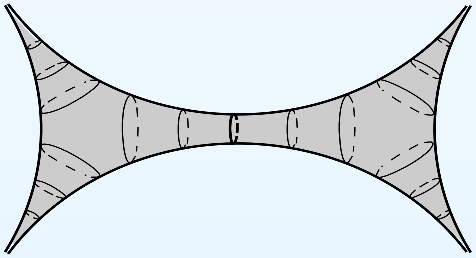

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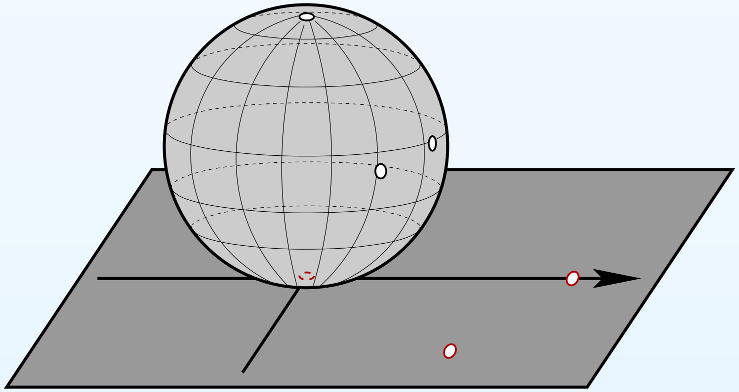

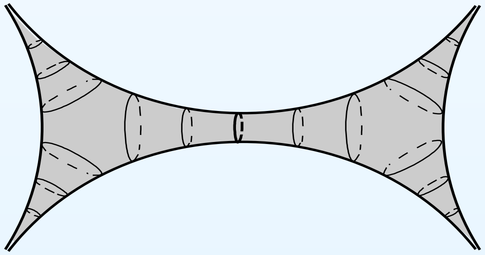

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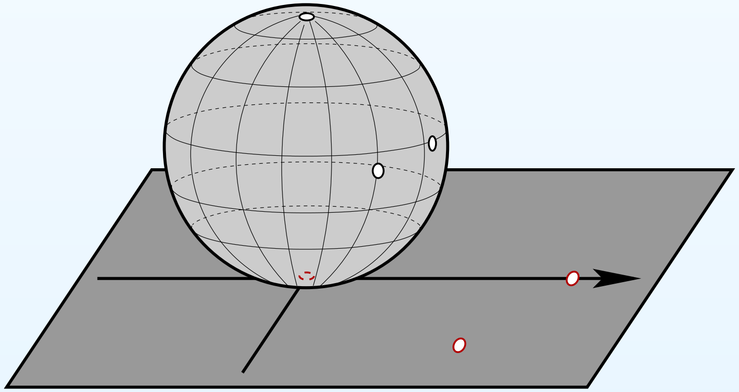

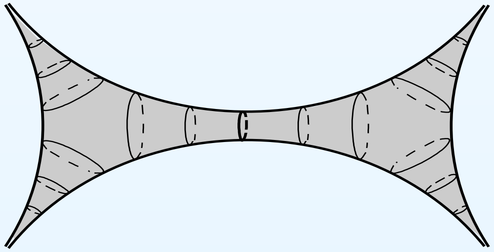

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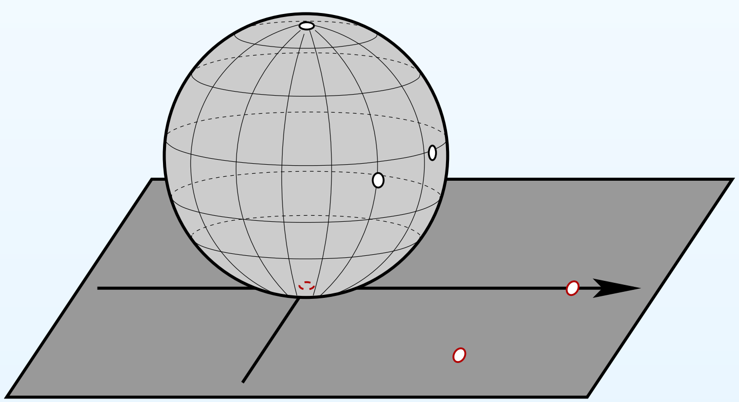

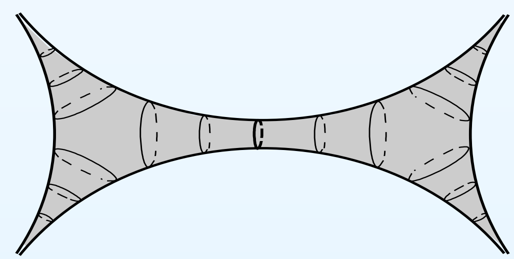

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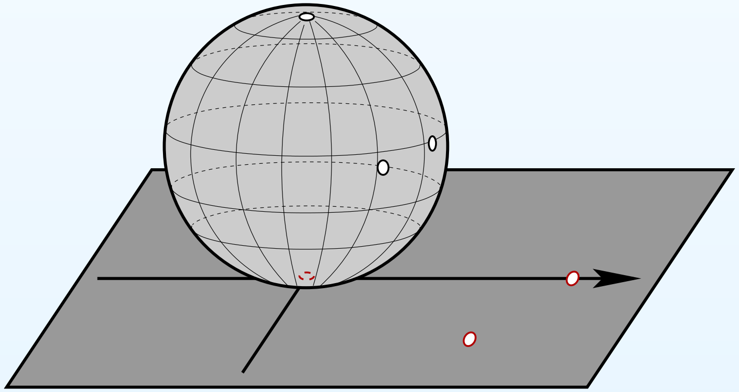

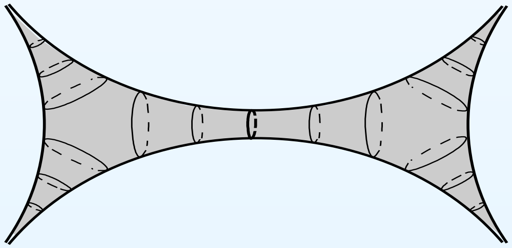

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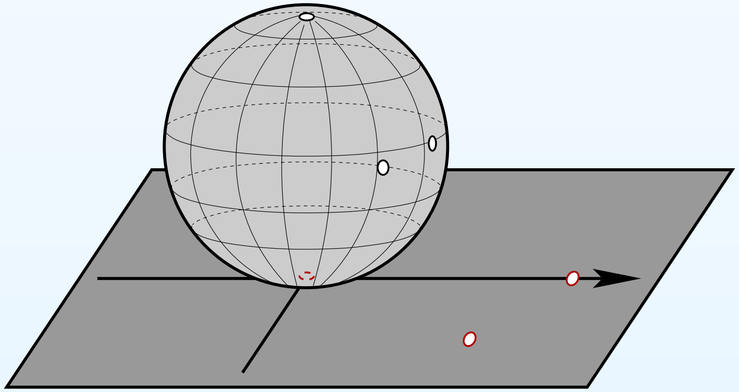

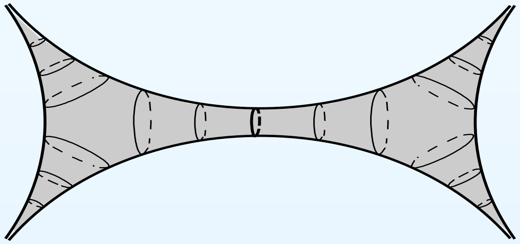

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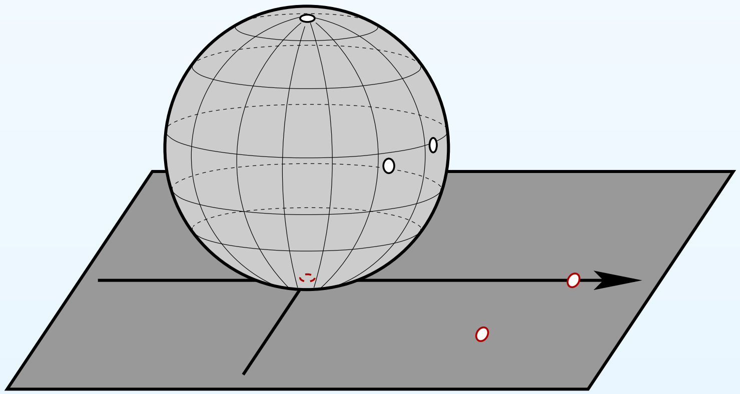

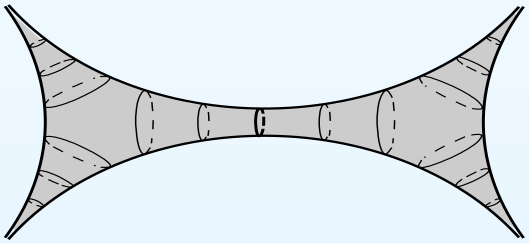

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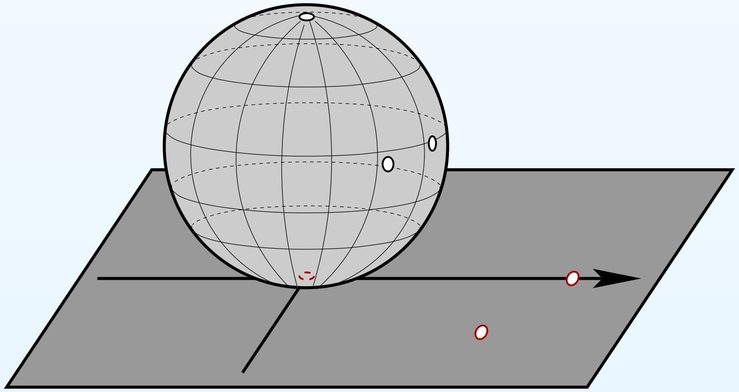

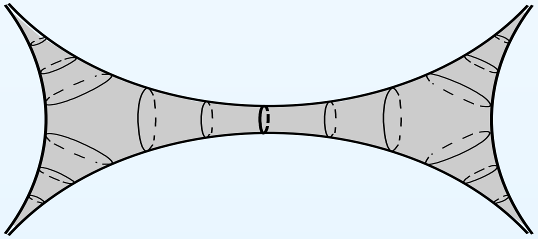

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Families of hyperbolic surfaces

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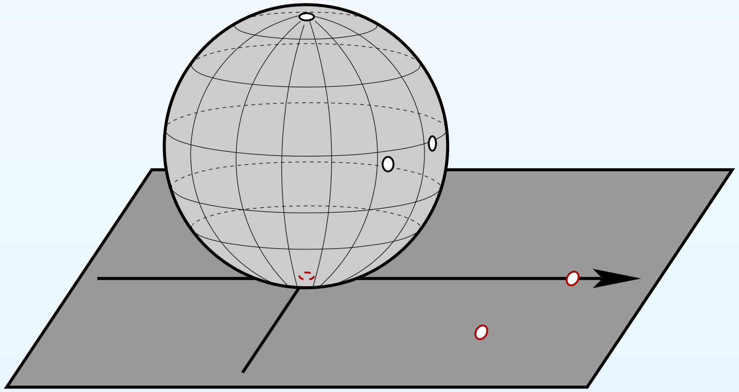

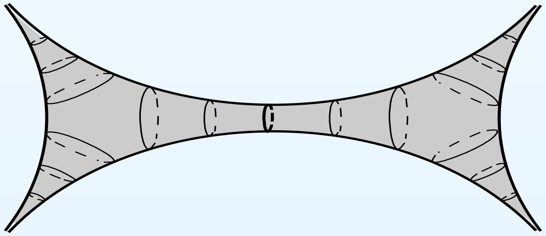

By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Space of multicurves

Topological types of simple closed curves

Let us say that two simple closed curves on a smooth surface have the same *topological type* if there is a diffeomorphism of the surface sending one curve to another.

It immediately follows from the classification theorem of surfaces that there is a finite number of topological types of simple closed curves. For example, if the surface does not have punctures, all simple closed curves which do not separate the surface into two pieces, belong to the same class.

One can consider more general *primitive multicurves*: collections of pairwise disjoint non-homotopic simple closed curves. For any fixed pair (g, n) the number of topological types of primitive multicurves on a surface of genus g with n punctures is also finite.

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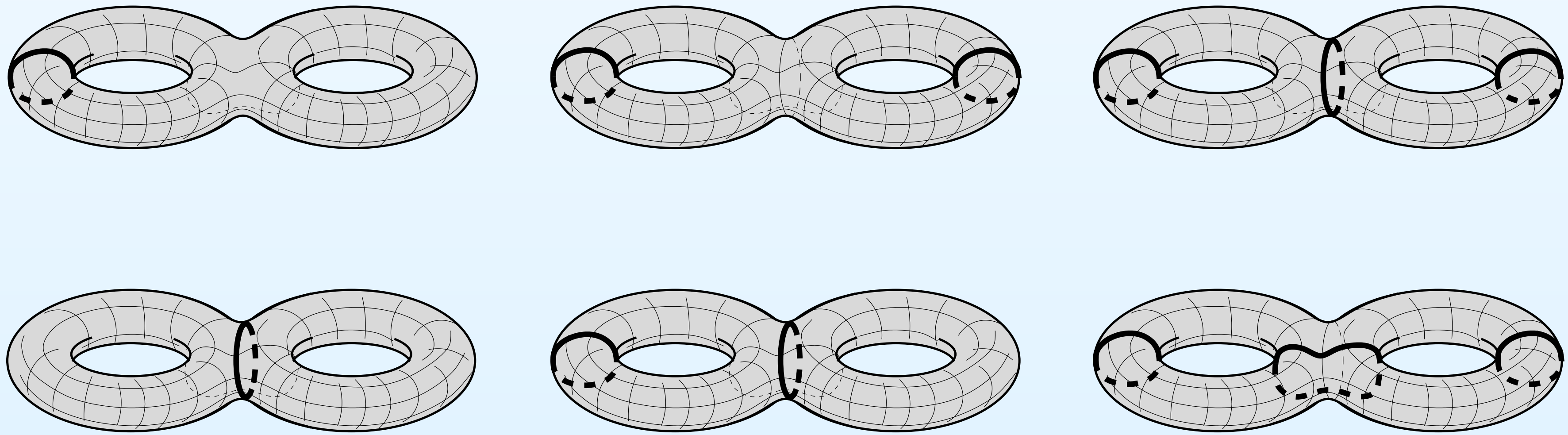
Example: primitive multicurves on a surface of genus two

The picture below illustrates all possible types of primitive multicurves on a surface of genus two without punctures.

Note that contracting all components of a multicurve we get a “stable curve” — a Riemann surface degenerated in one of the several regular ways. In this way the “topological types of primitive multicurves” on a smooth surface $S_{g,n}$ of genus g with n punctures are in the natural bijective correspondence with boundary classes of the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of pointed complex curves.



Mapping class group

The group of all diffeomorphisms of a closed smooth orientable surface of genus g quotient over diffeomorphisms homotopic to identity is called the *mapping class group* and is denoted by Mod_g.

When the surface has n marked points (punctures) we require that diffeomorphism sends marked points to marked points; the corresponding mapping class group is denoted $\mathrm{Mod}_{g,n}$.

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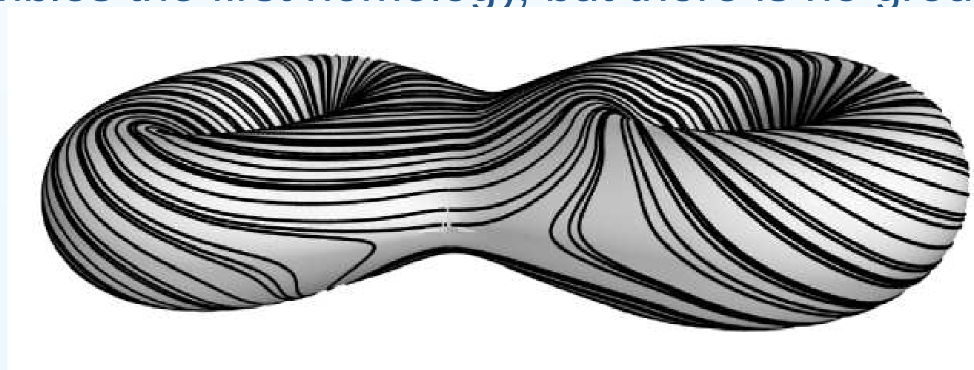
Simple closed multicurve, its topological type and underlying primitive multicurve

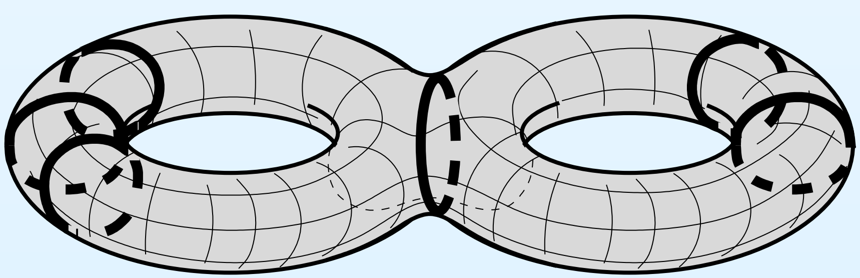
The first homology $H_1(M^2;\mathbb{Z})$ of the surface is great to study closed curves, but it ignores some interesting curves. The fundamental group $\pi_1(M^2)$ is also wonderful, but it is mainly designed to work with self-intersecting cycles. Thurston invented yet another structure to work with simple closed multicurves; in many aspects it resembles the first homology, but there is no group structure.

A general multicurve ρ:

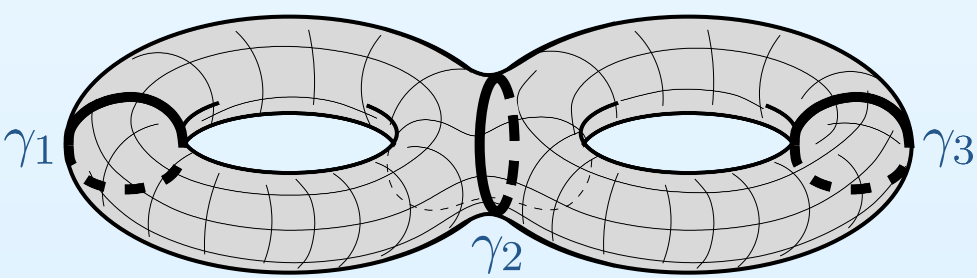


the canonical representative $\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$ in its orbit $\mathrm{Mod}_2 \cdot \rho$ under the action of the mapping class group and the associated *reduced* multicurve.

$\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$

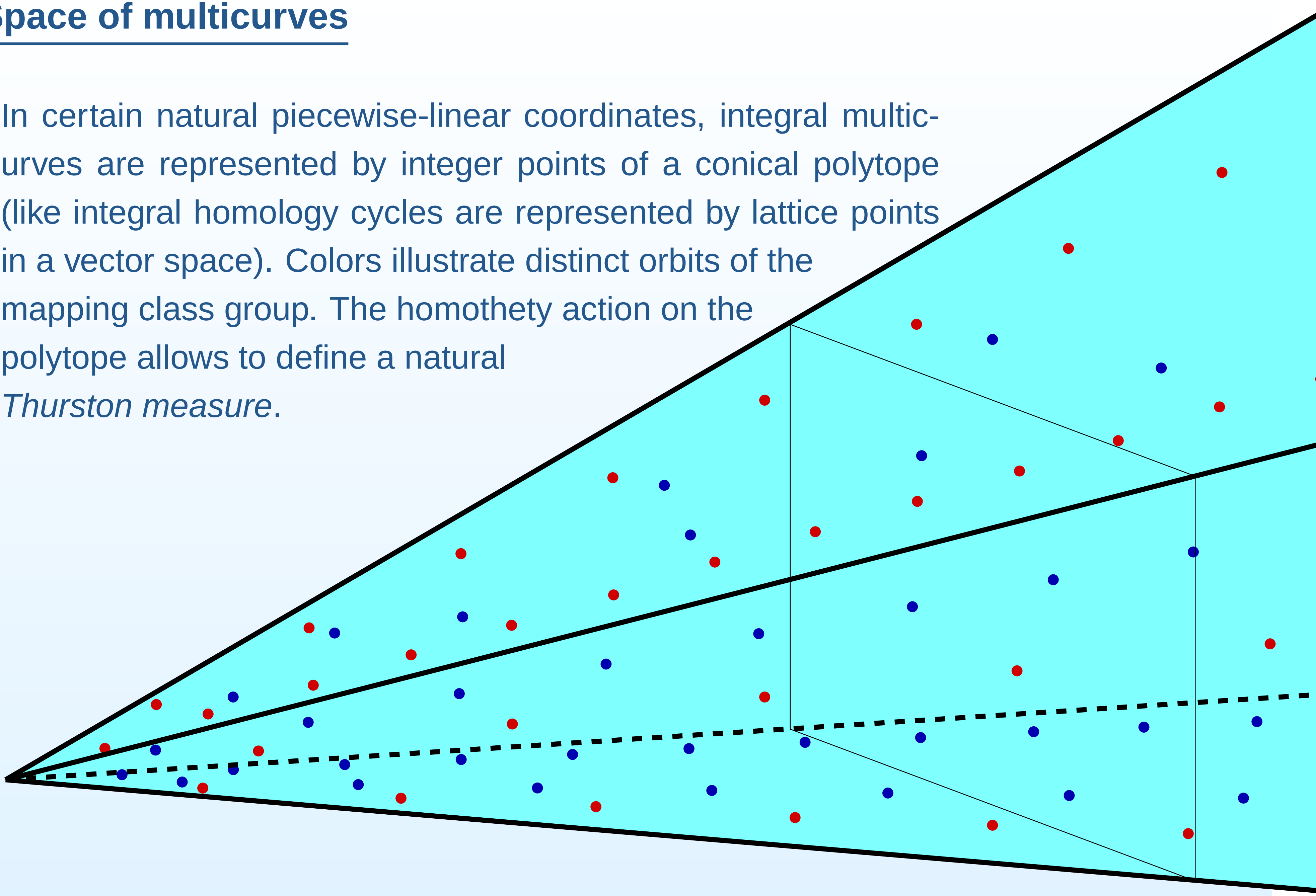


$\gamma_{reduced} = \gamma_1 + \gamma_2 + \gamma_3$



Space of multicurves

In certain natural piecewise-linear coordinates, integral multicurves are represented by integer points of a conical polytope (like integral homology cycles are represented by lattice points in a vector space). Colors illustrate distinct orbits of the mapping class group. The homothety action on the polytope allows to define a natural *Thurston measure*.



Space of measured laminations $\mathcal{ML}_{g,n}$. Ergodicity of the Thurston measure

In the presence of a hyperbolic metric the integral multicurves take the shape of simple closed geodesic multicurves. Moreover, every (not necessary integral) point of the conical polytope defines a *measured geodesic lamination*. The "natural coordiantes" are, for example, the *train tracks* coordinates.

Integral points in $\mathcal{ML}_{g,n}$ are in a one-to-one correspondence with the set of integral multi-curves, so the piecewise-linear action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ preserves the "integral lattice" $\mathcal{ML}_{g,n}(\mathbb{Z})$, and, hence, preserves the Thurston measure μ_{Th}.

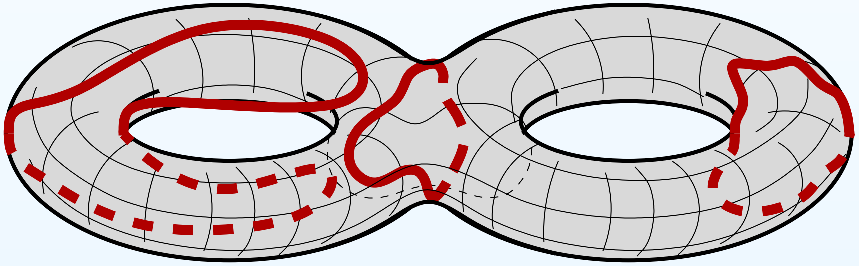
Theorem (H. Masur, 1985). *The action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ is ergodic with respect to the Lebesgue measure class (i.e. any measurable subset of $\mathcal{ML}_{g,n}$ invariant under $\mathrm{Mod}_{g,n}$ has measure zero or its complement has measure zero). Any $\mathrm{Mod}_{g,n}$-invariant measure in the Lebesgue measure class is just Thurston measure rescaled by some constant factor.*

Mirzakhani’s count of simple closed geodesics: statement of results

Picture by François Labourie taken at CIRM

Geodesic representatives of multicurves

Consider now several pairwise nonintersecting essential simple closed curves $\gamma_1, \ldots, \gamma_k$ on a smooth surface $S_{g,n}$ of genus g with n punctures. We have seen that in the presence of a hyperbolic metric X on $S_{g,n}$ the simple closed curves become simple closed geodesics.



Fact. *For any hyperbolic metric X the simple closed geodesics representing $\gamma_1, \ldots, \gamma_k$ do not have pairwise intersections.*

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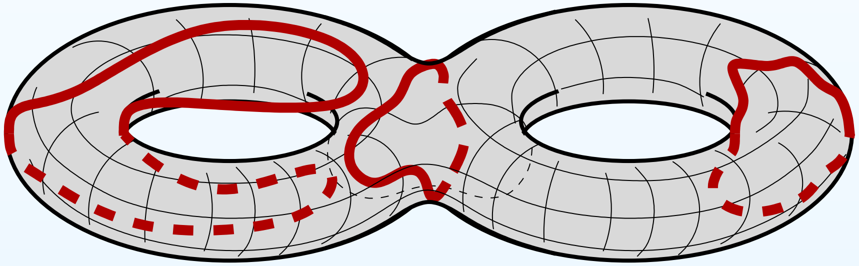

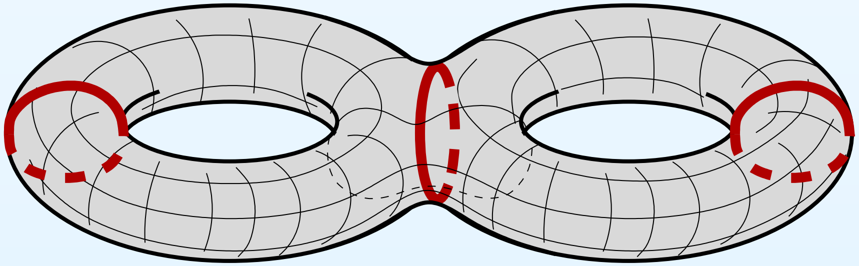

Fact. *For any hyperbolic metric X the simple closed geodesics representing $\gamma_1, \ldots, \gamma_k$ do not have pairwise intersections.*

Hyperbolic length of a multicurve

We can consider formal linear combinations $\gamma := \sum_{i=1}^{k} a_i\gamma_i$ of such simple closed curves with positive coefficients. When all coefficients a_i are integer (respectively rational), we call such γ integral (respectively rational) *multicurve*. In the presence of a hyperbolic metric X we define the hyperbolic length of a multicurve γ as $\ell_\gamma(X) := \sum_{i=1}^{k} a_i \ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

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Main counting results

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve γ and any hyperbolic surface X in $\mathcal{M}_{g,n}$ one has*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here the quantity $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X (it is the Thurstom measure of the unit ball B_X in the metric X); $b_{g,n}$ is a global constant depending only on g and n (which is the average value of $B(X)$ over $\mathcal{M}_{g,n}$); $c(\gamma)$ depends only on the topological type of γ (expressed in terms of the Witten–Kontsevich correlators).

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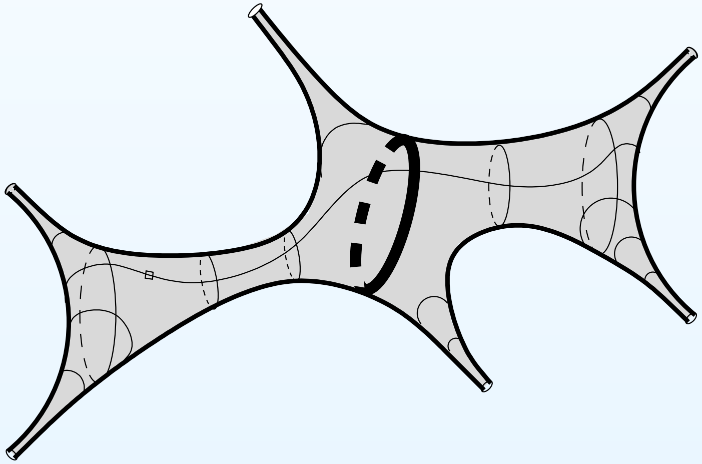
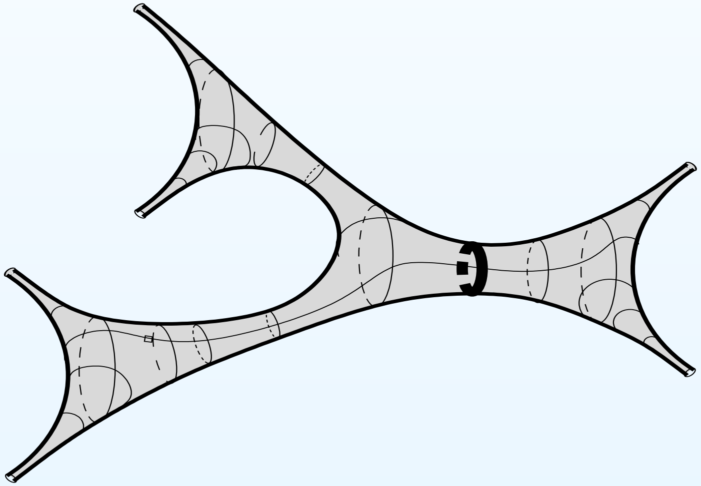
Corollary (M. Mirzakhani, 2008). *For any hyperbolic surface X in $\mathcal{M}_{g,n}$, and any two rational multicurves γ_1, γ_2 on a smooth surface $S_{g,n}$ considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.





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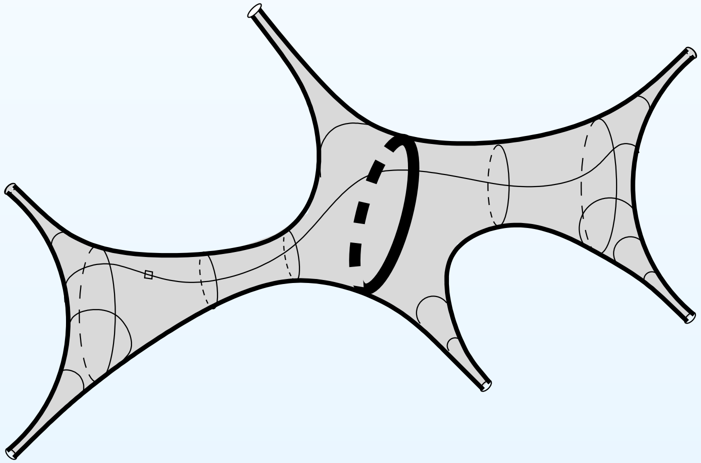
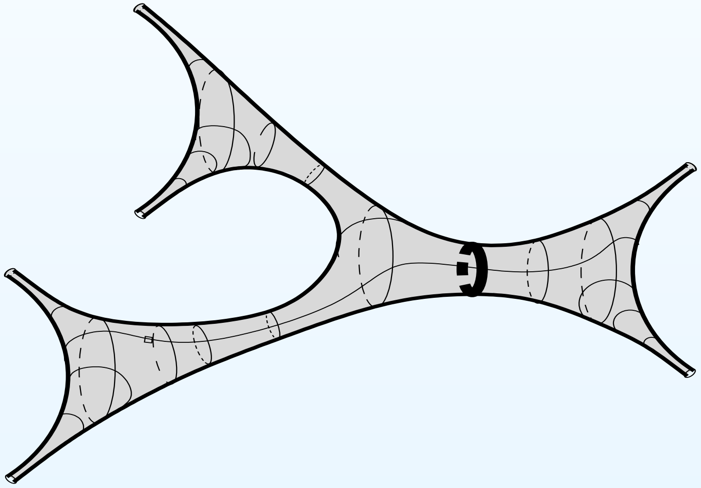
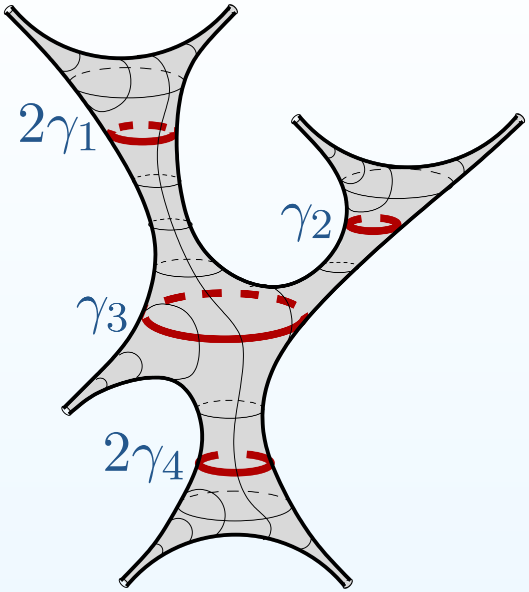


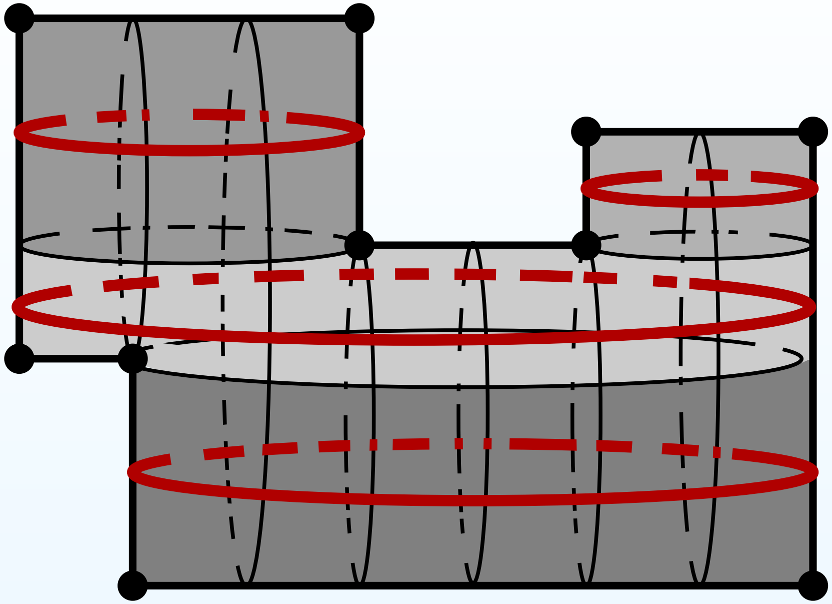


Example. (M. Mirzakhani, 2008); confirmed experimentally in 2017 by M. Bell; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by other means.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

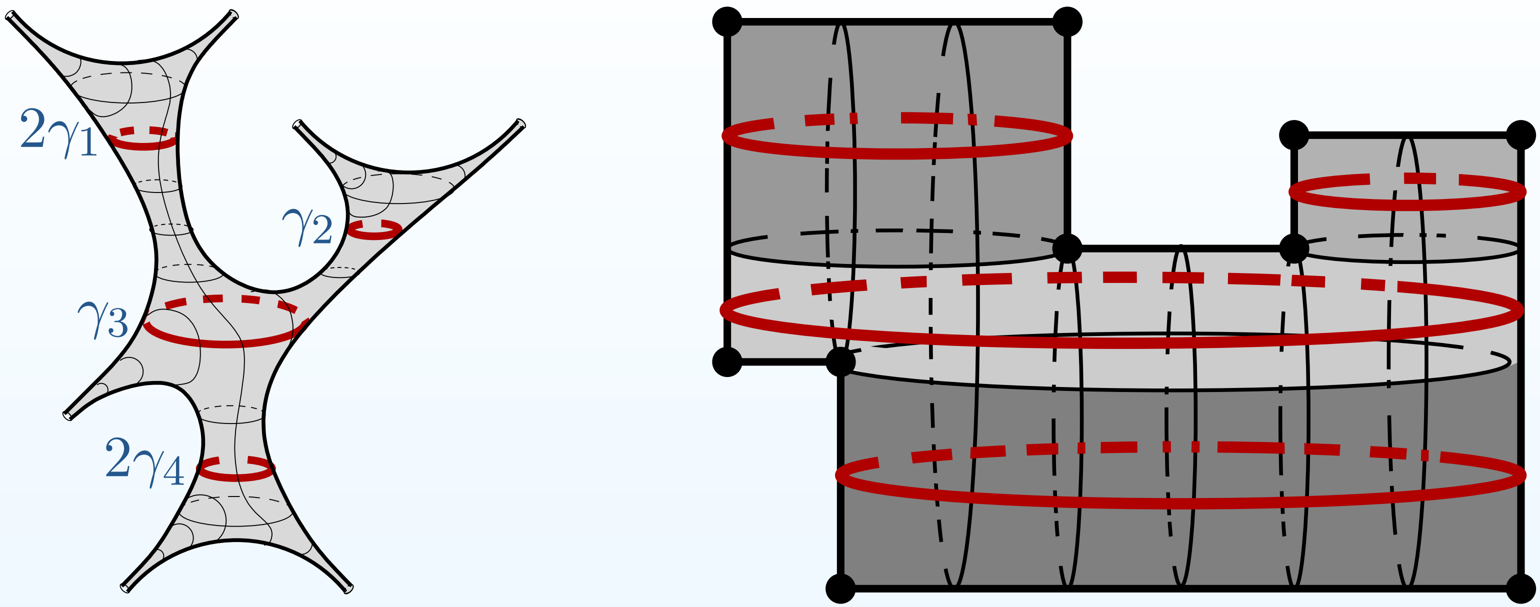
Hyperbolic and flat geodesic multicurves





Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$. Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders. Clearly, there are plenty of square-tiled surface realizing this multicurve.

Hyperbolic and flat geodesic multicurves

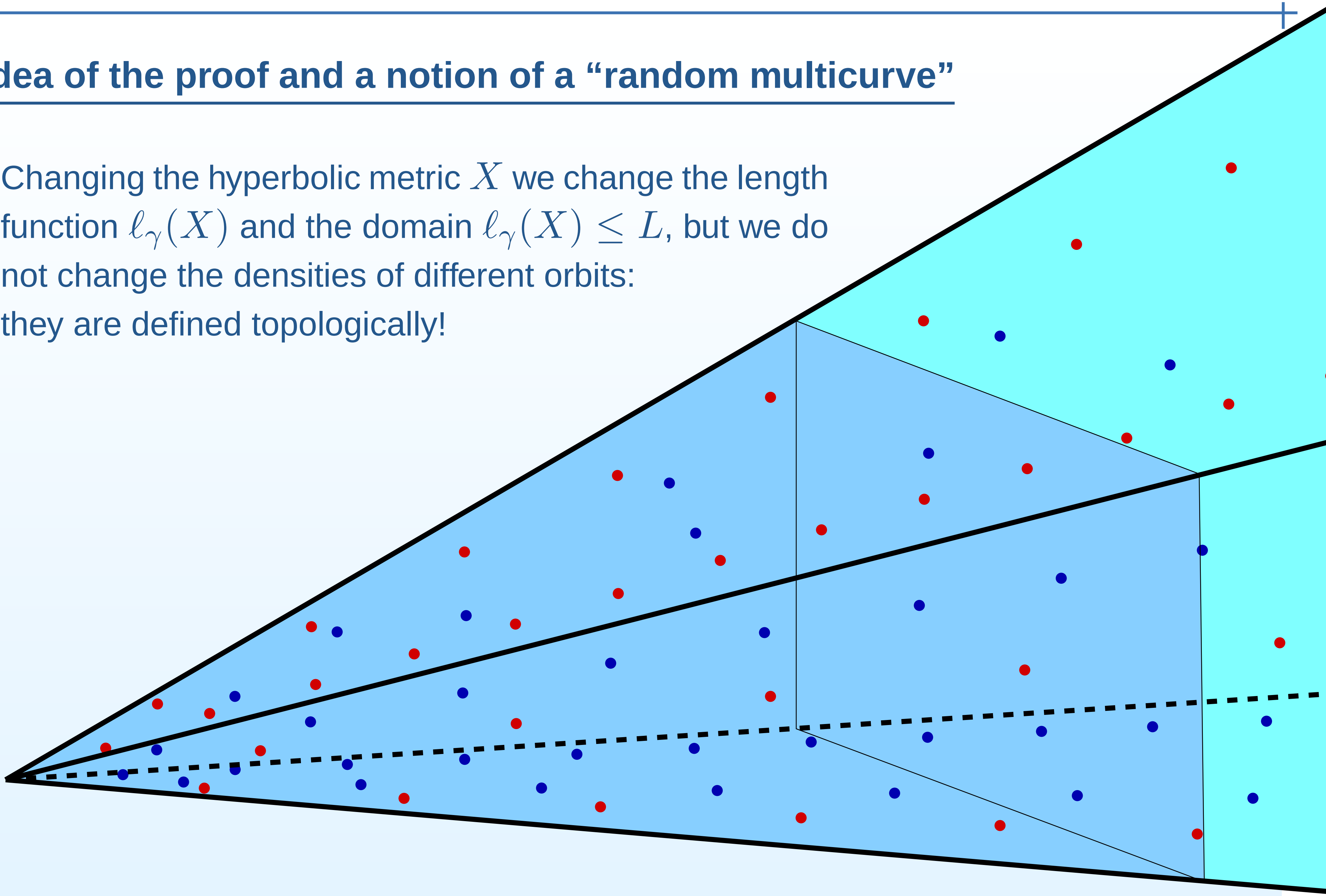


Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2018). *For any topological class γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\gamma)$ of* ***hyperbolic*** *multicurves coincides with the asymptotic frequency of simple closed* ***flat*** *geodesic multicurves of type γ represented by associated square-tiled surfaces.*

Remark. Francisco Arana Herrera recently found an alternative proof of this result. His proof uses more geometric approach.

Idea of the proof and a notion of a "random multicurve"

Changing the hyperbolic metric X we change the length function $\ell_\gamma(X)$ and the domain $\ell_\gamma(X) \leq L$, but we do not change the densities of different orbits: they are defined topologically!



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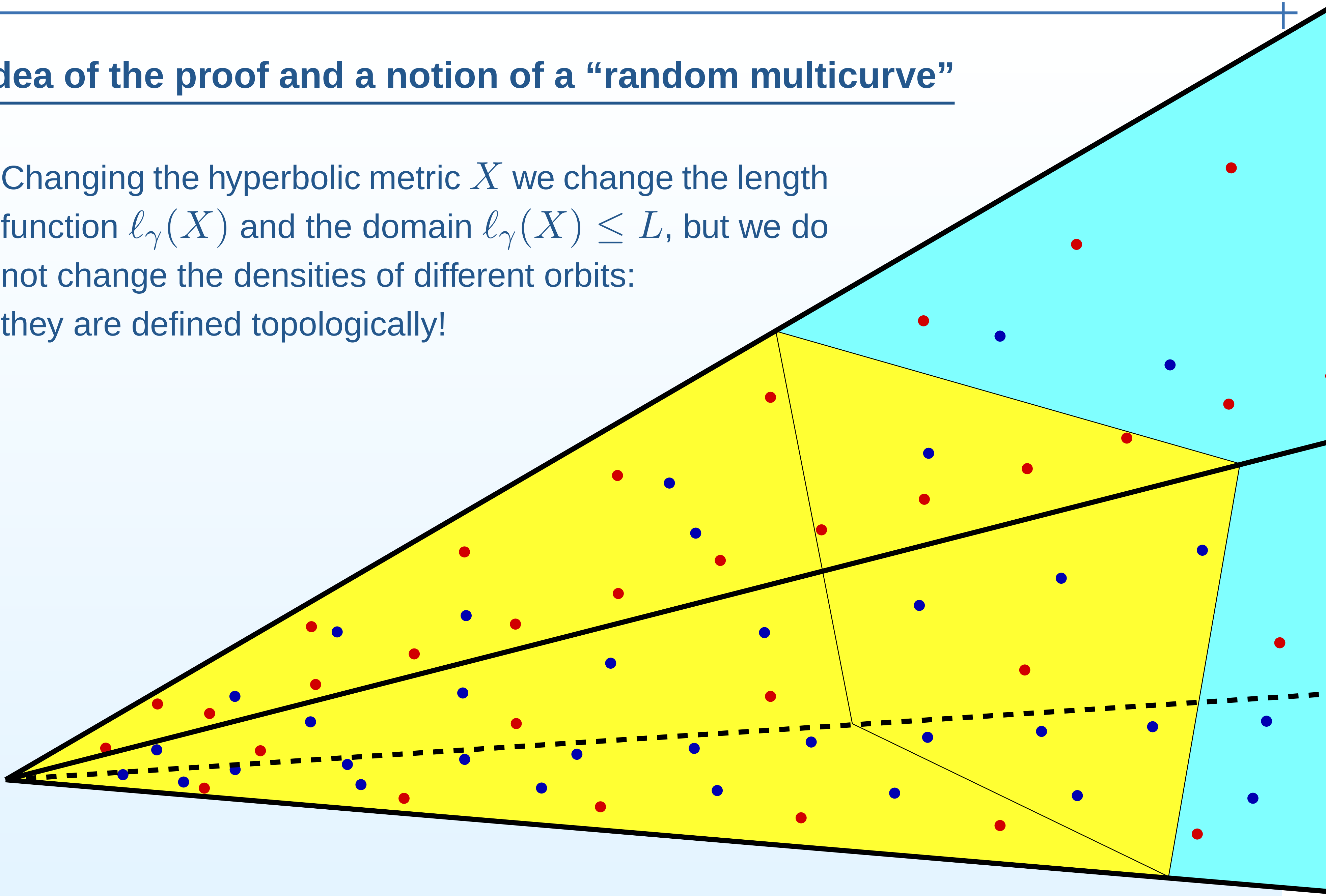

More honest idea of the proof

Recall that $s_X(L,\gamma)$ denotes the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L. Applying the definition of μ_γ to the “unit ball” B_X associated to hyperbolic metric X (instead of an abstract set B) and using proportionality of measures $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ we get

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = \lim_{L\to+\infty} \frac{\operatorname{card}\{L\cdot B_X \cap \operatorname{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}} = \mu_\gamma(B_X) = k_\gamma \cdot \mu_{\mathrm{Th}}(B_X)\,.$$

Finally, Mirzakhani computes the scaling factor k_γ as follows:

$$k_\gamma \cdot b_{g,n} = \int_{\mathcal{M}_{g,n}} k_\gamma \cdot \mu_{\mathrm{Th}}(B_X)\, dX = \int_{\mathcal{M}_{g,n}} \mu_\gamma(B_X)\, dX =$$

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so $k_\gamma = c(\gamma)/b_{g,n}$. Interchanging the integral and the limit we used the estimate of Mirzahani $\frac{s_X(L,\gamma)}{L^{6g-6+2n}} \le F(X)$, where F is integrable over $\mathcal{M}_{g,n}$.

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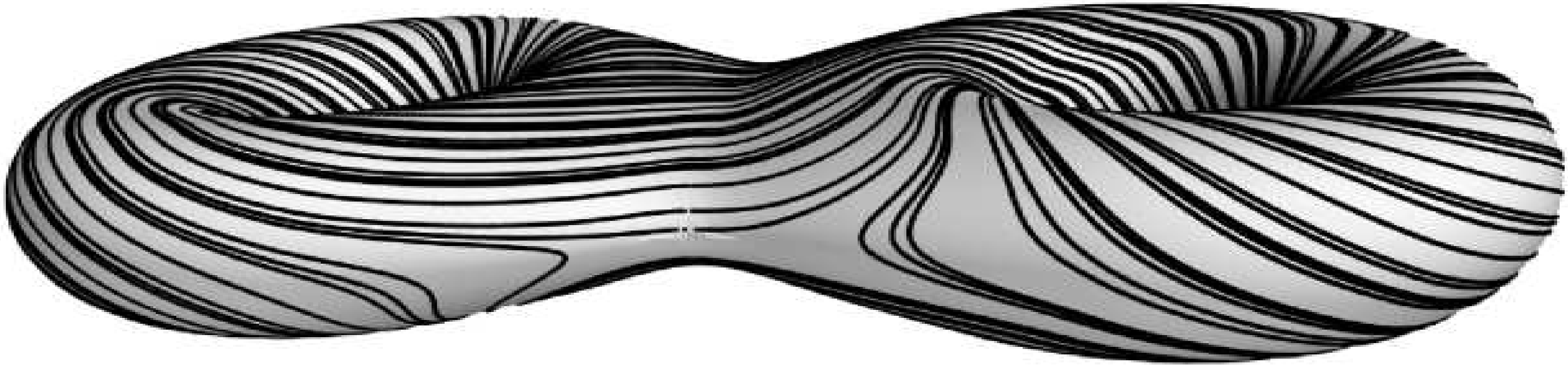
Shape of a random multicurve on a surface of genus two

What shape has a random simple closed multicurve?



Picture from a book of Danny Calegari

Questions.

● *Which simple closed geodesics are more frequent: separating or non-separating?*

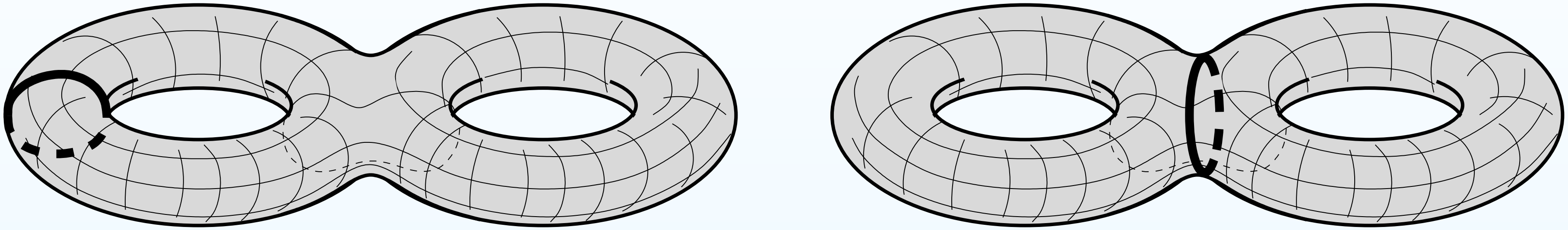
Take a random (non-primitive) multicurve $\gamma = m_1\gamma_1 + \cdots + m_k\gamma_k$. Consider the associated reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$.

● *With what probability that $\gamma_{reduced}$ slices the surface into $1, ..., 2g-2$ connected components?*

● *With what probability $\gamma_{reduced}$ has $k = 1, 2, \ldots, 3g-3$ primitive connected components $\gamma_1, \ldots, \gamma_k$?*

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (M. Mirzakhani, 2008). *Genus* $g = 2$



$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{6}$$

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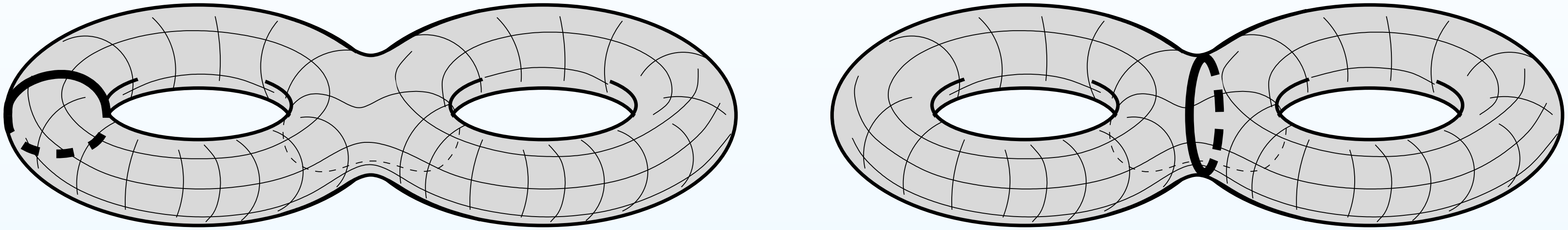

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after correction of a tiny bug in Mirzakhani's calculation.

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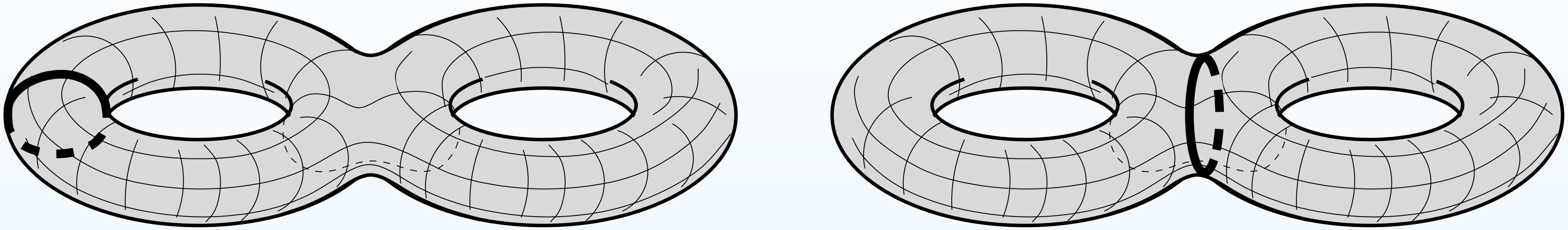
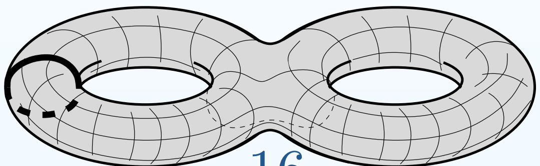


$$\lim_{L\to+\infty} \frac{\text{Number of \textbf{separating} simple closed geodesics of length at most } L}{\text{Number of \textbf{non-separating} simple closed geodesics of length at most } L} = \frac{1}{48}$$

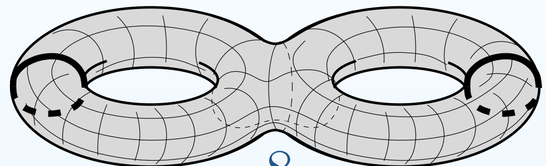
after further correction of another trickier bug in Mirzakhani's calculation. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell. Most recently it was independently confirmed by V. Erlandsson, K. Rafi, J. Souto and by A. Wright by methods independent of ours.

Multicurves on a surface of genus two and their frequencies

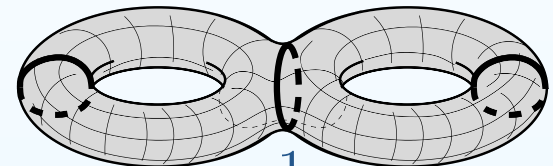
The picture below illustrates all topological types of primitive multicurves on a surface of genus two without punctures; the fractions give frequencies of non-primitive multicurves γ having a reduced multicurve $\gamma_{reduced}$ of the corresponding type.



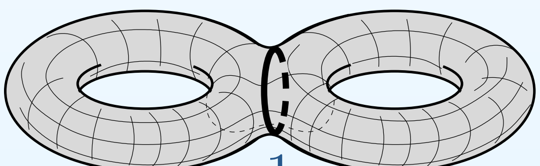
$\frac{16}{63}$



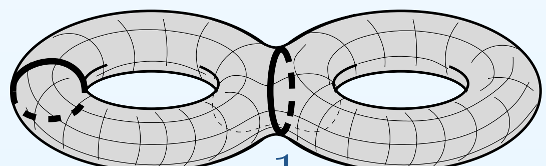
$\frac{8}{15}$



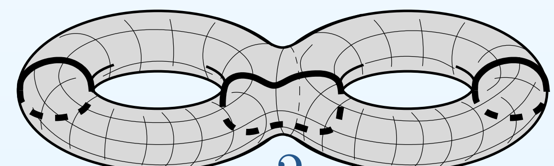
$\frac{1}{9}$



$\frac{1}{189}$



$\frac{1}{45}$



$\frac{2}{27}$

In genus 3 there are already 41 types of multicurves, in genus 4 there are 378 types, in genus 5 there are 4554 types and this number grows faster than exponentially when genus g grows. It becomes pointless to produce tables: we need to extract a reasonable sub-collection of most common types which ideally, carry all Thurston's measure when $g \to +\infty$.

Train track coordinates (after section 15.1 of the book of B. Farb and D. Margalit “A Primer on Mapping Class Groups”)

Train tracks carrying simple closed curves

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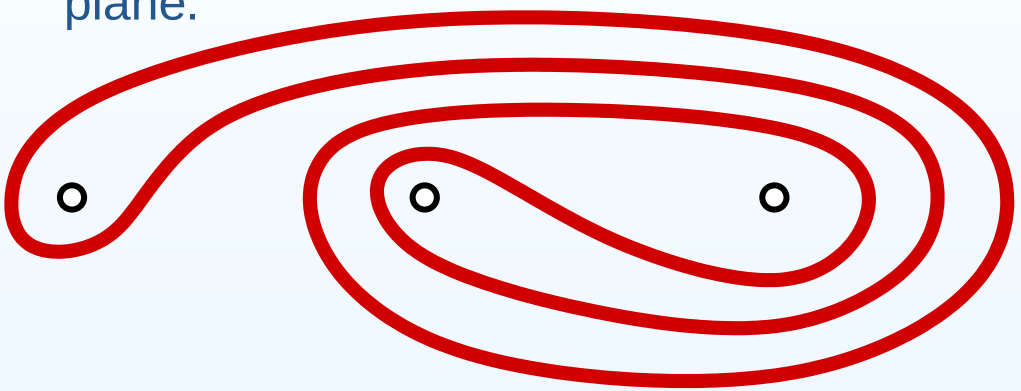

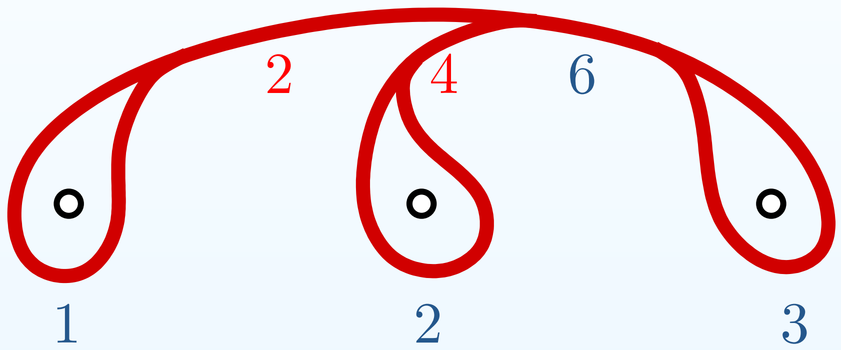

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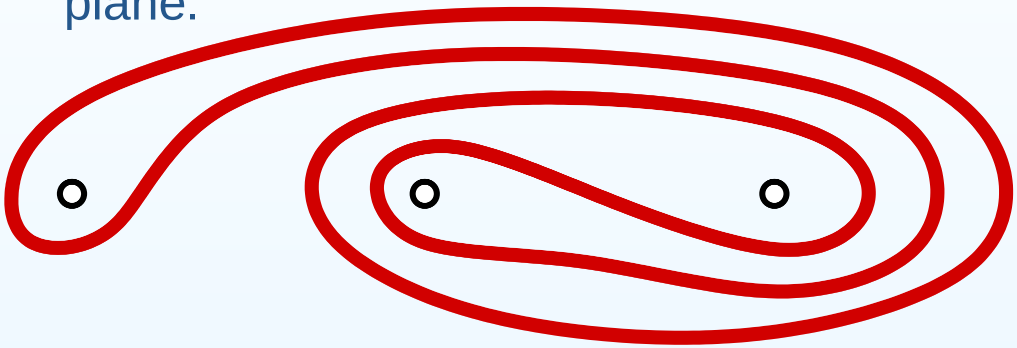

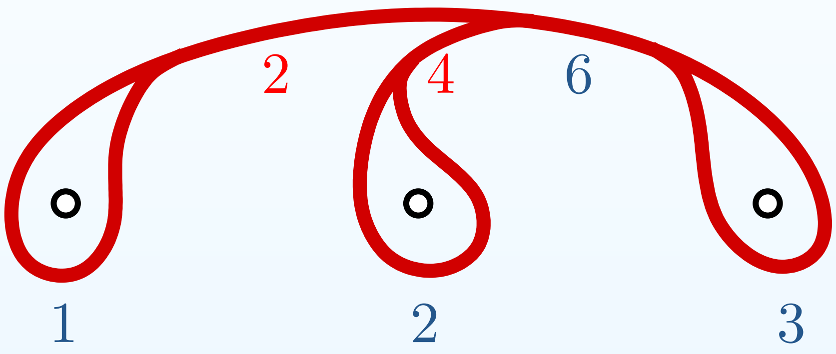

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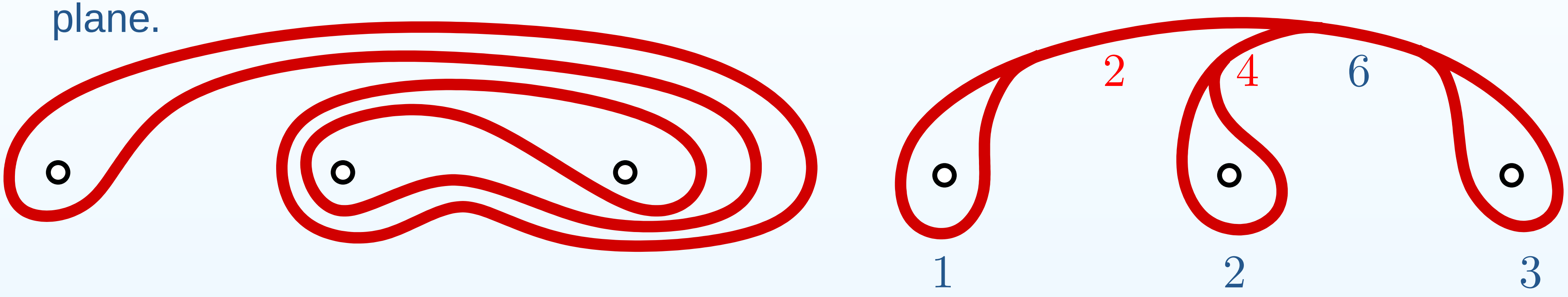

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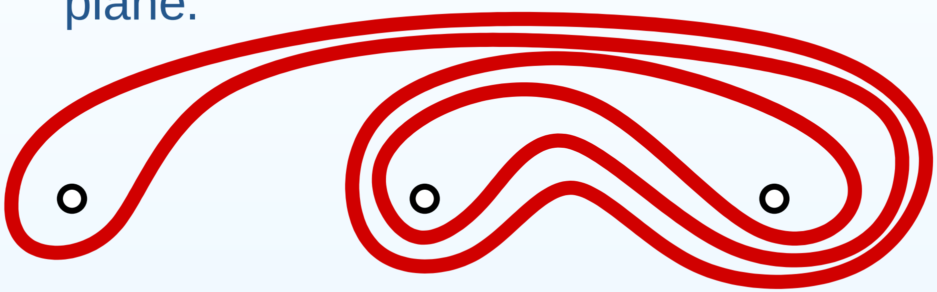

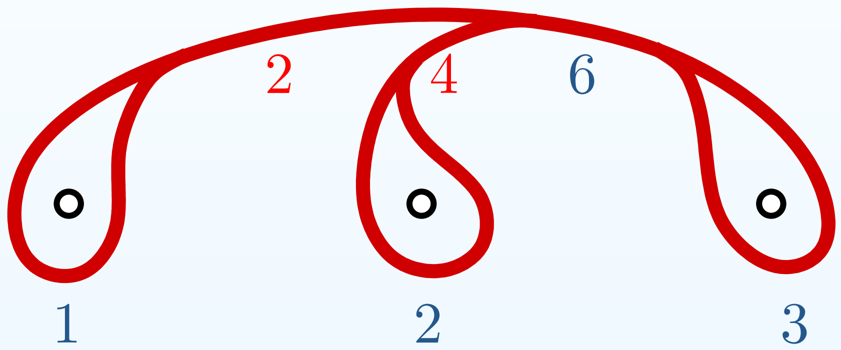

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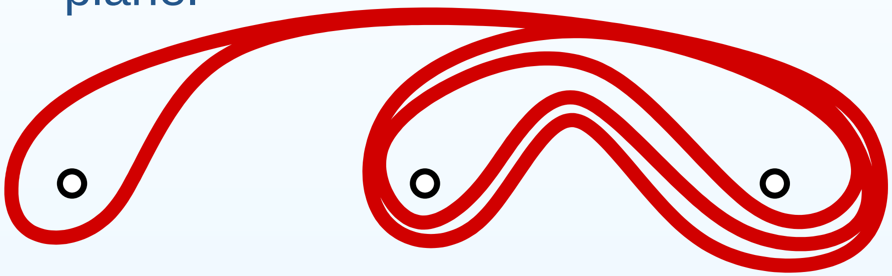

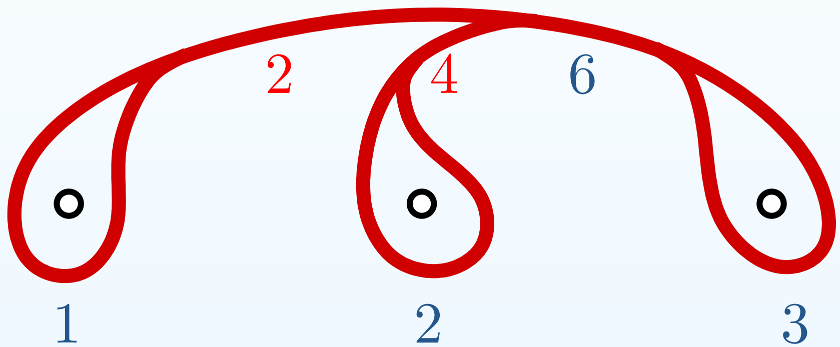

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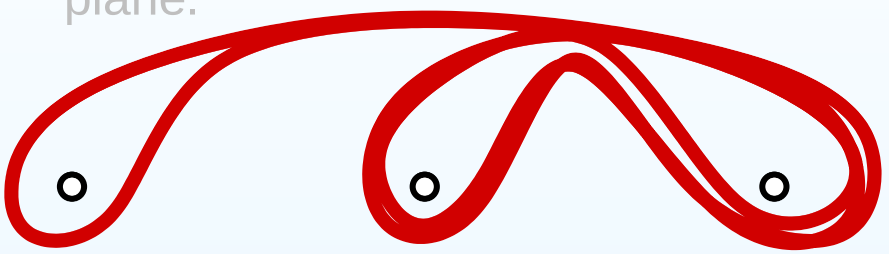

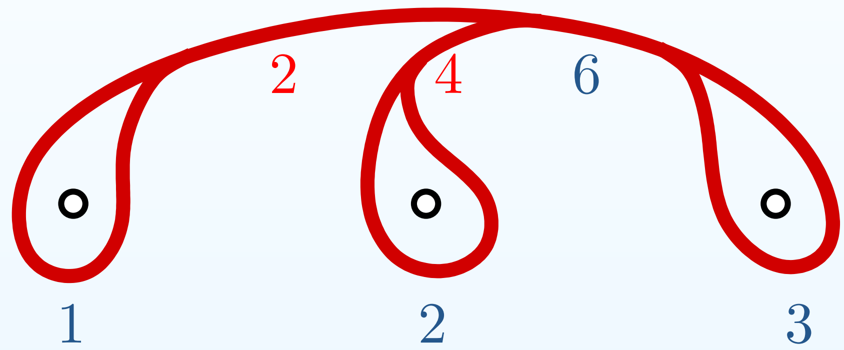

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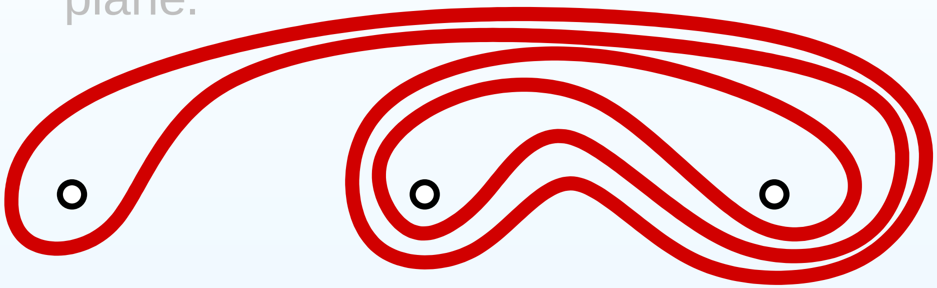

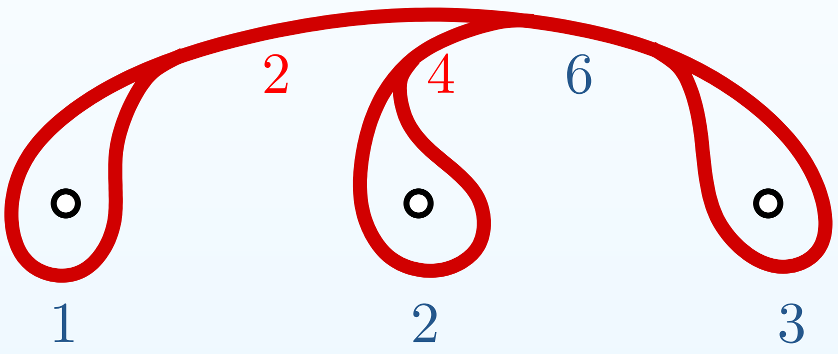

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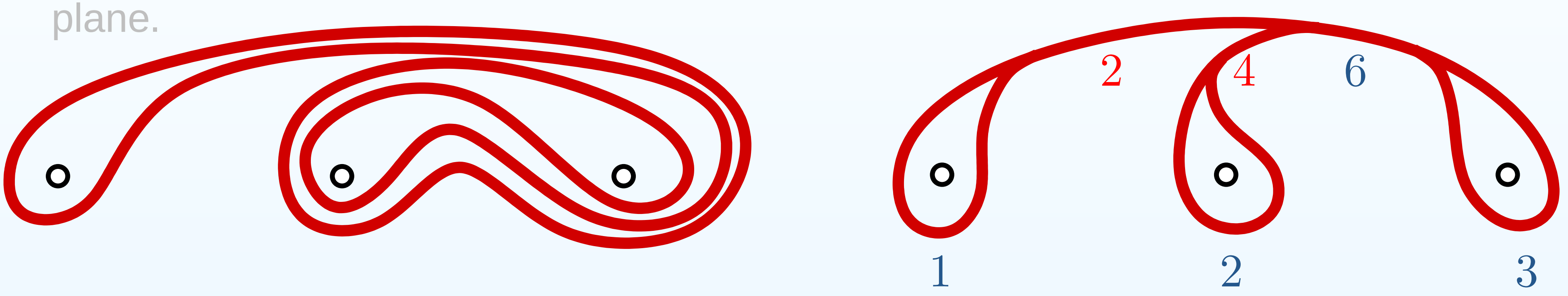

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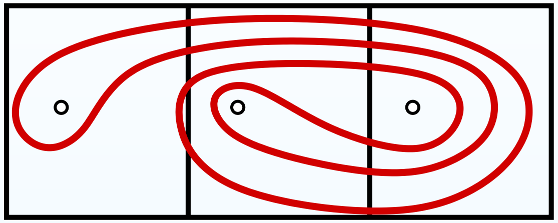
Four basic train tracks on $S_{0,4}$

Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:



By further isotopy, we eliminate bigons with the vertical edges of the three squares. Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

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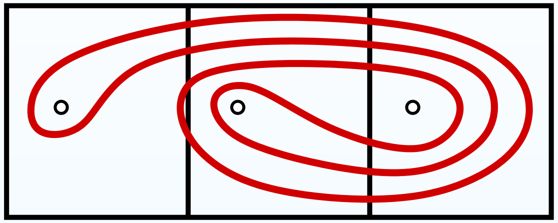

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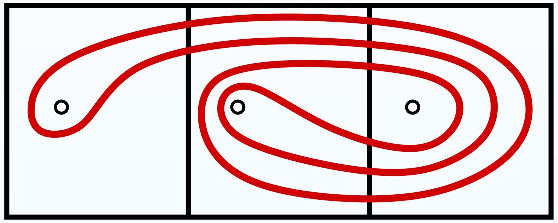

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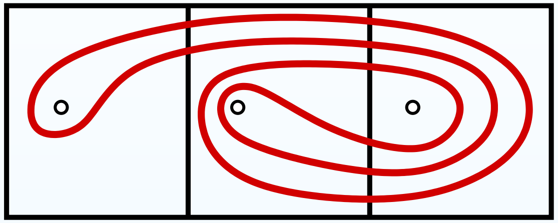

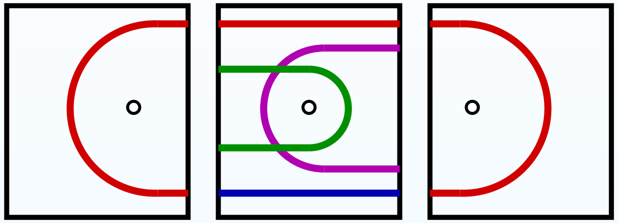

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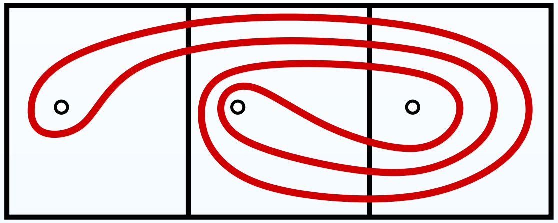

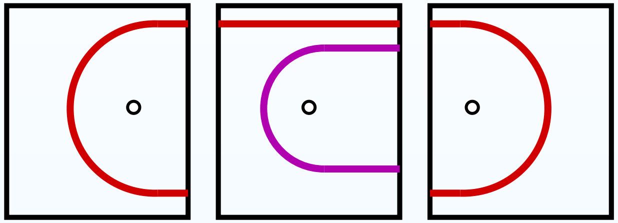

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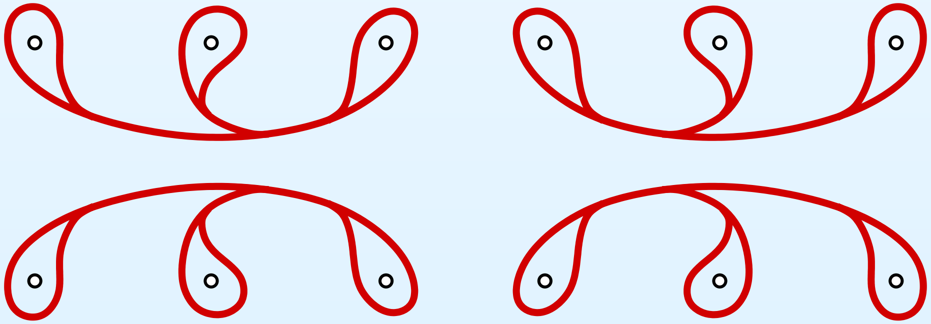
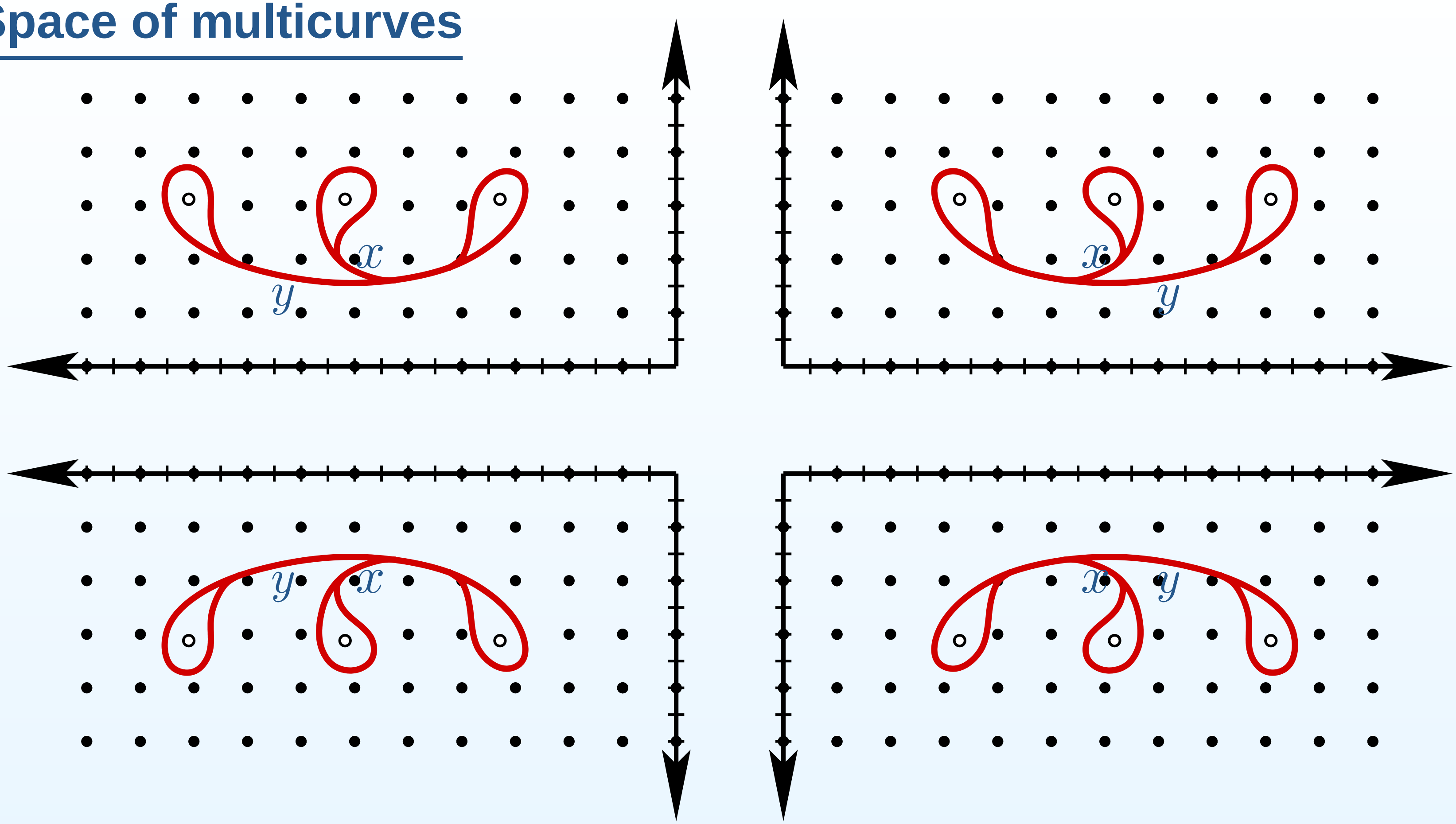
Conclusion: there are four types of simple closed curves in $S_{0,4}$, depending on which of each of the two pairs of arcs they use in the middle square. This is the same as saying that any simple closed curve in is carried by one of the following four train tracks:

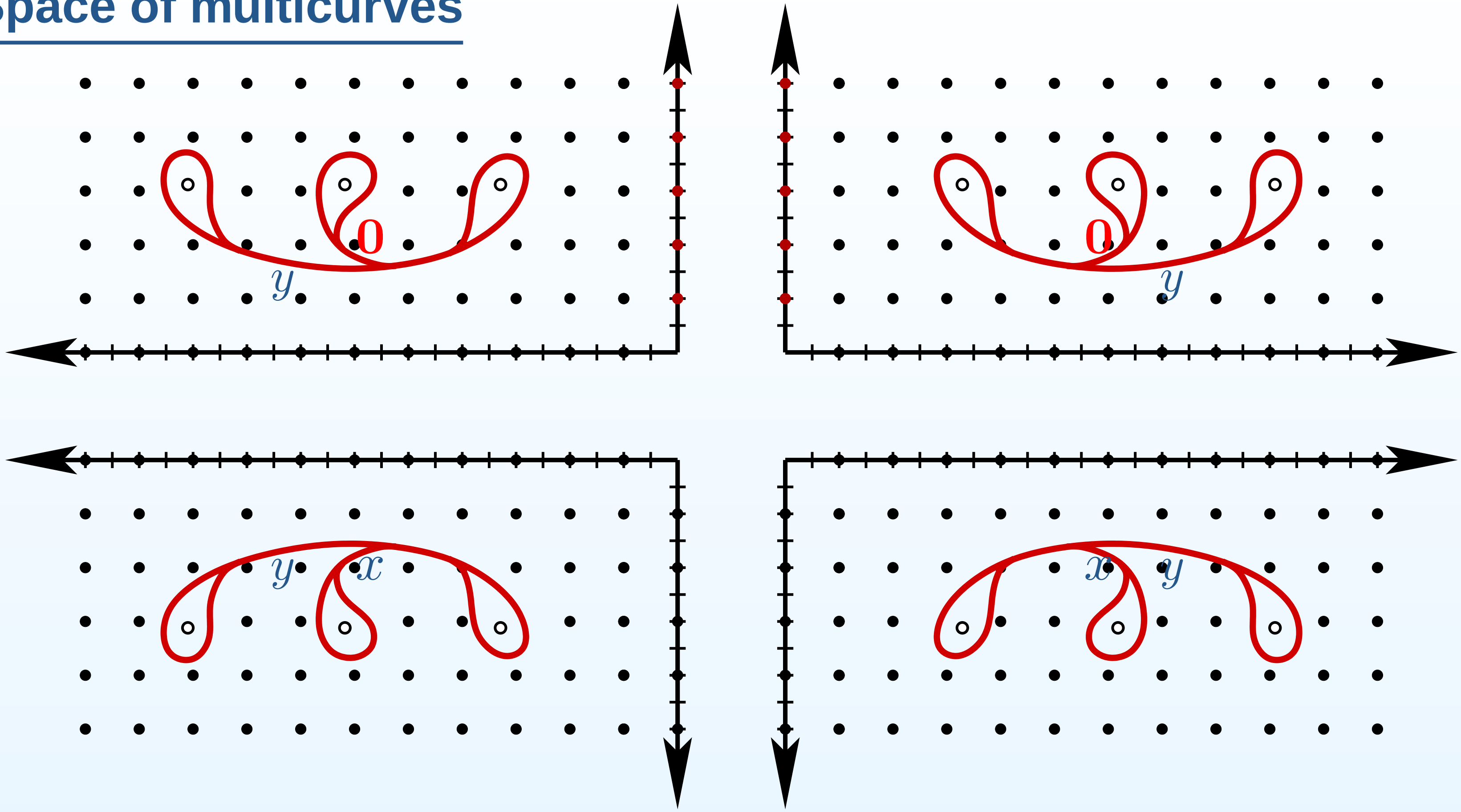


Space of multicurves



The four train tracks τ_1, τ_2, τ_3, τ_4 give four coordinate charts on the set of isotopy classes of simple closed curves in $S_{0,4}$. Each coordinate patch corresponding to a train track τ_i is given by the weights (x, y) of two chosen edges of τ_i. If we allow the coordinates x and y to be arbitrary nonnegative real numbers, then we obtain for each τ_i a closed quadrant in $\mathbb{R}^2$. Arbitrary points in this quadrant are measured train tracks.

Space of multicurves



Weight zero on an edge of a train track tells that such edge can be deleted. This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

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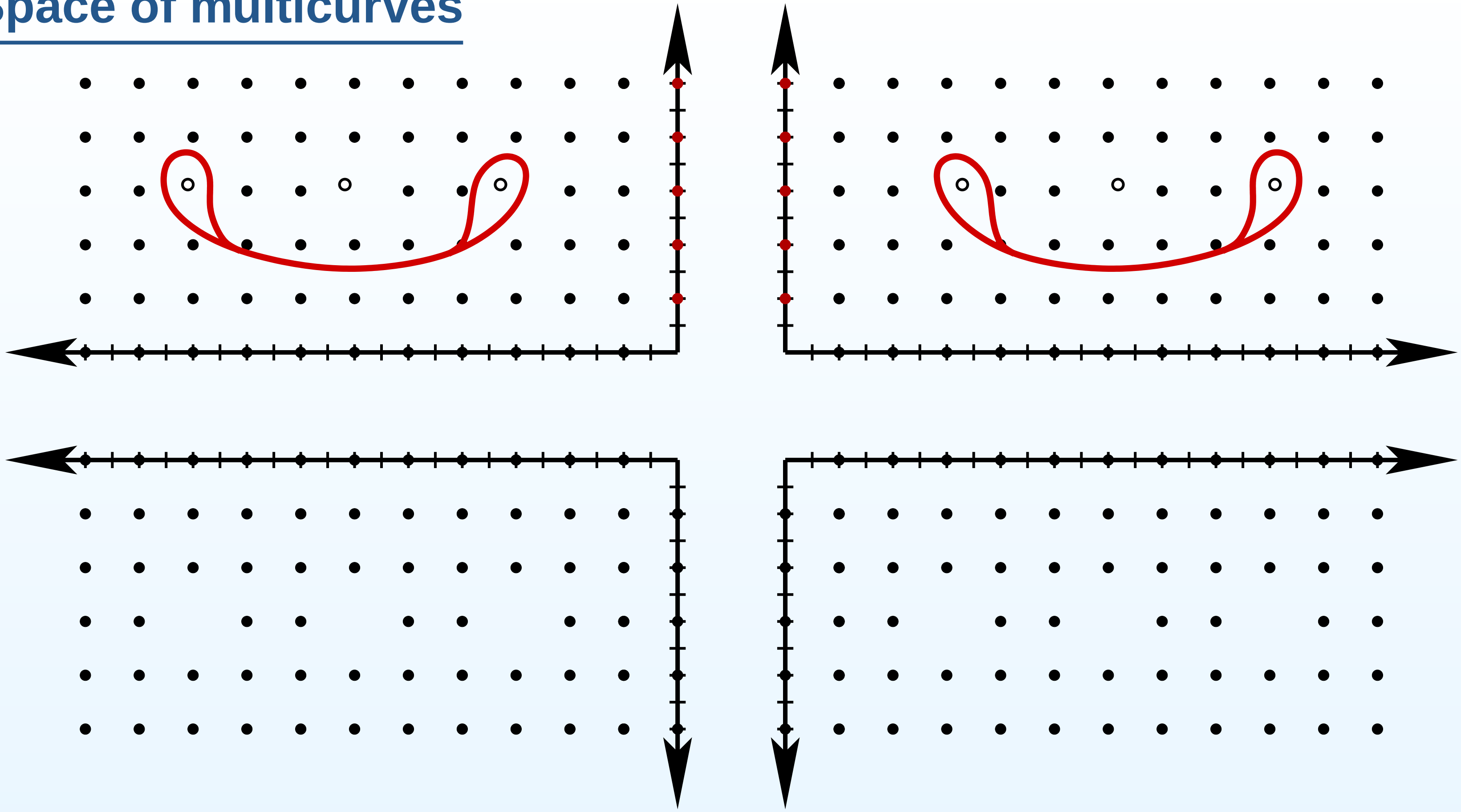

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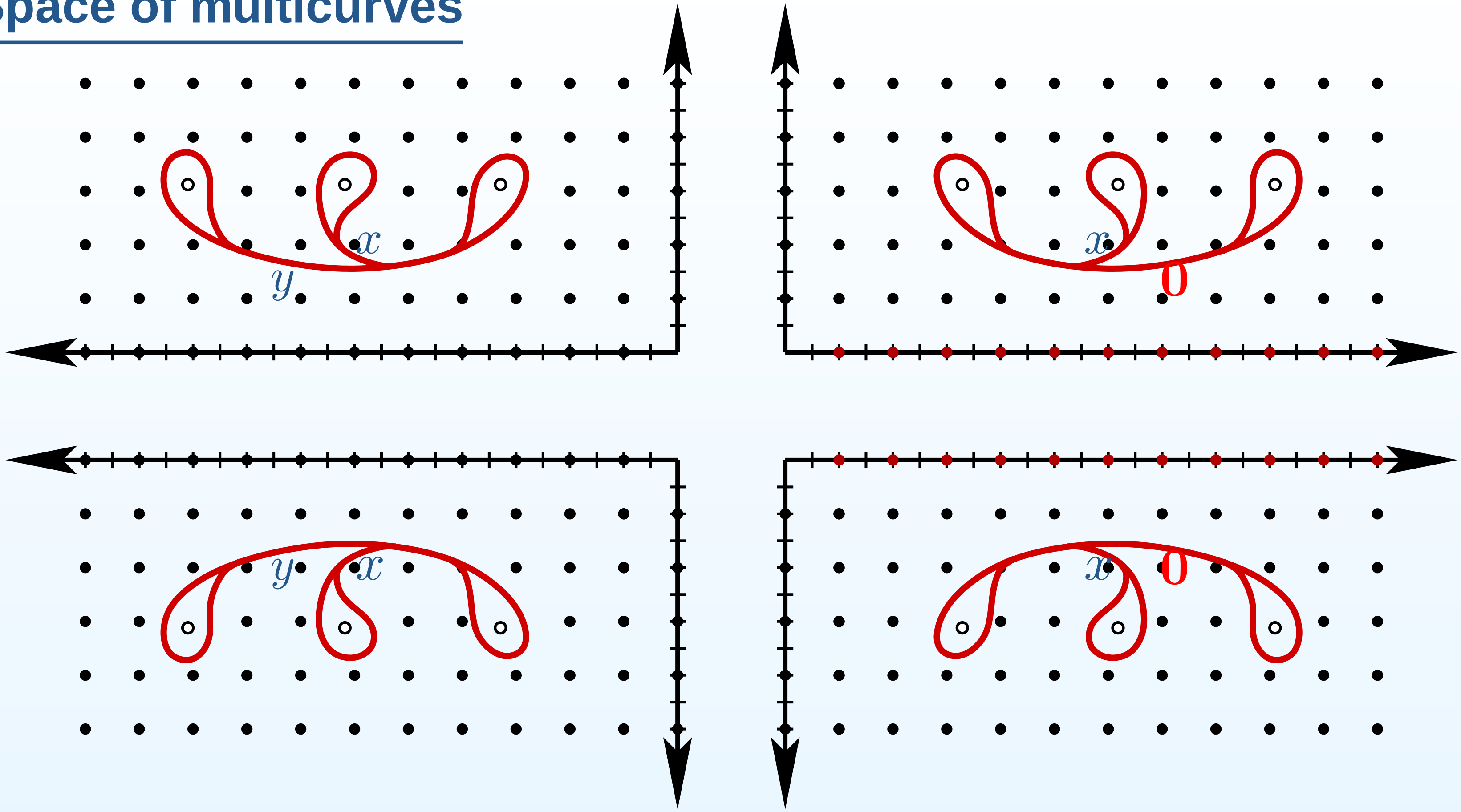

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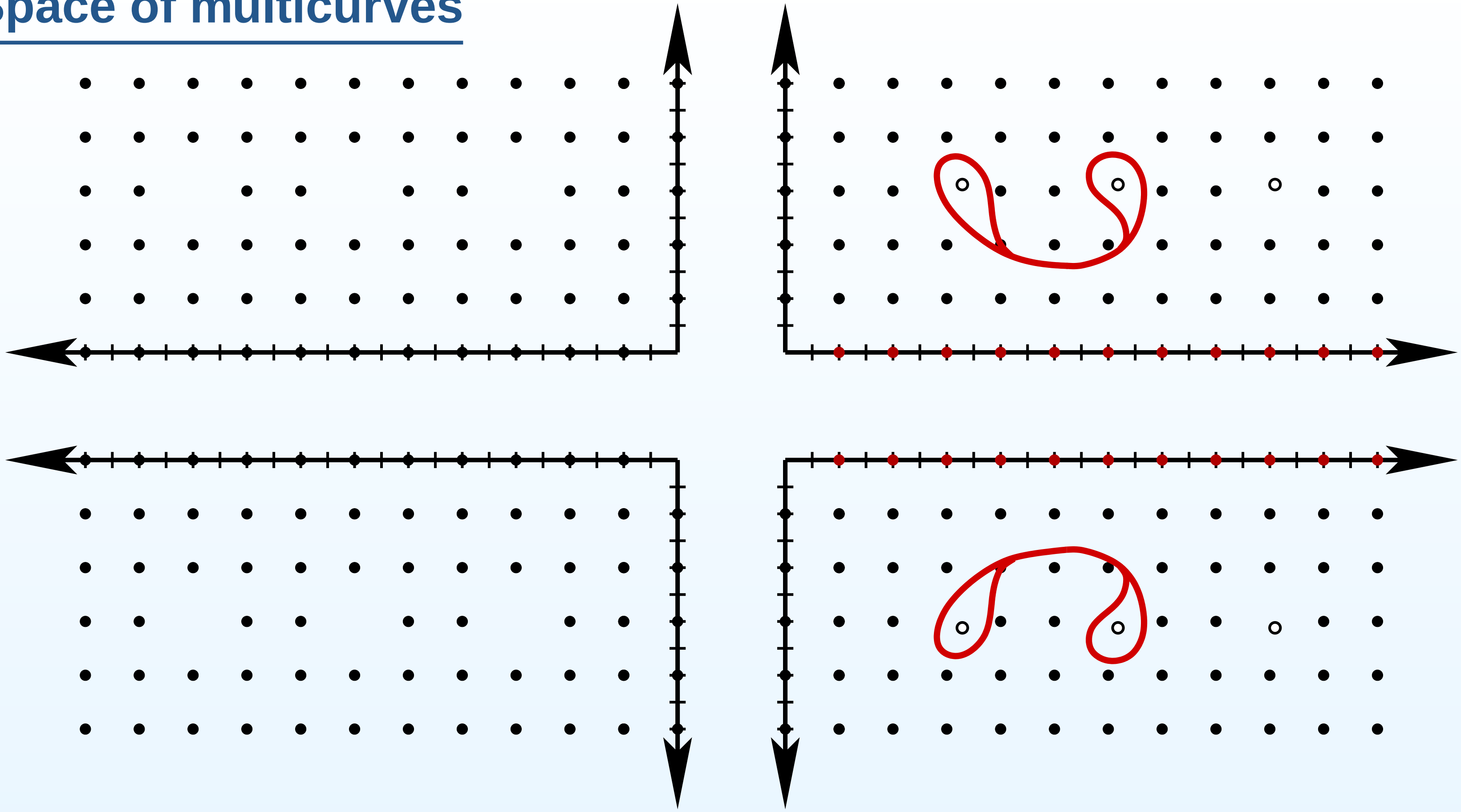

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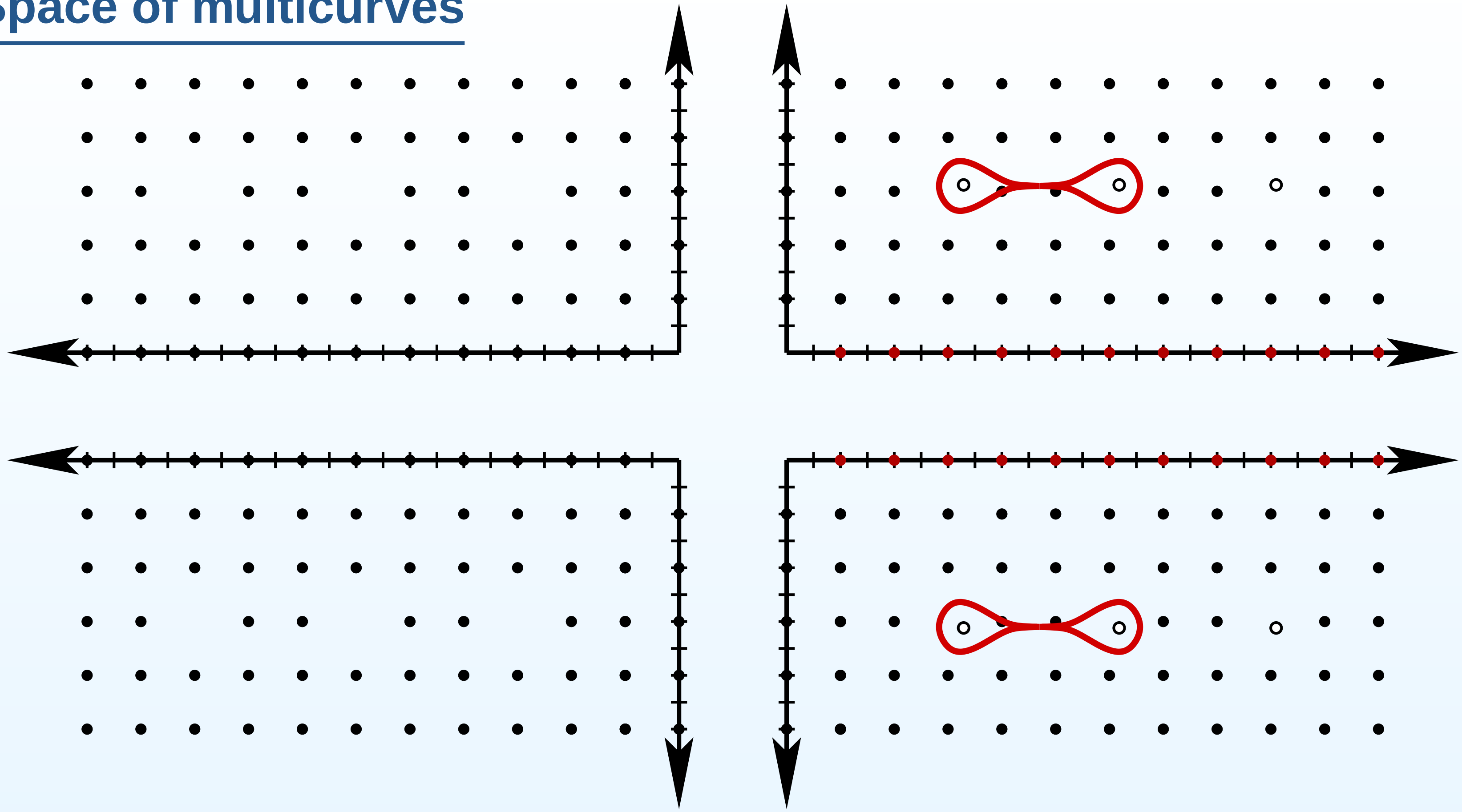

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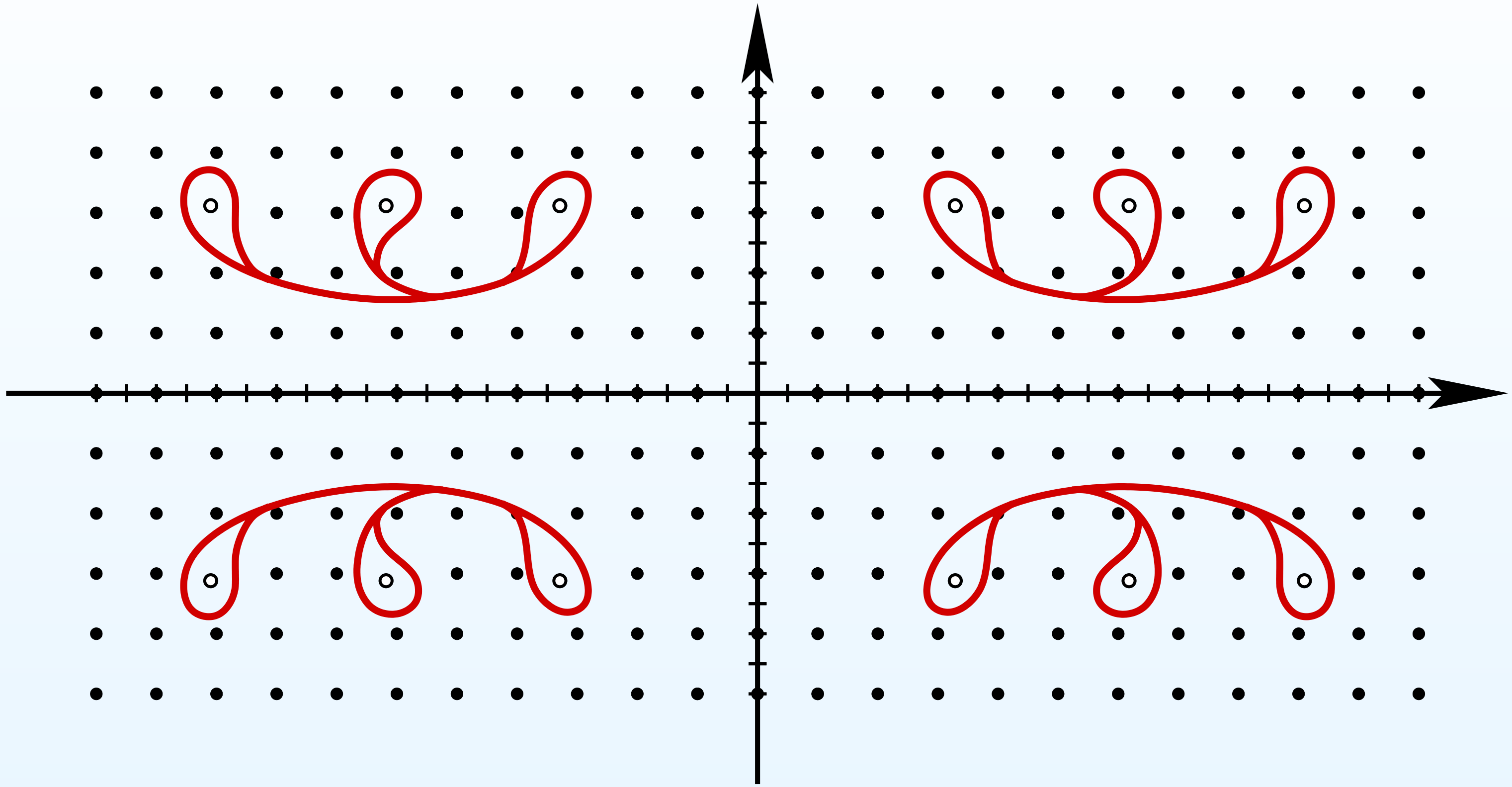

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Lecture 3. Large genus asymptotic geometry of random square-tiled surfaces and of random multicurves

Anton Zorich

(after a joint work with V. Delecroix, E. Goujard and P. Zograf);
(based on `arXiv:2007.04740`.)

School “Moduli Spaces, Combinatorics and Integrable Systems”
St. Petersburg, November 26, 2021

Reminder: count of square-tiled surfaces

Intersection numbers (Witten–Kontsevich correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \ldots, P_n \in C$ is a complex orbifold of complex dimension $3g-3+n$.
Choose index i in $\{1, \ldots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$.
The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g-3+n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \,.$$

The famous Witten's conjecture claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in 2-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; alternative proofs belong to A. Okounkov and R. Pandharipande, to M. Mirzakhani, to M. Kazarian and S. Lando (and there are more).

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

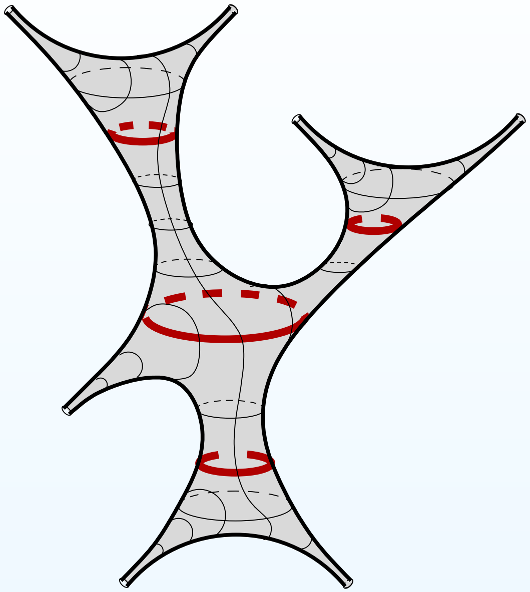
$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\, \mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

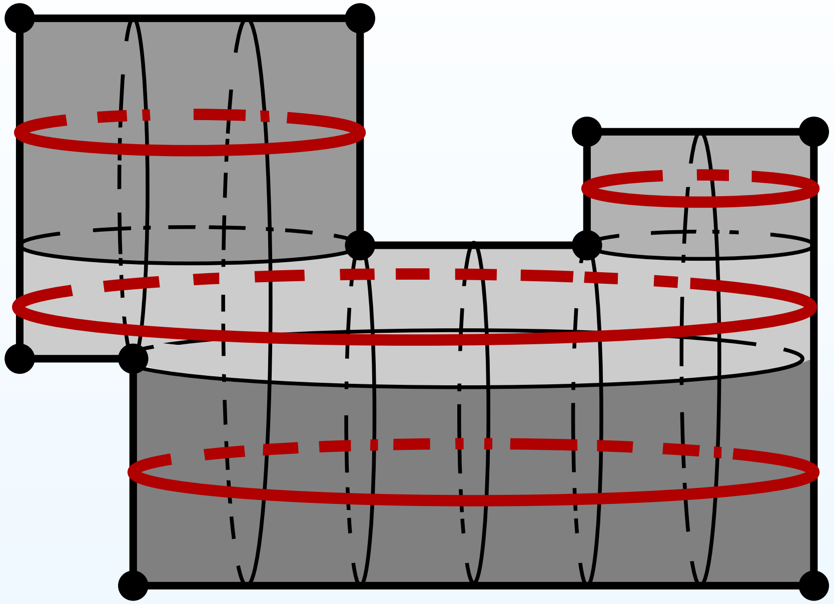
Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \;:\; \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) \,,$$

and extend it to symmetric polynomials in b_i by linearity.

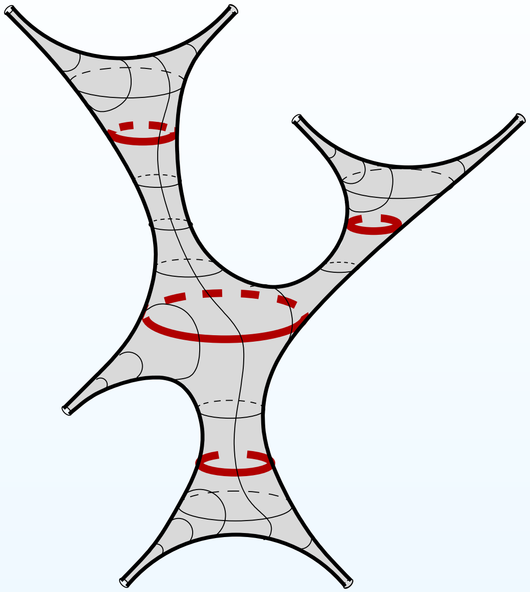
Stable graph associated to a square-tiled surface

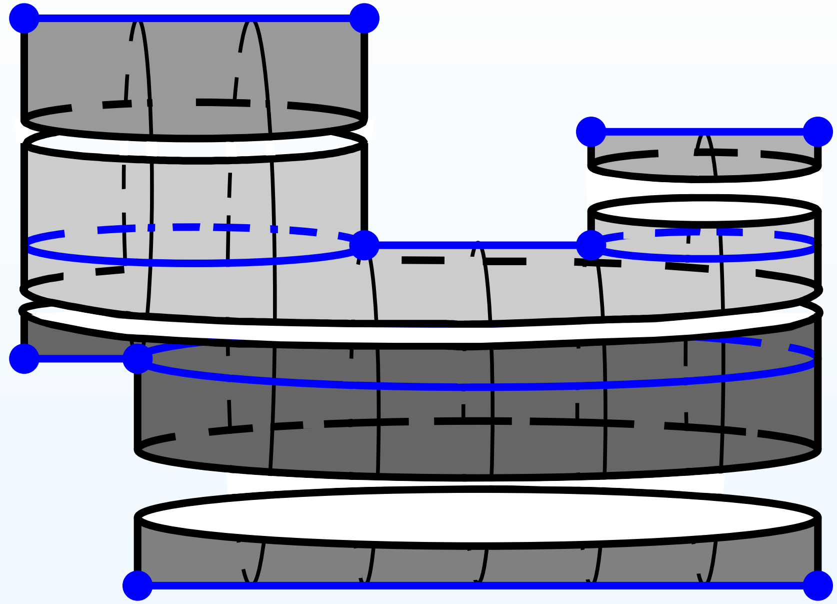


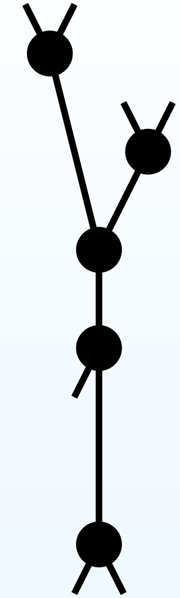


Having a square-tiled surface we associate to it a topological surface S on which we mark all “corners” with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points. We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders.

Stable graph associated to a square-tiled surface

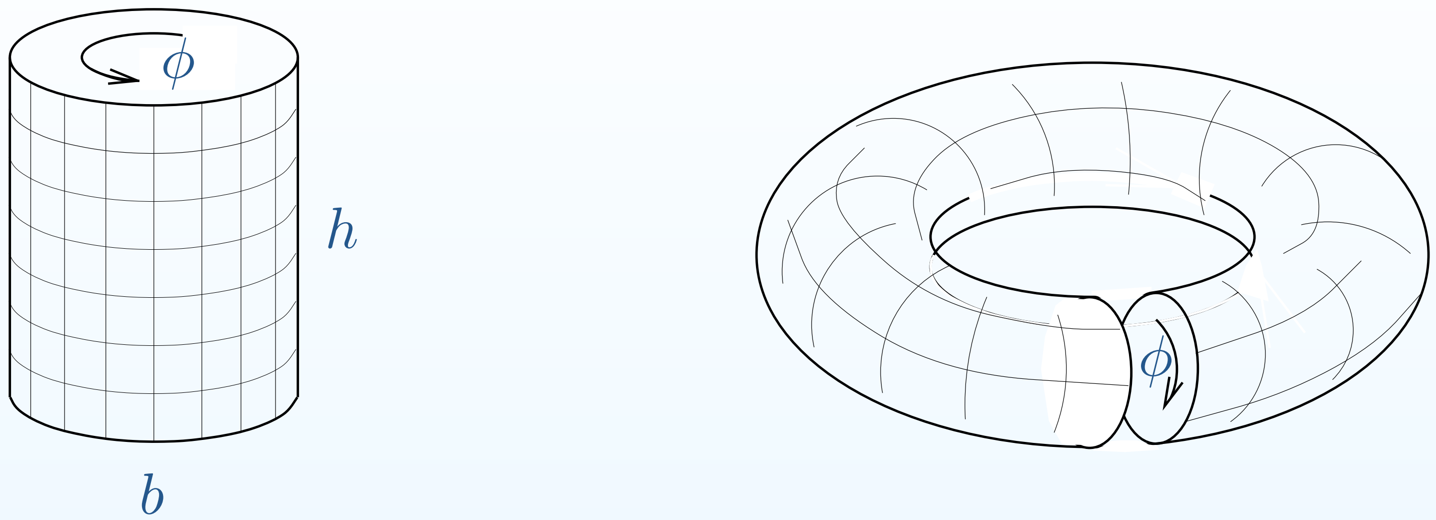






Having a square-tiled surface we associate to it a topological surface S on which we mark all “corners” with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points. We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders. The associated *stable graph* Γ is the dual graph to the multicurve. The vertices of Γ are in the natural bijection with metric ribbon graphs given by components of $S \setminus \gamma$. The edges are in the bijection with the waist curves γ_i of the cylinders. The marked points are encoded by “legs” — half-edges of the dual graph.

Number of square-tiled tori



The number of square-tiled tori tiled with at most N squares has asymptotics

$$\sum_{\substack{b,h\in\mathbb{N}\\ b\cdot h\le N}} b = \sum_{\substack{b,h\in\mathbb{N}\\ b\le \frac{N}{h}}} b \sim \sum_{h\in\mathbb{N}} \frac{1}{2}\cdot\left(\frac{N}{h}\right)^2 = \frac{N^2}{2}\sum_{h\in\mathbb{N}}\frac{1}{h^2} = \frac{N^2}{2}\cdot\frac{\pi^2}{6} = \frac{N^2}{2}\zeta(2) =$$

$$= \frac{N^2}{2}\mathcal{Z}(b)\,, \quad \text{where} \quad \mathcal{Z} : \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n}\big(m_i!\cdot\zeta(m_i+1)\big)\,.$$

Volume of $\mathcal{Q}_{g,n}$

Theorem (Delecroix, Goujard, Zograf, Zorich). *The Masur–Veech volume* $\operatorname{Vol}\mathcal{Q}_{g,n}$ *of the moduli space of meromorphic quadratic differentials with* n *simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot \sum_{\substack{\text{Weighted graphs }\Gamma\\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of }\Gamma-1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices of }\Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right)\,,$$

The partial sum for fixed number k *of edges gives the contribution of* k*-cylinder square-tiled surfaces.*

Shape of a random multicurve on a surface of large genus. Shape of a random square-tiled surface of large genus.

Statistics of prime decompositions: random integer numbers

The Prime Number Theorem states that an integer number n taken randomly in a large interval $[1, N]$ is prime with asymptotic probability $\frac{\log N}{N}$.

Actually, one can tell much more about prime decomposition of a large random integer. Denote by $\omega(n)$ the number of prime divisors of an integer n counted without multiplicities. In other words, if n has prime decomposition $n = p_1^{m_1} \dots p_k^{m_k}$, let $\omega(n) = k$. By the Erdős–Kac theorem, the centered and rescaled distribution prescribed by the counting function $\omega(n)$ tends to the normal distribution:

Erdős–Kac Theorem (1939)

$$\lim_{N\to+\infty} \frac{1}{N} \operatorname{card}\left\{n \le N \,\middle|\, \frac{\omega(n) - \log\log N}{\sqrt{\log\log N}} \le x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt\,.$$

The subsequent results of of A. Selberg (1954) and of A. Rényi and P. Turán (1958) describe the rate of convergence.

Statistics of prime decompositions: random permutations

Denote by $\mathrm{K}_n(\sigma)$ the number of disjoint cycles in the cycle decomposition of a permutation σ in the symmetric group S_n. Consider the uniform probability measure on S_n. A random permutation σ of n elements has exactly k cycles in its cyclic decomposition with probability $\mathbb{P}\big(\mathrm{K}_n(\sigma) = k\big) = \frac{s(n,k)}{n!}$, where $s(n,k)$ is the unsigned Stirling number of the first kind. It is immediate to see that $\mathbb{P}\big(\mathrm{K}_n(\sigma) = 1\big) = \frac{1}{n}$. V. L. Goncharov computed the expected value and the variance of K_n as $n \to +\infty$:

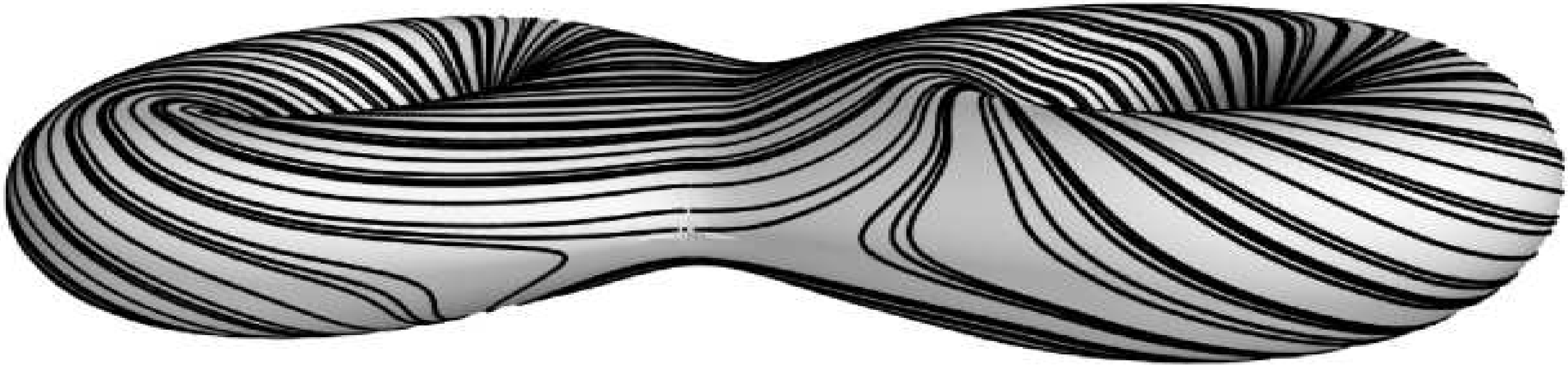
$$\mathbb{E}(\mathrm{K}_n) = \log n + \gamma + o(1)\,, \qquad \mathbb{V}(\mathrm{K}_n) = \log n + \gamma - \zeta(2) + o(1)\,,$$

and proved the following central limit theorem:

Theorem (V. L. Goncharov, 1944)

$$\lim_{n\to+\infty} \frac{1}{n!}\,\mathrm{card}\left\{\sigma \in S_n \,\middle|\, \frac{\mathrm{K}_n(\sigma) - \log n}{\sqrt{\log n}} \leq x\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt\,.$$

What shape has a random simple closed multicurve?



Picture from a book of Danny Calegari

Questions.

• *Which simple closed geodesics are more frequent: separating or non-separating?*

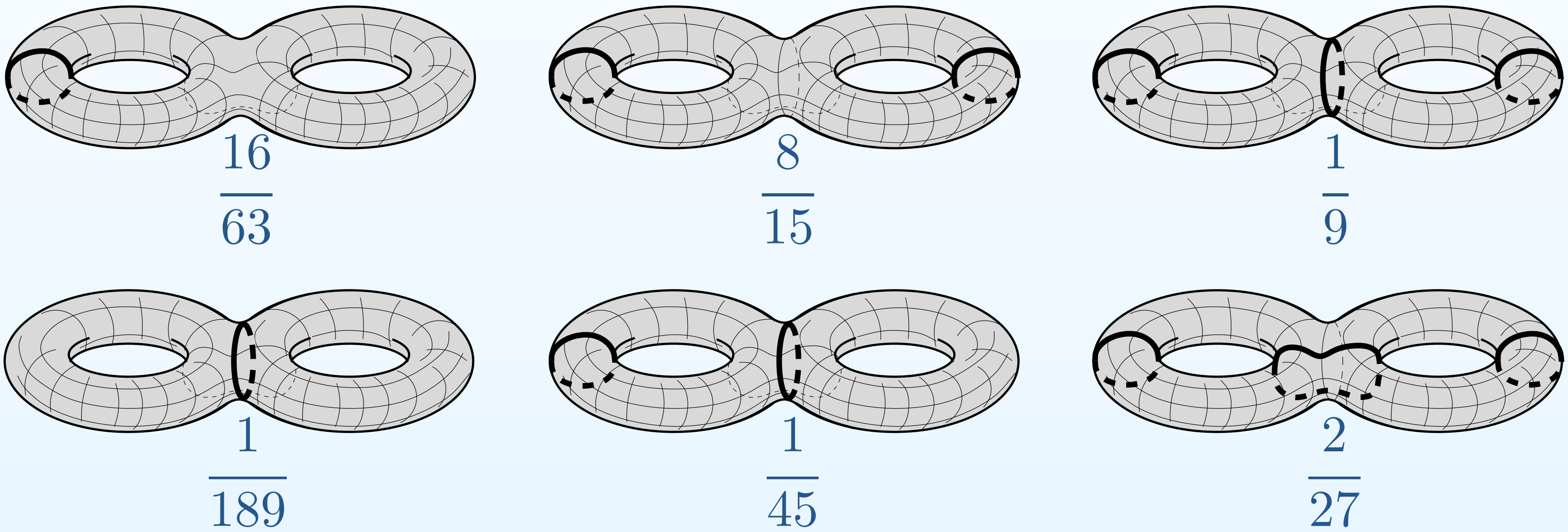
Take a random (non-primitive) multicurve $\gamma = m_1\gamma_1 + \cdots + m_k\gamma_k$. Consider the associated reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$.

• *With what probability that $\gamma_{reduced}$ slices the surface into $1, ..., 2g-2$ connected components?*

• *With what probability $\gamma_{reduced}$ has $k = 1, 2, \ldots, 3g-3$ primitive connected components $\gamma_1, \ldots, \gamma_k$?*

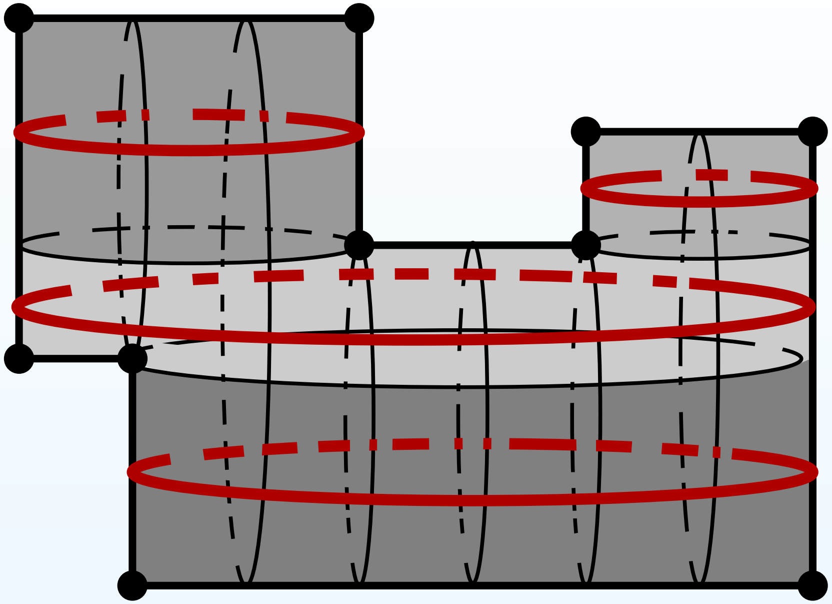
Multicurves on a surface of genus two and their frequencies

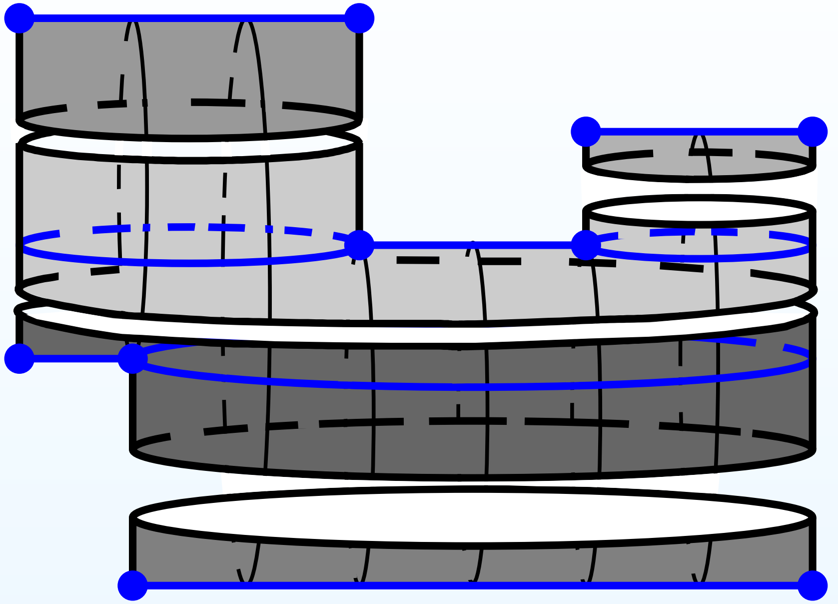
The picture below illustrates all topological types of primitive multicurves on a surface of genus two without punctures; the fractions give frequencies of non-primitive multicurves γ having a reduced multicurve $\gamma_{reduced}$ of the corresponding type.



In genus 3 there are already 41 types of multicurves, in genus 4 there are 378 types, in genus 5 there are 4554 types and this number grows faster than exponentially when genus g grows. It becomes pointless to produce tables: we need to extract a reasonable sub-collection of most common types which ideally, carry all Thurston's measure when $g \to +\infty$.

Shape of a random square-tiled surface of large genus?





Questions.

- *With what probability a random square-tiled surface* S *of genus* g *has* $1, 2, 3, \ldots$ *singular horizontal leaves (in blue on the right picture)?*
- *With what probability a random square-tiled surface* S *of genus* g *has* $K_g(S) = 1, 2, 3, \ldots, 3g-3$ *maximal horizontal cylinders (represented by red waist curves on the left picture)?*
- *What are the typical heights* $h_1, \ldots, h_k$ *of the cylinders?*
- *What is the shape of a random square-tiled surface of large genus?*

Random multicurves and random square-tiled surfaces

Denote by $K_g(\gamma)$ the number of components k of the multicurve $\gamma = \sum_{i=1}^{k} m_i \gamma_i$ on a surface of genus g counted without multiplicities.

Denote by $K_g(S)$ the number of maximal horizontal cylinders in the cylinder decomposition of a square-tiled surface S of genus g. We will always consider square-tiled surfaces without cone-angles π, i.e. the ones corresponding to holomorphic quadratic differentials.

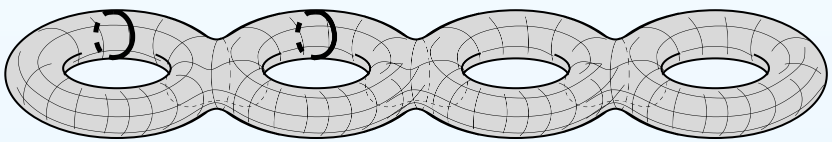
Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *For any genus $g \geq 2$ and for any $k \in \mathbb{N}$, the probability $p_g(k)$ that a random multicurve γ on a surface of genus g has exactly k components counted without multiplicities coincides with the probability that a random square-tiled surface S of genus g has exactly k maximal horizontal cylinders:*

$$p_g(k) = \mathbb{P}\big(K_g(\gamma) = k\big) = \mathbb{P}\big(K_g(S) = k\big)\,.$$

In other words, $K_g(\gamma)$ and $K_g(S)$, considered as random variables, determine the same probability distribution $p_g(k)$, where $k = 1, 2, \ldots, 3g-3$.

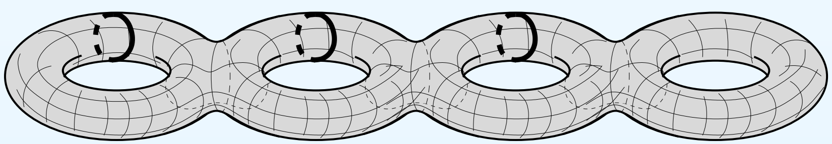
Shape of a random multicurve (random square-tiled surface) on a surface of large genus in simple words

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *The reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$ associated to a random integral multicurve $m_1\gamma_1 + \ldots m_k\gamma_k$ separates the surface with probability which tends to zero as genus g grows. For large g, $\gamma_{reduced}$ has about $(\log g)/2$ components and has one of the following topological types*



$0.09 \log(g)$ components

$\ldots$ $\qquad$ $\ldots$ $\quad$ $\ldots$ $\quad$ $\ldots$



$0.62 \log(g)$ components

$$\mathbb{P}\Big(0.09 \log g < K_g(\gamma) < 0.62 \log g\Big) = 1 - O\left((\log g)^{24} g^{-1/4}\right) .$$

A random square-tiled surface (without conical points of angle π) of large genus has about $\frac{\log(g)}{2}$ cylinders, and all its conical points sit at the same level.

Weights of a random multicurve (heights of cylinders of a ransom square-tiled surface)

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *A random integer multicurve $m_1\gamma_1 + \cdots + m_k\gamma_k$ with bounded number k of primitive components is reduced (i.e., $m_1 = \cdots = m_k = 1$) with probability which tends to 1 as $g \to +\infty$. In other terms, if we consider a random square-tiled surface with at most K cylinders, the heights of all cylinders would be very likely equal to 1 for $g \gg 1$.*

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *A general random integer multicurve $m_1\gamma_1 + \cdots + m_k\gamma_k$ is reduced (i.e., $m_1 = \cdots = m_k = 1$) with probability which tends to $\frac{\sqrt{2}}{2}$ as genus grows. More generally, all weights $m_1, \dots m_k$ of a random multicurve are bounded from above by an integer m with probability which tends to $\sqrt{\frac{m}{m+1}}$ as $g \to +\infty$.*

In other words, for more 70% of square-tiled surfaces of large genus, the heights of all cylinders are equal to 1.

However, the mean value of $m_1 + \ldots + m_k$ is infinite in any genus g.

Main Theorem (informally)

Main Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *As g grows, the probability distribution p_g rapidly becomes, basically, indistinguishable from the distribution of the number of cycles in a (very explicitly nonuniform) random permutation. In particular, for any $k \in \mathbb{N}$ the difference of the k-th moments of the two distributions is of the order $O(g^{-1})$.*

Actually, we have an explicit asymptotic formula for all cumulants. For example

$$\mathbb{E}(K_g) = \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 + o(1)\,,$$
$$\mathbb{V}(K_g) = \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 - \frac{3}{4}\zeta(2) + o(1)\,,$$

where $\gamma = 0.5772\ldots$ denotes the Euler–Mascheroni constant.

Let $\lambda_{3g-3} = \log(6g-6)/2$. We have uniformly in $0 \le k \le 1.233 \cdot \lambda_{3g-3}$

$$\mathbb{P}\big(K_g(\gamma) = k{+}1\big) = e^{-\lambda_{3g-3}} \cdot \frac{\lambda_{3g-3}^k}{k!} \cdot \left(\frac{\sqrt{\pi}}{2\Gamma\left(1 + \frac{k}{2\lambda_{3g-3}}\right)} + O\left(\frac{k}{(\log g)^2}\right)\right)\,.$$

Keystone underlying results

Our results are strongly based on the following conjecture which we stated in August 2019, and which Amol Aggarwal proved in April 2020.

Theorem (Aggarwal). *The following* ***uniform*** *asymptotic formula is valid:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} =$$
$$= \frac{1}{24^g} \cdot \frac{(6g-5+2n)!}{g!\,(3g-3+n)!} \cdot \frac{d_1! \dots d_n!}{(2d_1+1)! \cdots (2d_n+1)!} \cdot \big(1+\varepsilon(\boldsymbol{d})\big)\,,$$

where $\varepsilon(\boldsymbol{d}) = O\left(1 + \frac{(n+\log g)^2}{g}\right)$ ***uniformly*** *for all* $n = o(\sqrt{g})$ *and all partitions* $\boldsymbol{d}$, $d_1 + \cdots + d_n = 3g-3+n$, *as* $g \to +\infty$.

Another Keystone result

Denote by $\mathcal{ST}(\mathcal{Q}_{g,n}, N)$ the set of square-tiled surfaces of genus g having exactly n conical points of angle π and tiled with at most N squares.
Reformulating a theorem of H. Masur of 1982 one concludes that for $2g + n > 3$ the following limit is a strictly positive (finite) number:

$$\frac{\operatorname{Vol}\mathcal{Q}_{g,n}}{2(6g-6+2n)} = \lim_{N\to+\infty} \frac{\operatorname{card}(\mathcal{ST}(\mathcal{Q}_{g,n}), 2N)}{N^d}\,.$$

Another Conjecture which we stated in August 2019 was also proved by Amol Aggarwal in April 2020.

Theorem (Aggarwal). *The Masur–Veech volume of the moduli space of holomorphic quadratic differentials has the following large genus asymptotics:*

$$\operatorname{Vol}\mathcal{Q}_g \sim \frac{4}{\pi}\cdot\left(\frac{8}{3}\right)^{4g-4} \quad \text{as } g\to+\infty\,.$$

Random permutations

Let $\theta = \{\theta_k\}_{k\geq 1}$ be positive real numbers. Given a permutation $\sigma \in S_n$ with cycle type $(1^{\mu_1} 2^{\mu_2} \ldots n^{\mu_n})$ we define its *weight* as

$$w_\theta(\sigma) := \theta_1^{\mu_1} \theta_2^{\mu_2} \cdots \theta_n^{\mu_n}.$$

To every sequence $\theta = \{\theta_k\}_{k\geq 1}$ we associate a probability measure on the symmetric group S_n by setting

$$\mathbb{P}_{\theta,n}(\sigma) := \frac{w_\theta(\sigma)}{W_{\theta,n}} \qquad \text{where} \qquad W_{\theta,n} := \sum_{\sigma \in S_n} w_\theta(\sigma).$$

Constant weights $\theta_i = 1$ correspond to the uniform measure on S_n; the probability measures on S_n obtained from constant weights $\theta_i = \alpha \neq 1$ are called *Ewens measure*.

Probability that a random permutations has k cycles

The following Lemma identifies normalized weighted multi-variate harmonic sums as total contributions of permutations having exactly k cycles to the total sum $W_{\theta,n}$.

Lemma. *Let $\theta = \{\theta_k\}_{k\geq 1}$ be non-negative real numbers and consider the associated probability measure $\mathbb{P}_{\theta,n}$ on the symmetric group S_n for some n. Then*

$$\frac{1}{n!} \cdot \sum_{\substack{\sigma\in S_n \\ \mathrm{K}_n(\sigma)=k}} w_\theta(\sigma) = \frac{1}{k!} \cdot \sum_{i_1+\cdots+i_k=n} \frac{\theta_{i_1}\theta_{i_2}\cdots\theta_{i_k}}{i_1\cdots i_k},$$

where $\mathrm{K}_n(\sigma)$ is the number of cycles in the cycle decomposition of σ and the sum in the right hand-side is taken over positive integers $i_1, \ldots, i_k$. In other words, we have the identity in the ring $\mathbb{Q}[[t, z]]$ of formal power series in t and z

$$\sum_{n\geq 1} \sum_{\sigma\in S_n} w_\theta(\sigma) t^{\mathrm{K}_n(\sigma)} \frac{z^n}{n!} = \exp\left(t \sum_{k\geq 1} \theta_k \frac{z^k}{k} \right).$$

Schematic idea of the proof.

● Observe that square-tiled surfaces corresponding to stable graphs with more than one vertex taken together contribute only $O\left(\frac{1}{g}\right)$ to the count of all square-tiled surfaces of genus g (this conjecture of ours was proved by A. Aggarwal).

● Using large genus asymptotics for the Witten–Kontsevich correlators (conjectured by us and proved by A. Aggarwal) compute the contribution of square-tiled surfaces of genus g represented by the stable graph with exactly one vertex and with j loops. Recognize in the resulting expression the multivariate harmonic sum as in the above Lemma corresponding to parameters $\theta_k = \zeta(2k)/2$, where $k = 1, 2, \ldots$.

● Apply the analytic technique developed by H. Hwang for random permutations to prove mod-Poisson convergence of the resulting distribution of the number of cycles $\mathrm{K}_n(\sigma)$ of a random permutation σ, where "randomness" is defined using parameters $\theta_k = \zeta(2k)/2$, where $k = 1, 2, \ldots$.