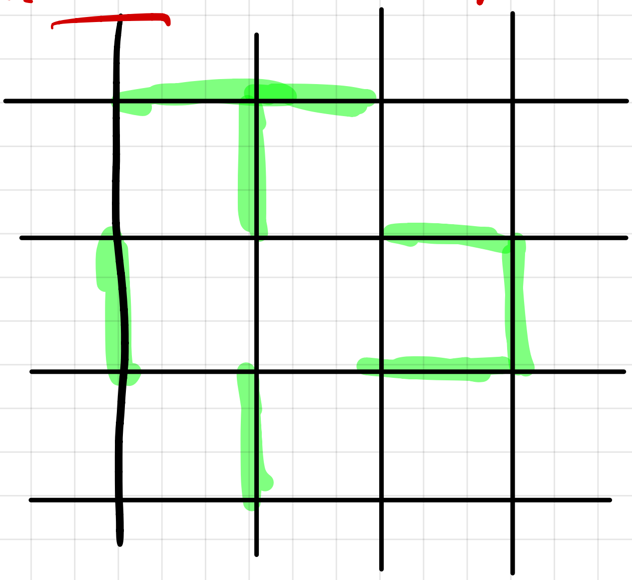
# Lattice models of statistical mechanics

## 1. Def. Chapter I. Percolation



**Hypercubic lattice:**

- vertices:
  $$\mathbb{Z}^d := \{(x_1,\dots,x_d) : x_i \in \mathbb{Z}\}$$
- edges:
  $$\mathbb{E}^d := \{(x,y) : |x-y|_1 = 1\}$$
  $$= \{x-y = (0,\dots,0,\pm 1,0,\dots,0)\}$$

[x, y differ in one coordinate]

Let $G = (V,E) \subset \mathbb{Z}^d$ - finite subgraph.

Take $p \in [0,1]$

**Percolation model:**

each edge is open with probability $p$, independently of the others:

$$\mathbb{P}(e \text{ is open}) = p.$$

**Def:**

Space of configurations:

$$\Omega := \{\underset{\text{closed}}{0}, \underset{\text{open}}{1}\}^E$$

$\Omega = \{ (\omega_e)_{e \in E} : \omega_e \in \{0,1\} \}$

On every edge, we have a Bernoulli rand. var.:

$$\omega_e \begin{cases} \nearrow 1 \text{ (open)} & \text{w. prob. } p \\ \searrow 0 \text{ (closed)} & \text{— " — } 1-p. \end{cases}$$

Probability measure on $\Omega$: product of Bernoulli distributions:

$$\mathbb{P}_p(\omega) = p^{\#\text{open}(\omega)} (1-p)^{\#\text{closed}(\omega)}$$

Parameter $p$:

$p \sim$ density of open edges

If $p \sim 0$, then almost no open edges

If $p \sim 1$, then almost all edges are open.

By the Law of Large Numbers: the expected number of open edges is $p \cdot |E|$.

Configurations $\Longleftrightarrow$ subgraphs.

$\omega \in \{0,1\}^{E} \longleftrightarrow H = (V, \{e : w_e = 1\})$

subgraph on open edges.

Def (on $(\mathbb{Z}^d, \mathbb{E}^d)$)

$\Omega = \{0,1\}^{\mathbb{E}^d}$

$\sigma$-algebra $\mathcal{F}$: generated by cylinder events [events that depend on finitely many edges]

Percolation measure on $\Omega$ is the product of Bernoulli rand. var. on edges $e \in \mathbb{E}^d$ $\mathbb{P}_p$

[Kolmogorov's extension thm. Carathéodory's thm]

Basic question:
is there an infinite connected component on open edges?

Or that the component of 0 is infinite? $0 \longleftrightarrow \infty$

Depends on $p$!

$\mathbb{P}_p(0 \leftrightarrow \infty) = 0$ $\qquad$ $\mathbb{P}_p(0 \leftrightarrow \infty) > 0$

$0 \qquad p < p_c \qquad p_c \qquad p > p_c \qquad 1$

Monotonicity in $p$?

## 2. Monotonicity in $p$.

Notations:

Let $x, y \in \mathbb{Z}^d$. Events:

- $\{x \leftrightarrow y\} := \{\exists$ path on open edges from $x$ to $y\}$
- $\{x \leftrightarrow \infty\} := \{x$ belongs to the infinite component (cluster)$\}$

Exerc $\{x \leftrightarrow y\}$, $\{x \leftrightarrow \infty\}$ are in the $\sigma$-algebra $\mathcal{F}$.

Def:

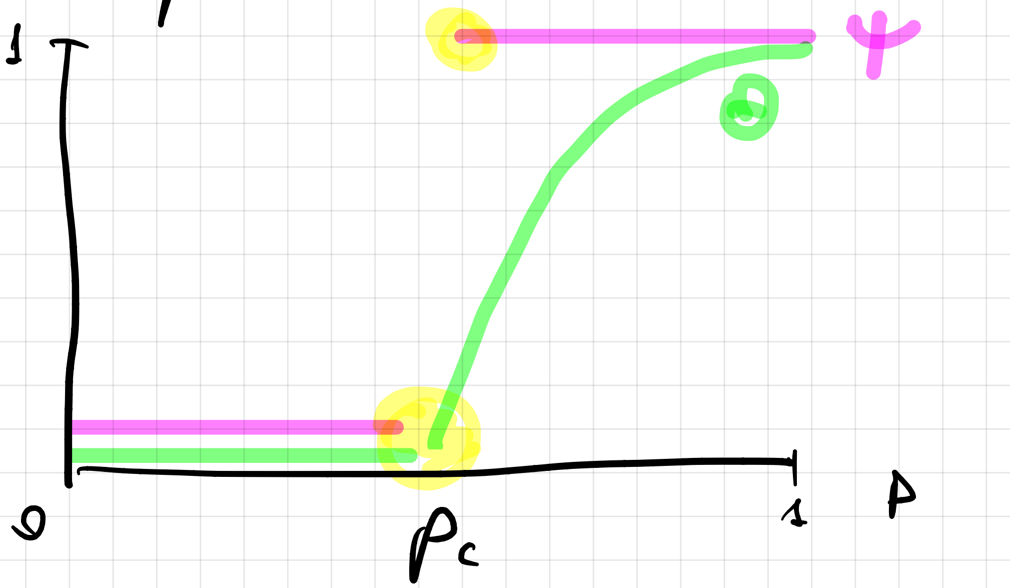
- connectivity function:

$$\theta(p) := \mathbb{P}_p(0 \leftrightarrow \infty)$$

- Existence of an infinite cluster:

$$\psi(p) := \mathbb{P}_p\Big(\bigcup_{x \in \mathbb{Z}^d} \{x \leftrightarrow \infty\}\Big)$$

Expected behaviour:




**Big open problem:**

$$\theta(p_c) = 0 \text{ in } \mathbb{Z}^3.$$

**Prop. (monotonicity)**
$\psi, \theta$ are non-decreasing in $p$.

Proof:
We'll construct a coupling for percolation measures with different $p$.
On every edge, we put a uniform rand. var. $U_e$ on $[0,1]$, that are indep. of each other!

$\{U_e\}_{e \in E^d}$ : $U_e \sim$ uniform $[0,1]$

$U_e$ and $U_f$ are indep. if $e \neq f$.

Let $p \in [0,1]$

Define $\omega$ (percolation config.):

- $\omega_e = 1$ if $U_e \leq p$
- $\omega_e = 0$ if $U_e > p$

<u>Claim</u>: $\omega \sim \mathbb{P}_p$.

<u>Proof</u>:

$\mathbb{P}(U_e \leq p) = p$.

So $\mathbb{P}(\omega_e = 1) = p$.

and $\mathbb{P}(\omega_e = 0) = 1 - p$.

Every $\omega_e$ is a function of $U_e$.

Since $(U_e)_{e \in E^d}$ are indep.,

so are $(\omega_e)_{e \in E^d}$

$\blacksquare$

Let $q \in [p, 1]$.

Define $\omega'$:

- $\omega'_e = 1$ if $U_e \leq q$
- $\omega'_e = 0$ if $U_e > q$.

As before, $\omega' \sim \mathbb{P}_\varphi$.

Crucially:

$\omega_e \leq \omega'_e$ at every $e \in \mathbb{E}^d$,

Indeed, if $\omega_e = 1$, then $U_e \leq p \leq \varphi$, hence $\omega'_e = 1$.

Then, if $\omega \in \{0 \leftrightarrow \infty\}$, then also $\omega' \in \{0 \leftrightarrow \infty\}$.

Hence,

$$\mathbb{P}(0 \overset{\omega}{\longleftrightarrow} \infty) \leq \mathbb{P}(0 \overset{\omega'}{\longleftrightarrow} \infty)$$

$$\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty) \qquad \mathbb{P}_\varphi(0 \leftrightarrow \infty) = \theta(\varphi)$$

For $+$ – analogously $\blacksquare$

## 3. Phase transition

**Def:** Critical density:

$$p_c := \inf\{p : \theta(p) > 0\}.$$

Exrc

**Thm**

(i) $p_c(\mathbb{Z}) = 1$. [no phase transition]

(ii) $d \geq 2$. Then

$$0 < p_c(\mathbb{Z}^d) < 1$$

**Proof:**

(ii) Part 1: $p_c > 0$.

We want to prove that when $p$ is small enough,

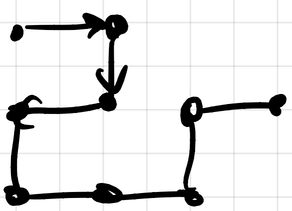
$$\mathbb{P}_p(0 \leftrightarrow \infty) = 0.$$

We will use a counting argument:

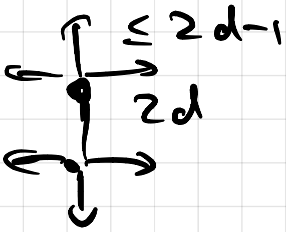
the weight of a long path will dominate the number of such paths.

**Paths:** (simple) self-avoiding walks

$SAW_n :=$ set of all self-avoiding walks of length $n$, starting from $0$.



$$\mathbb{P}_p(0 \leftrightarrow \infty) = \mathbb{P}_p(\forall n \; \exists \gamma \in SAW_n : \text{all edges in } \gamma \text{ are open})$$




$\forall n$

$$\leq \sum_{\gamma \in SAW_n} \underbrace{\mathbb{P}_p(\gamma \text{ is open})}_{= p^n}$$

$$= p^n \cdot |SAW_n|$$

$$\leq \underbrace{p^n \cdot 2d \cdot (2d-1)^{n-1}}_{\to 0 \text{ if } p < \frac{1}{2d-1}}$$

Thus, for any $p < \frac{1}{2d-1}$:

$$\mathbb{P}_p(0 \leftrightarrow \infty) = 0.$$

Hence $p_c \geq \frac{1}{2d-1}$.