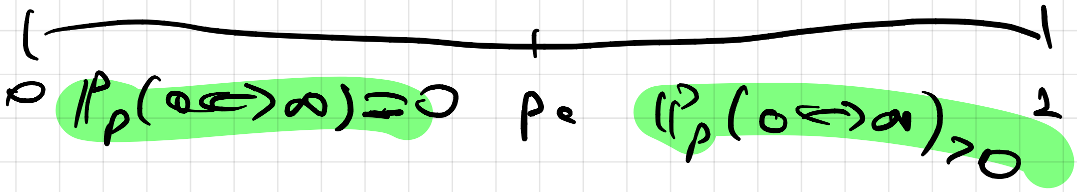
Lecture 5.

Recap:

$\mathbb{P}_p$ – percolation measure on $\mathbb{Z}^d$, $d \geq 2$



We follow the proof of Duminil-Copin and Tassion – for sharpness of the phase transition.

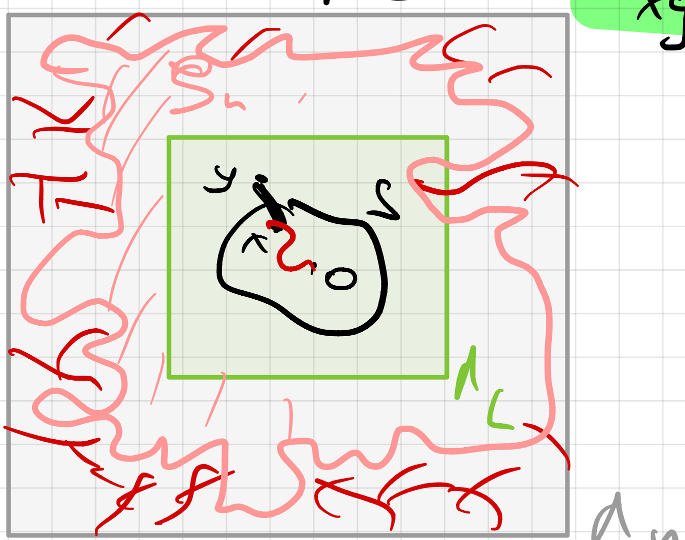
In exercises:

- $\theta_n(p) := \mathbb{P}_p(0 \leftrightarrow \partial \Lambda_n)$

  Take $L < n$, $S \subset \Lambda_{L-1}$.

  Then,

$$\theta_n(p) \leq \theta_{n-L}(p) \cdot p \sum_{xy \in \Delta S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} x)$$



$\Delta S := \{xy \in E : x \in S, y \notin S\}$

- $\theta_n'(p) = \frac{1}{1-p} \mathbb{E}_p \sum_{xy \in \Delta S_n} \mathbb{P}_p(0 \overset{S_n}{\leftrightarrow} x)$,

where $S_n := \{x \in \Lambda_n : x \not\leftrightarrow \partial \Lambda_n\}$

<u>Key idea:</u>

Define p_c differently!
We will use a finite criterion.

Def:
For $S \subset \mathbb{Z}^d$ finite set of vertices and $p \in (0,1)$, define

$$\varphi_p(S) := p \sum_{xy \in \Delta S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} x)$$

Take

$$\tilde{p}_c := \sup\{p \in [0,1] : \exists\, 0 \in S \subset \mathbb{Z}^d \text{ finite s.t. } \varphi_p(S) < 1\}$$

<u>Rem.</u> If $0 \notin S$, then $\varphi_p(S) = 0$.

<u>Proof (Sharpness):</u>

We prove it for $\tilde{p}_c$.

Part II: exp. decay below $\tilde{p}_c$.

Take any $p < \tilde{p}_c$.

By definition, there exists $0 \in S \subset \mathbb{Z}^d$ - finite, s.t.

$$\varphi_p(S) < 1.$$

Take L, s.t. $S \subset \Lambda_{L-1}$.

Use our inequality for $n = k \cdot L$:

$$\theta_{kL}(p) \le \theta_{(k-1)L}(p) \cdot \varphi_p(S)$$

iterate

$$\le \theta_{(k-2)L}(p) \cdot [\varphi_p(S)]^2$$

$$\cdots$$

$$\le \theta_L(p) \cdot [\varphi_p(S)]^{k-1}$$

$$\le \varphi_p(S)^{k/2} = [\varphi_p(S)]^{n/2L}$$

Since $\varphi_p(S) < 1$, there exists $c > 0$, s.t.

$$\varphi_p(S) \le e^{-c \cdot 2L}$$

Then, $\theta_n(p) \le e^{-c \cdot n}$.

Part II:

Take $p > \hat{p}_c$. $\quad \mathcal{S}_n := \{x \in \Lambda_n : x \leftrightarrow \partial\Lambda_n\}$

We know:

$$\theta_n'(p) = \frac{1}{(1-p)p} \mathbb{E}_p\left[\varphi_p(\mathcal{S}_n)\right]$$

That is $\begin{cases} \text{equals } 0 & \text{if } 0 \notin \mathcal{S}_n \\ \geq 1 & \text{if } 0 \in \mathcal{S}_n \end{cases}$

$$\varphi_p(\mathcal{S}_n) \geq \mathbb{1}_{0 \in \mathcal{S}_n}$$

Then:

$$\theta_n'(p) \geq \frac{1}{p(1-p)} \mathbb{E}_p\left[\mathbb{1}_{0 \in \mathcal{S}_n}\right]$$

$$= \frac{1}{p(1-p)} \mathbb{P}_p(0 \in \mathcal{S}_n)$$

$$= \frac{1}{p(1-p)} (1 - \theta_n(p))$$

This is a differential ineq. on $\theta_n(p)$!

Then:

$$\frac{\theta_n'(p)}{1-\theta_n(p)} \geq \frac{1}{p(1-p)}$$

$$\log\left(\frac{1}{1-\theta_n(p)}\right)' \geq \log\left(\frac{p}{1-p}\right)'$$

(*1)

Integrate it between $\tilde{p}_c$ and p:

$$\frac{1}{1-\theta_n(p)} - \underbrace{\frac{1}{1-\theta_n(\tilde{p}_c)}}_{\geq 1} \geq \frac{p}{1-p} - \frac{\tilde{p}_c}{1-\tilde{p}_c}$$

$$\frac{\theta_n(p)}{1-\theta_n(p)} \geq \frac{p-\tilde{p}_c}{(1-p)(1-\tilde{p}_c)}$$

$$\theta_n(p) \geq \frac{p-\tilde{p}_c}{(1-p)(1-\tilde{p}_c)+p-\tilde{p}_c} \geq c(p-\tilde{p}_c)$$

Tend $n \to \infty$ and get the bound.

Put together Parts I & II:

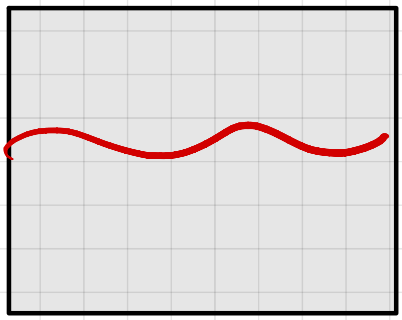
$$p_c = \tilde{p}_c$$

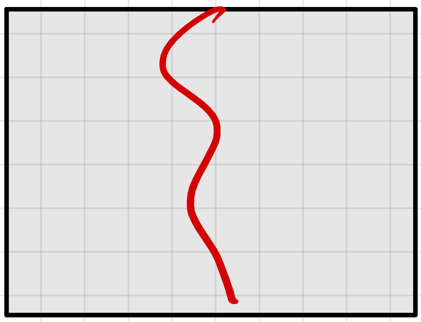
8. Harris-FKG inequality.

Edges in percolation are open/closed independently. Thus, for events A and B that are determined by disjoint sets of edges, we have

$$\mathbb{P}_p(A \cap B) = \mathbb{P}_p(A) \cdot \mathbb{P}_p(B).$$

What if A and B depend on the same edges? For instance:

$A :=$  $= \{\exists \text{ left-right crossing}\}$

$B :=$  $= \{\exists \text{ top-bottom crossing}\}$

Can we estimate $\mathbb{P}_p(A \cap B)$?

Yes – if A and B are increasing!

Thm (Harris '60)

Let A, B be two increasing events. Then,

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A) \cdot \mathbb{P}_p(B).$$

More generally, if f, g are increasing functions that are bounded, then they are positively correlated!

$$\mathbb{E}_p(f \cdot g) \geq \mathbb{E}_p(f) \cdot \mathbb{E}_p(g).$$

Meaning:
knowing that one increasing event occurs, increases the probability that any other increases event occurs.

Indeed, if $\mathbb{P}_p(B) > 0$, then dividing by $\mathbb{P}_p(B)$, we get:

$$\mathbb{P}_p(A|B) := \frac{\mathbb{P}_p(A \cap B)}{\mathbb{P}_p(B)} \geq \mathbb{P}_p(A)$$

where $\mathbb{P}_p(A|B)$ is the conditional probability of A given B".

Not surprising:
since B occurs, there are a lot of open edges.
This edges help A to occur

<u>Rem.</u>
This holds for a big family of models - it was shown for the first time by Fortuin-Kasteleyn-Ginibre (FKG inequality)

<u>Proof</u> (Harris' inequality):
It's enough to prove it only for increasing bdd f, g:
we can take $f := \mathbb{1}_A$, $g := \mathbb{1}_B$.

$$\underbrace{\mathbb{E}_p(\mathbb{1}_A \cdot \mathbb{1}_B)}_{= \mathbb{P}_p(A \cap B)} \geq \underbrace{\mathbb{E}_p(\mathbb{1}_A)}_{= \mathbb{P}_p(A)} \cdot \underbrace{\mathbb{E}_p(\mathbb{1}_B)}_{= \mathbb{P}_p(B)}$$

Sketch:

- reduce to the case when f, g depend on finite sets of edges: n edges
- $n = 1$: exact computation
- $n-1 \to n$: induction step via conditional expectation

Number all edges:

$$E = \{e_i\}_{i \geq 1}$$

For a configuration $\omega \in \{0,1\}^E$:

$$\omega_i := \omega_{e_i}$$

Define a family of σ-algebras:

$$\mathcal{F}_n := \sigma(\{\omega_1, \dots, \omega_n\})$$

This is a filtration:

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \dots$$

Step 1: Enough to prove for f, g that depend on finitely many edges.

Proof (step 1):

Define $f_n := \underbrace{\mathbb{E}_p(f \mid \mathcal{F}_n)}$.

this is a rand. var. which is $\mathcal{F}_n$-measurable.

In other words:

$$f_n(\xi_1, \dots, \xi_n) = \mathbb{E}_p(f \mid \omega_1 = \xi_1, \dots, \omega_n = \xi_n)$$

Clearly, f_n is a martingale:

$$\mathbb{E}_p(f_{n+1} \mid \mathcal{F}_n) = \mathbb{E}_p(\mathbb{E}_p(f \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n)$$

$$= \mathbb{E}_p(f \mid \mathcal{F}_n) = f_n$$

Since f is bounded, so is f_n.

Then, by the martingale convergence theorem,

f_n converges a.s. $\underbrace{\text{to } f}_{\text{exercise}}$.

Similarly, $g_n := \mathbb{E}_p(g \mid \mathcal{F}_n) \xrightarrow[n \to \infty]{} g$ a.s.

Finally, we also have

$$f_n g_n \xrightarrow[n\to\infty]{} fg \quad \text{a.s.}$$

Assume that, for any n,

$$\begin{array}{ccccc} \mathbb{E}(f_n g_n) & \geq & \mathbb{E}(f_n) & \cdot & \mathbb{E}(g_n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{E}(fg) & & \mathbb{E}(f) & & \mathbb{E}(g) \end{array}$$

$\boxtimes$

Step 2:

Show that if $f, g : [0,1] \to \mathbb{R}$ are increasing, then

$$\mathbb{E}_p(fg) \geq \mathbb{E}_p(f) \cdot \mathbb{E}_p(g).$$

Proof:

Shifting f, g by a constant does not change the inequality.

Then we can assume that

$$f(0) = g(0) = 0.$$

Since f, g are increasing, we have

$$f(1), g(1) \geq 0.$$

Now. let us do the computation:

$$\mathbb{E}_p(fg) = p\cdot f(1)g(1) + \underbrace{(1-p)f(0)g(0)}_{=\,0}$$

$$\mathbb{E}_p(f) = p\cdot f(1) + \underbrace{(1-p)f(0)}_{0}$$

$$\mathbb{E}_p(g) = p\cdot g(1)$$

Then,

$$\mathbb{E}_p(fg) - \mathbb{E}_p(f)\cdot \mathbb{E}_p(g)$$

$$= p\cdot f(1)g(1) - p\cdot f(1)\cdot p\cdot g(1)$$

$$= \underset{0}{\underset{\vee\!\mid}{f(1)}}\,\underset{0}{\underset{\vee\!\mid}{g(1)}}\cdot \underset{0}{\underset{\vee\!\mid}{(p-p^2)}} \geq 0$$

$\square$

Step 3:

Assume that we know the inequality for functions that depend on $n-1$ edges. Show it for

$$f, g : \{0,1\}^{\{e_1,\dots,e_n\}} \to \mathbb{R}.$$

Proof:

We first fix

$$(\omega_1, \dots, \omega_{n-1}) = \xi \in \{0,1\}^{\{e_1, \dots, e_{n-1}\}}$$

and then we average over them:

$$\mathbb{E}_p(fg) = \sum_{\substack{\xi \in \{0,1\}^{\{e_1, \dots, e_{n-1}\}} \\ \xi_n \in \{0,1\}}} f(\xi, \xi_n) g(\xi, \xi_n) \underbrace{\mathbb{P}(\xi, \xi_n)}_{\substack{\parallel \\ \mathbb{P}_p(\xi) \cdot \mathbb{P}_p(\xi_n)}}$$

$$= \sum_{\xi} \underbrace{\left(\sum_{\xi_n \in \{0,1\}} f(\xi, \xi_n) g(\xi, \xi_n) \mathbb{P}_p(\xi_n) \right)}_{\underbrace{\mathbb{E}_p^{\xi_n}}_{\text{average over } \xi_n} \Big(\underbrace{f(\xi, \xi_n)}_{\text{increasing in } \xi_n} \cdot \underbrace{g(\xi, \xi_n)}_{-\text{u}-} \Big)} \cdot \mathbb{P}_p(\xi)$$

ξ - fixed $\geq$ apply Harris' ineq. for $n = 1$

$$\underbrace{\mathbb{E}_p^{\xi_n}(f(\xi, \xi_n))}_{\substack{\parallel \\ f_{n-1}(\xi)}} \cdot \underbrace{\mathbb{E}_p^{\xi_n}(g(\xi, \xi_n))}_{\substack{\parallel \\ g_{n-1}(\xi)}}$$

$\boxed{\text{exercise}}$

Thus, we have:

$$\mathbb{E}_p(f\cdot g) \geq \sum_{\xi\in\{0,1\}^{\{1,\dots,n-1\}}} f_{n-1}(\xi)\cdot g_{n-1}(\xi)\cdot \mathbb{P}_p(\xi)$$

$$= \mathbb{E}_p(f_{n-1}\cdot g_{n-1})$$

Harris's ineq. for $n-1$

$$\geq \mathbb{E}_p(f_{n-1})\cdot \mathbb{E}_p(g_{n-1})$$

$$= \mathbb{E}_p(f)\cdot \mathbb{E}_p(g)$$

∎