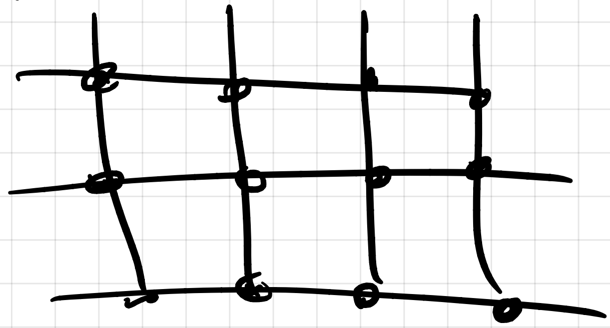
Lecture 2.

Last time:

Lattice: hypercubic
$(\mathbb{Z}^d, \mathbb{E}^d)$



Percolation model:
- on a finite graph $G=(V,E) \subset (\mathbb{Z}^d, \mathbb{E}^d)$

$$\mathbb{P}_p(\omega) = p^{\#\text{open}}(1-p)^{\#\text{closed}}$$

$\omega \in \{0, 1\}^E$
0 → closed, 1 → open

$p \in [0,1]$ — density

- extend to $(\mathbb{Z}^d, \mathbb{E}^d)$
Kolmogorov's theorem

Monotonicity in p:

$$\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty) \nearrow \text{in } p$$

$$p_c := \inf\{p : \theta(p) > 0\}$$

Thm

(i) $p_c(\mathbb{Z}) = 1$. [no phase transition]

(ii) $d \geq 2$. Then

$$0 < p_c(\mathbb{Z}^d) < 1$$

Proof:

(i) - Exercise

(ii). We've proved part I ; $p_c > 0$,

so that

$$\mathbb{P}_p(0 \leftrightarrow \infty) = 0,$$

when p is small enough

Part II : $p_c < 1$.

First step - reduce to $d = 2$.

Claim: $p_c(\mathbb{Z}^d) \geq p_c(\mathbb{Z}^{d+1})$

Proof:

Let $\mathbb{P}_p^d$ and $\mathbb{P}_p^{d+1}$ be percolation measures on $\mathbb{Z}^d$ and $\mathbb{Z}^{d+1}$

$\boxed{\mathbb{Z}^d \subset \mathbb{Z}^{d+1}}$

$$\underbrace{\mathbb{P}_p^d(0 \leftrightarrow \infty)}_{\theta^d(p)} = \mathbb{P}_p^{d+1}(0 \overset{\mathbb{Z}^d}{\longleftrightarrow} \infty)$$

$$\leq \underbrace{\mathbb{P}_p^{d+1}(0 \leftrightarrow \infty)}_{\theta^{d+1}(p)}$$

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Then, $\theta^d(p) \le \theta^{d+1}(p)$

Hence, $\inf\{p: \theta^d(p)>0\}$

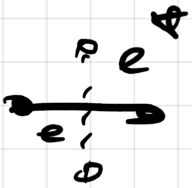
$$\geq \inf\{p: \theta^{d+1}(p)>0\}. \quad \blacksquare$$

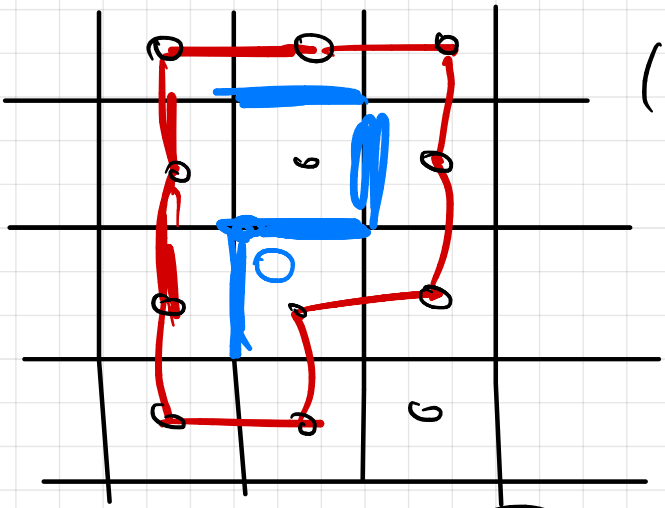
So it's enough to consider $d=2$.

Instead of constructing the infinite path directly, we will prove that when p is large enough, nothing can block us.

Duality idea

The block is a circuit on a dual graph;





(V^*, E^*):

$V^* = \{(\frac{1}{2}+x, \frac{1}{2}+y), \; x, y \in \mathbb{Z}\}$

E^*: $u^* \sim v^*$ if $|u^* - v^*| = 1$

So (V^*, E^*) is also a square grid!

We say that $(\mathbb{Z}^2, \mathbb{E}^2)$ is **self-dual**: $(V^*, E^*) \simeq (V, E)$

Let ω be a percolation configuration.

Define the dual config. ω^*:

$$\omega^*_{e^*} = 1 - \omega_e$$

e^* - open $\iff$ e - closed.

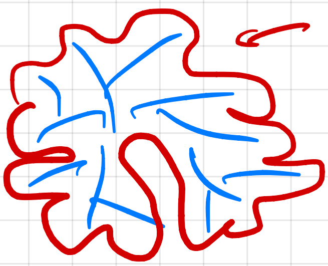
The key: $\omega^* \sim \mathbb{P}_{1-p}$ on (V^*, E^*)

* $\mathbb{P}(e^* \text{ open}) = \mathbb{P}(e \text{ closed}) = 1-p$
* and the edges are indep.

this defines $\mathbb{P}_{1-p}$ on (V^*, E^*)

Topological statement:

$0 \overset{\omega}{\not\leftrightarrow} \infty \iff \omega^*$ contains a circuit of open edges around 0.



← blocking circuit

Now again: the counting argument
(Peierls' argument 1939)

$\mathcal{C}_n :=$ set of dual circuits of length n around 0.

$$\mathbb{P}(0 \overset{\omega}{\not\leftrightarrow} \infty) = \mathbb{P}(\exists n\; \exists \gamma^* \in \mathcal{C}_n : \gamma^* \text{ is open in } \omega^*)$$

$$1-\theta(p) = \mathbb{P}(0 \overset{\omega}{\not\leftrightarrow} \infty)$$

$$\mathbb{P}(\exists n\; \exists \gamma^* \in \mathcal{C}_n : \gamma^* \text{ is open in } \omega^*) = \underbrace{\mathbb{P}_{1-p}(\exists n\, \exists \gamma^* \in \mathcal{C}_n \text{ - open})}_{}$$

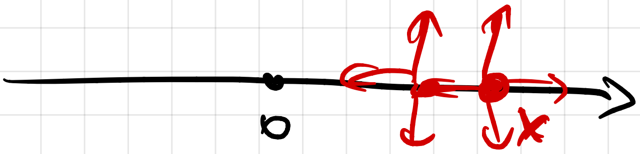
$$\le \sum_n \mathbb{P}_{1-p}(\exists \gamma^* \in \mathcal{C}_n - \text{open})$$

$$\le \sum_n \sum_{\gamma^* \in \mathcal{C}_n} \underbrace{\mathbb{P}_{1-p}(\gamma^* - \text{open})}_{=(1-p)^n}$$

$$= \sum_n (1-p)^n \cdot |\mathcal{C}_n|$$

Step 1: 4 choices
Step $k>1$: ≤ 3 choices

Every circuit in $\mathcal{C}_n$ must have a point on $[0,n)\times\{0\}$



We estimate number of circuits passing through x: $\le 4\cdot 3^{n-1}$

To summarize:

$$1-\Theta(p) \le \sum_n (1-p)^n \cdot |C_n|$$

$$\le \sum_n (1-p)^n \cdot \underbrace{n}_{\text{choices for pt } x} \cdot 4 \cdot 3^{n-1}$$

exercise

bound on p_c?

< 1 when $(1-p)$ is small enough

Then $\Theta(p) > 0$ for some $p < 1$.

Hence, $p_c < 1$. $\blacksquare$

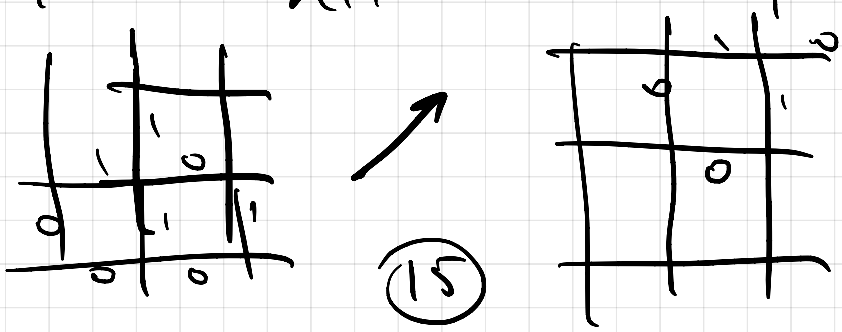
4. Ergodicity

Let $x \in \mathbb{Z}^d$.

Notations:

- τ_x - translation of the lattice by vector x

This translation defines the shift on the config. space.



• for event $A \subset \{0,1\}^{\mathbb{E}}$ define

$$T_x A := \{\omega \in \{0,1\}^{\mathbb{E}} : T_x^{-1}\omega \in A\}$$

Def:
A is invariant to translations, if, for all $x \in \mathbb{Z}^d$,

$$T_x A = A.$$

Example: $A := \{\exists \text{ infinite cluster}\}$

Non-translation-invar.:
$B := \{0 \longleftrightarrow \infty\}$

Def:
Measure μ on $\{0,1\}^{\mathbb{E}}$ is invariant to translations if, for all $x \in \mathbb{Z}^d$,

$$\mu(T_x A) = \mu(A)$$

Measure μ is ergodic, if for any event A that is invariant to translations,

$$\mu(A) \in \{0,1\}.$$

Example:

$\{\exists$ infinite clusters$\} \in$ tail σ-alg.

$\{\exists$ 2 infinite clusters$\}$

invariant to all translations

not a tail event

Tail σ-algebra:

σ-algebra of events whose occurrence cannot be changed by changing finitely many edges.

Prop.

Measure $\mathbb{P}_p$ is invariant to translations and ergodic.

Proof:

Exercise

Any event in $\mathcal{F}$ can be approximated by events depending on finitely many edges;

σ-alg.

$\forall \varepsilon > 0,\ A \in \mathcal{F}\ \exists\ E \subset \mathbb{E}$ - finite and $B \in \{0,1\}^E$, s.t.

$$\mathbb{P}_p (A \triangle B) < \varepsilon,$$

where $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

Invariance:

It's enough to prove for events depending on finitely many edges:

for $A \in \mathcal{F}$ find $B \in \{0,1\}^E$:

(*) $\boxed{\text{details next time.}}$

If $B \in \{0,1\}^E$, where E - finite, then the invariance follows from the formula for $\mathbb{P}_p$:

$$\mathbb{P}_p (\omega|_{F_1} = 1,\ \omega|_{F_2} = 0) = p^{|F_1|} (1-p)^{|F_2|}$$