Lecture 4.

Recap:

hypercubic lattice

Percolation measure on $\mathbb{Z}^d$:

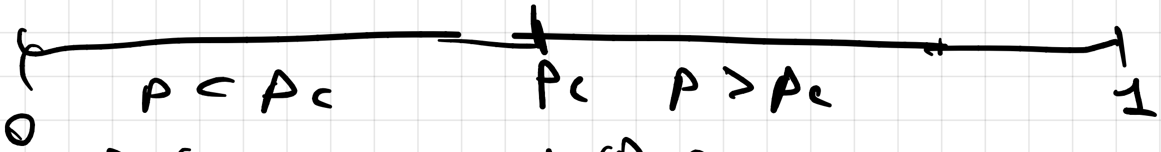
open $\quad \mathbb{P}_p(\omega_e = 1) = p$

closed $\quad \mathbb{P}_p(\omega_e = 0) = 1-p$,

independently for every edge.

$$\omega \in \{0,1\}^{\mathbb{E}^d}$$

Phase transition when $d \geq 2$:



$p < p_c$	$p > p_c$
$\mathbb{P}_p(0 \leftrightarrow \infty) = 0$	$\mathbb{P}_p(0 \leftrightarrow \infty) > 0$
no inf. cluster	unique infinite cluster

6. Margulis - Russo formula

We will show that clusters are very small (exponential tails) when $p < p_c$:

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_k) < e^{-ck},$$

$c > 0$ fixed.

To achieve this we will need differential inequalities. Margulis–Russo formula

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Let $G=(V,E)$ - finite graph.

Let $f: \{0,1\}^E \to \{0,1\}$ boolean function

Define (notation) $\mathbb{F}(p) := \mathbb{E}_p(f(\omega))$.

Lemma

$$F'(p) = \frac{1}{p(1-p)} \cdot \sum_{e\in E} \operatorname{cov}(f, \omega_e)$$

Proof:

Take $|\omega| := \sum_{e\in E} \omega_e$ = #{open edges}

Then:

$$F(p) = \sum_{\omega} f(\omega) \cdot \mathbb{P}_p(\omega)$$

$$= \sum_{\omega} f(\omega) \cdot p^{|\omega|} (1-p)^{|E|-|\omega|}$$

Differentiate in p:

$$F'(p) = \sum_{\omega} f(\omega) p^{|\omega|} (1-p)^{|E|-|\omega|} \cdot \left(\frac{|\omega|}{p} - \frac{|E|-|\omega|}{1-p}\right)$$

this is because

$$(p^{|\omega|})' = |\omega| \cdot p^{|\omega|-1} = p^{|\omega|} \cdot \frac{|\omega|}{p}.$$

$$\left((1-p)^{|E|-|\omega|}\right)' = -(|E|-|\omega|) \cdot (1-p)^{|E|-|\omega|-1}$$

$$= -(1-p)^{|E|-|\omega|} \cdot \frac{|E|-|\omega|}{1-p}$$

$$= \mathbb{E}_p\Big[f(\omega) \cdot \underbrace{\left(\frac{|\omega|}{p} - \frac{|E|-|\omega|}{1-p}\right)}_{\frac{1}{p(1-p)} \cdot \underbrace{(|\omega|(1-p) - (|E|-|\omega|)p)}_{|\omega| - |E| \cdot p}}\Big]$$

(*)

$$= \frac{1}{p(1-p)} \mathbb{E}_p\big[f(\omega) \cdot (|\omega| - |E| \cdot p)\big]$$

Note that

$$|\omega| - |E| p = \sum_{e \in E} (\omega_e - p)$$

Insert this in (*):

$$F'(p) = \frac{1}{p(1-p)} \sum_{e \in E} \underbrace{\mathbb{E}_p[f(\omega) \cdot (\omega_e - p)]}_{\text{since } p = \mathbb{E}_p(\omega_e) \ \ = \ \operatorname{cov}(f, \omega_e)}$$

$\boxtimes$

Rem. here we've used:

$$\operatorname{cov}(X, Y) = \mathbb{E}(X \cdot (Y - \mathbb{E}(Y)))$$

(31) $$= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

This formula has a geometric interpretation when $f = \mathbb{1}_A$, for an increasing event A.

Def

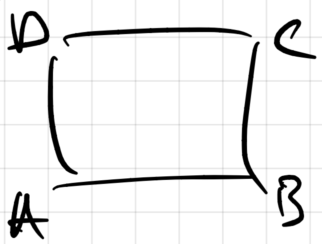
Define on $\{0,1\}^E$ a partial order: $\omega \preceq \omega'$ iff

$$\omega_e \le \omega'_e \text{ for } \forall e \in E.$$

Event $A \in \{0,1\}^E$ is called increasing if the following holds: for every $\omega \in A$ and $\omega' \succeq \omega$, we have $\omega' \in A$.

In other words $\mathbb{1}_A$ is increasing w.r.t. (with respect to) this partial order.

Eg. $\{\exists$ infinite cluster$\}$

$\{$  $AB \leftrightarrow CD \}$

It can happen that the status of an event A depends on the status of a certain edge – then this edge is called **pivotal**.

Def: Let $\omega \in \{0,1\}^E$, $e \in E$.
Denote by ω^o and ω^c two config. that coincide with ω on edges in $E \setminus \{e\}$ and: $\omega^o_e = 1$ open

$\omega^c_e = 0$ closed.

We say that e is pivotal for event A in ω, if

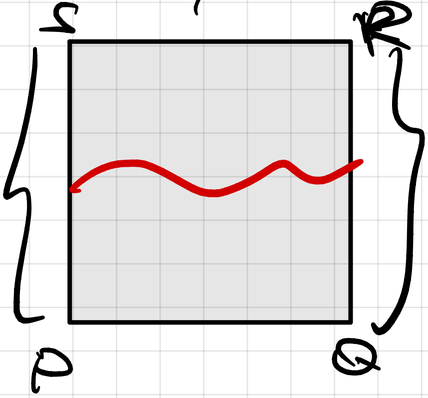
$$\omega^o \in A, \quad \omega^c \notin A$$

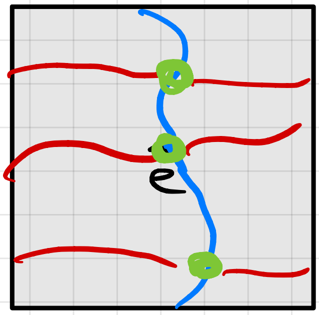
This event is $Piv_e(A)$.

Rem: $Piv_e(A)$ does not depend on the status of e.

Eg. 1) $A := \{\omega_e = 1\}$
$Piv_e(A) = \{0,1\}^E$.

If $f \neq e$, then $Piv_f(A) = \emptyset$.

2) $A :=$  $= \{PS \leftrightarrow QR\}$

$Piv_e(A)$ in $\mathbb{Z}^2$ $:=$ 

$Piv_e(A) = \{\omega : \omega^0 \in A, \omega^c \notin A\}$

coincide with ω on $E \setminus \{e\}$

Thm (Margulis '74, Russo '78)

Let $p \in [0,1]$; E - finite set of edges.

Consider event $A \subset \{0,1\}^E$, which is increasing. Then:

$$\mathbb{P}_p(A)' = \sum_{e \in E} \mathbb{P}_p(Piv_e(A))$$

Meaning:

derivative of $\mathbb{P}_p(A)$ is the expectation of the number of pivotal points.

Proof:

Substitute $f := \mathbb{1}_A$ in the Lemma

$$F(p) = \mathbb{P}_p(A)$$

$$(**) \quad F'(p) = \frac{1}{p(1-p)} \cdot \sum_{e \in E} \operatorname{cov}(\mathbb{1}_A, \omega_e)$$

Note that

$$\operatorname{cov}(\mathbb{1}_A, \omega_e) = \mathbb{E}_p(\mathbb{1}_A(\omega_e - p))$$

$$\overset{(*)}{=} \mathbb{E}_p(\mathbb{1}_A(\omega_e - p)\mathbb{1}_{Piv_e(A)})$$

Proof of $(*)$:

- $A \cap Piv_e^c(A)$ does not depend on e.

Indeed consider ω^o and ω^c.

Either $\omega^o, \omega^c \in Piv_e(A)$ (1)

or $\omega^o, \omega^c \notin Piv_e(A)$ (2)

If (1), then $\omega^o, \omega^c \notin A \cap Piv_e^c(A)$

If (2), then by definition of $Piv_e(A)$

either $\omega^o, \omega^c \in A$ (3)

or $\omega^o, \omega^c \notin A$. (4)

If (3), then $\omega^o, \omega^c \in A \cap Piv_e^c(A)$

If (4), then $\omega^o, \omega^c \notin A \cap Piv_e^c(A)$.

- Then:

$$\mathbb{E}\left(\mathbb{1}_A (\omega_e - p)\, \mathbb{1}_{Piv_e(A)^c} \right)$$

$$= \mathbb{E}\Big(\underbrace{\mathbb{1}_{A \cap Piv_e(A)^c}}_{\text{does not depend on } \omega_e} \cdot \underbrace{(\omega_e - p)}_{\text{determined by } \omega_e} \Big)$$

disjoint support

$$= \mathbb{E}\left(\mathbb{1}_{A \cap Piv_e(A)^c} \right) \cdot \underbrace{\mathbb{E}(\omega_e - p)}_{=0}$$

Now note that

$$A \cap Piv_e(A) = Piv_e(A) \cap \{\omega_e = 1\}$$

Insert this info (*):

$$\mathrm{cov}(\mathbb{1}_A, \omega_e) = \mathbb{E}\left((\omega_e - p) \cdot \mathbb{1}_{Piv_e(A)}\, \mathbb{1}_{\omega_e = 1} \right)$$

$$= (1-p)\, \mathbb{P}_p\Big(\underbrace{Piv_e(A)}\, \cap\, \underbrace{\{\omega_e = 1\}} \Big)$$

independent

$$= (1-p)\, \mathbb{P}_p(Piv_e(A)) \cdot \underbrace{\mathbb{P}_p(\omega_e = 1)}_{= p}$$

$$= p(1-p)\, \mathbb{P}_p(Piv_e(A))$$

Insert in (**)

Exercise: prove this formula directly through the coupling.

7. Sharpness of the phase transition.

Thm (Menshikov '86, Aizenman–Barsky '87)

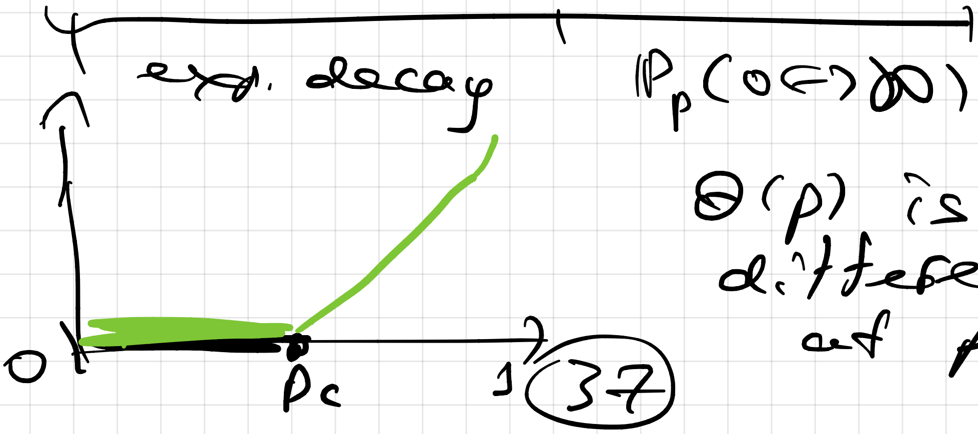
Fix $d \geq 2$ and $p < p_c(d)$. Then there exists $c_p > 0$, such that, for all $n \geq 1$,

$$\mathbb{P}_p(0 \leftrightarrow \partial \Lambda_n) < e^{-c_p n}$$

Furthermore, there exists $C > 0$, s.t. for all $p > p_c(d)$:

$$\mathbb{P}_p(0 \leftrightarrow \infty) \geq (p - p_c) \cdot C$$

Meaning: sharp transition



$\mathbb{P}_p(0 \leftrightarrow \infty) \geq (p-p_c) C$

$\theta(p)$ is not differentiable at p_c.