Lecture 3

Recap:

$\mathbb{P}_p$ - percolation measure on $(\mathbb{Z}^d, \mathbb{E}^d)$

$p \in [0, 1]$.

For every edge $e \in \mathbb{E}^d$:

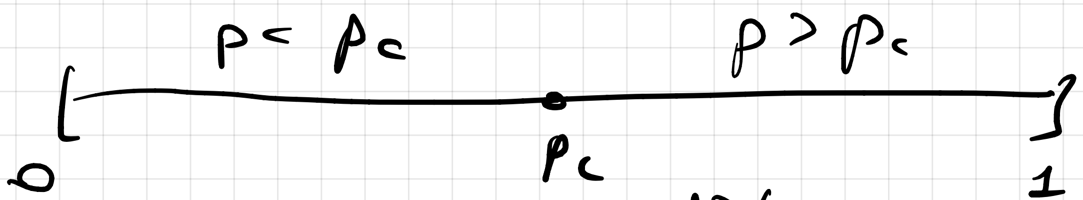
$$\left.\begin{aligned} \mathbb{P}(\omega_e = 1) &= p \\ \mathbb{P}(\omega_e = 0) &= 1-p \end{aligned}\right| \text{ indep. of other edges}$$

$\omega \in \{0, 1\}^{\mathbb{E}^d}$ - percolation config.

closed (0), open (1)

Consider the subgraph on open edges.

When $d \geq 2$, there exists $0 < p_c < 1$,



$\mathbb{P}(0 \leftrightarrow \infty) = 0$ $\qquad$ $\mathbb{P}(0 \leftrightarrow \infty) > 0$

$\mathbb{P}(\exists \text{ inf. cluster}) = 1$

[Kolmogorov's zero-one law]

$\mathbb{P}_p$ is: • translation-invariant

for every event $A \in \mathcal{F}$

$$\mathbb{P}_p(A) = \mathbb{P}_p(T_x A),$$

where T_x is translation by $x \in \mathbb{Z}^d$.

• ergodic

For every event $A \in \mathcal{F}$ that is translation-inv., we have $\mathbb{P}_p(A) \in \{0,1\}$.

Today:

5. Uniqueness of the infinite cluster.

Thm (Aizenman-Kesten-Newman '87)

Let $p > p_c$. Then

$$\boxed{\mathbb{P}_p\left(\begin{array}{c}\text{exists a unique}\\ \text{infinite cluster}\end{array}\right) = 1}$$

We will follow a beautiful argument due to Burton and Keane.

It is way more general! Instead of independence, it uses the finite energy property:

$\forall n \in \mathbb{N}\ \exists c > 0\ \forall S_0, S_1 \subset \mathbb{E}^d$, if $S_0 \cap S_1 = \emptyset$ and $|S_0|, |S_1| \leq n$,

then $\mathbb{P}_p(\omega|_{S_1} = 1, \omega|_{S_0} = 0) \geq c$.

The meaning: at any finite set of edges, we can fix any config. at a finite cost – for any outside config.

Example:

- Percolation on $\mathbb{E}^d$ has this property
- Percolation on lines of edges does not have it.

(side remark)

Proof:
Define a rand. var.:

$N := \#\{\text{infinite clusters}\}$

Step 1 (easy):
Show that either $\mathbb{P}(1 \text{ cluster}) = 1$
or $\mathbb{P}(\infty \text{ clusters}) = 1$

In particular $\mathbb{P}(1 < N < \infty) = 0$.

Proof:
Event $\{N = k\}$ is translation invariant.

Hence, by ergodicity,
$$\mathbb{P}(N = k) \in \{0, 1\}$$

Then, for some $k \in \mathbb{N} \cup \{\infty\}$,
$\mathbb{P}(N = k) = 1$, and

$\mathbb{P}(N = \ell) = 0$, for $\ell \neq k$.

Now, assume that $k > 1$ and k is finite.

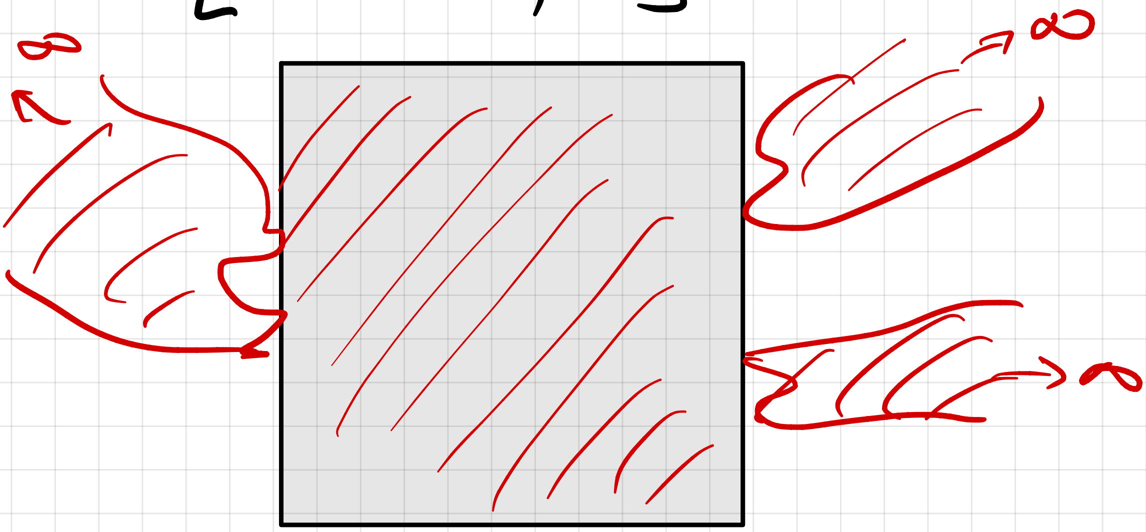
$$\boxed{\text{Idea: connect the clusters and get a contradiction.}}$$

For $L \in \mathbb{N}$ large enough, we have

Exrs

$$\mathbb{P}_p\left(\text{all infinite clusters intersect } \Lambda_L\right) \geq \frac{1}{2}$$

$$\Lambda_L := [-L, L]^d$$



these events are indep.

Open all edges in Λ_L:

$$\mathbb{P}_p\left(\text{all edges in } \Lambda_L \text{ are open}\right) = p^{|\Lambda_L|}$$

By independence:

$$\mathbb{P}_p\Big(\underbrace{\{\text{all infinite clusters intersect } \Lambda_L\} \cap \{\Lambda_L \text{ is open}\}}_{\text{in this case the infinite cluster is unique.}}\Big) \geq \frac{p^{|\Lambda_L|}}{2}$$

Thus, $\mathbb{P}$(unique inf. cluster) > 0.

By ergodicity then

$\mathbb{P}$(unique inf. cluster) = 1.

Contradiction.

Step 2. Exclude infinite number of infinite clusters.

„Idea"

In $\mathbb{Z}^d$ there is not enough place for infinitely many infinite clusters.

Idea:

We will prove that there exists many trifurcation points (where 3 infinite clusters meet)

These points will form a binary tree — which grows faster that $\mathbb{Z}^d$.

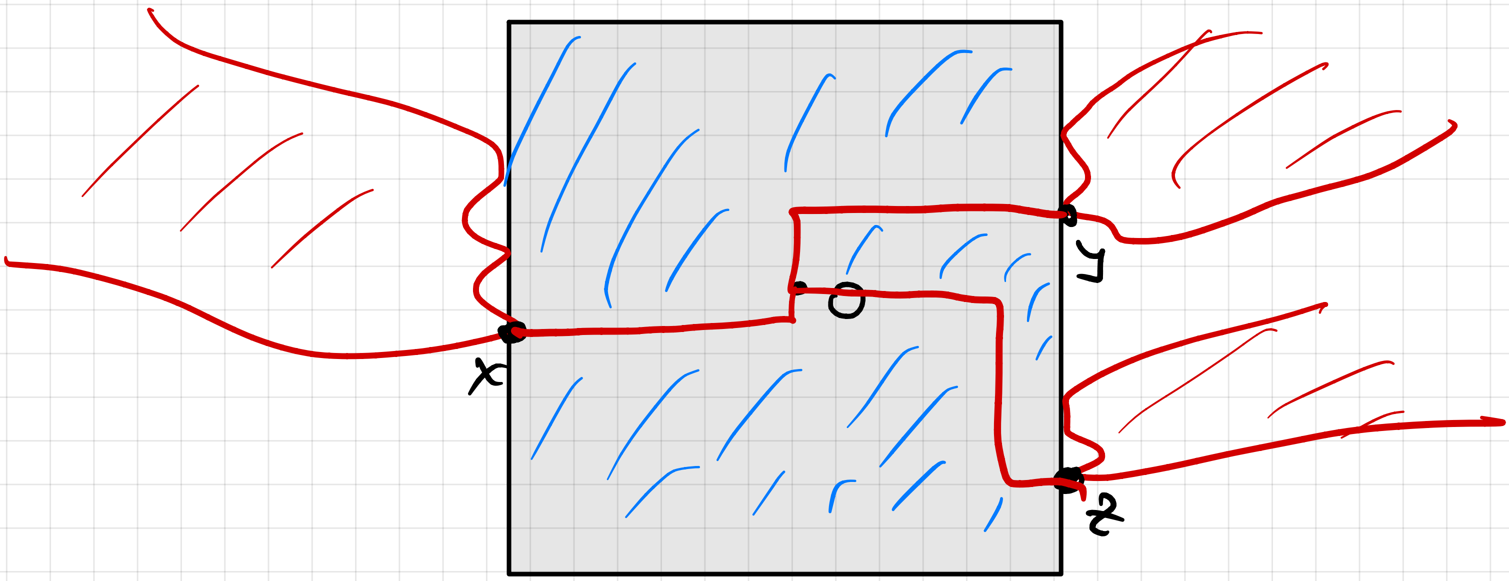
Proof:

Assume

$$\mathbb{P}(N = \infty) = 1.$$

For large enough L:

$$\mathbb{P}\left(\begin{array}{c}3 \text{ infinite clusters} \\ \text{intersect } \Lambda_L\end{array}\right) \geq \frac{1}{2}$$



On this event, there exist points $x, y, z \in \partial\Lambda_L$, s.t. in $\mathbb{Z}^d \setminus \Lambda_L$ the following connections hold:

- $x \leftrightarrow \infty$, $y \leftrightarrow \infty$, $z \leftrightarrow \infty$
- $x \not\leftrightarrow y$, $y \not\leftrightarrow z$, $z \not\leftrightarrow x$.

In Λ_L:

disconnected

Open paths from 0 to x, y, z

And close all other edges in Λ_L.

This has a positive probability $\geq (p(1-p))^{|\Lambda_L|}$

$$\mathbb{P}_p\Big(\underbrace{\text{in } \mathbb{Z}^d \setminus \{0\} \text{ exists three infinite clusters that are connected to } 0 \text{ by open edges}}_{\mathrm{Tri}_0}\Big) \geq \underbrace{\frac{(p(1-p))^{|\Lambda_L|}}{2}}_{c}$$

On the event Tri_0, we call 0 a trifurcation point.

For every $x \in \mathbb{Z}^d$, define

$\mathrm{Tri}_x = \tau_x \mathrm{Tri}_0$

By translational invariance of $\mathbb{P}_p$:

$$\forall x \in \mathbb{Z}^d \quad \mathbb{P}_p(\mathrm{Tri}_x) = \mathbb{P}_p(\mathrm{Tri}_0) \geq c.$$

Fix K very large - size of the box

Let $T := \#\{$trifurcation points in $\Lambda_K\}$

Estimate the expected value of T:

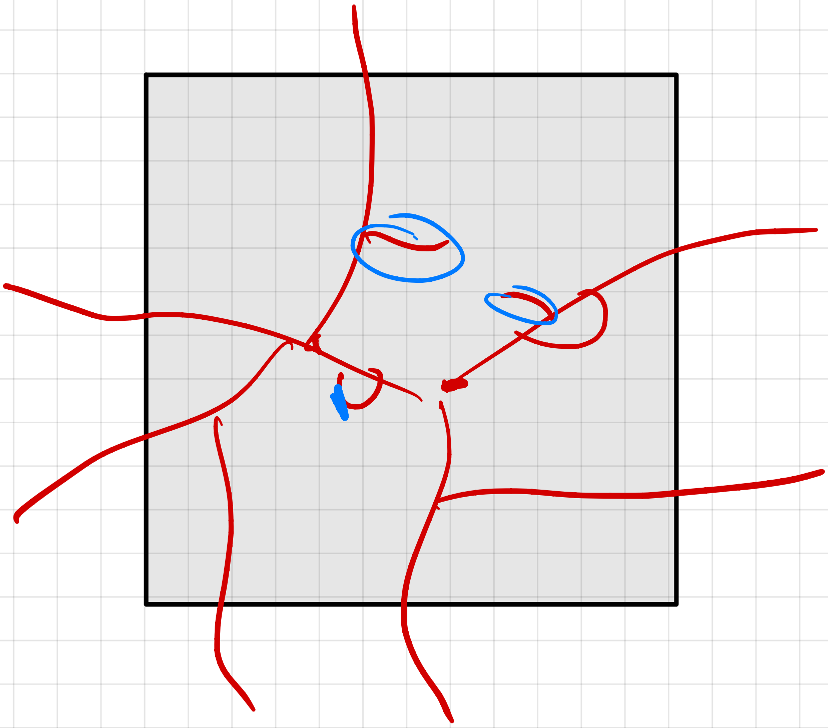
$$\mathbb{E}_p(T) = \mathbb{E}_p\Big(\sum_{x \in \Lambda_K} \mathbb{1}_{\mathrm{Tri}_x}\Big)$$

$$= \sum_{x \in \Lambda_K} \mathbb{E}_p(\mathbb{1}_{\mathrm{Tri}_x})$$

$$= \sum_{x \in \Lambda_K} \mathbb{P}(\mathrm{Tri}_x)$$

$$\geq c \cdot |\Lambda_K|$$

To get a contradiction we will prove a deterministic upper bound.



Vaguely:

- destroy all cycles
- remove all extra branches

This doesn't break any trifurcation point.

$\Downarrow$

Get a forest:

- all trifurcation points have degree ≥ 3
- all leaves are on $\partial\Lambda_k$.

Then $T \leq |\partial \Lambda_k|$ Exos

Hence $\mathbb{E}_p(T) \leq |\partial\Lambda_k| \sim k^{d-1}$

$|k|^d \sim c\,|\Lambda_k| \leq$ contradiction