Thus, we have:

$$\mathbb{E}_p(f\cdot g) \geq \sum_{\xi\in\{0,1\}^{\{e_1,\dots,e_{n-1}\}}} f_{n-1}(\xi)\cdot g_{n-1}(\xi)\cdot \mathbb{P}_p(\xi)$$

$$= \mathbb{E}_p(f_{n-1}\cdot g_{n-1})$$

Harris' ineq. for $n-1$

$$\geq \mathbb{E}_p(f_{n-1})\cdot \mathbb{E}_p(g_{n-1})$$

$$= \mathbb{E}_p(f)\cdot \mathbb{E}_p(g)$$

∎

Lecture 6

9. $p_c(\mathbb{Z}^2) = \frac{1}{2}$.

Kesten's theorem from 1980.

Proof: exercises.

Part of a more general phenomenon:

- in percolation:

$$p_c(G) + p_c(G^*) = 1$$

- in other (bond) models critical point on $\mathbb{Z}^2$ is self-dual.

[not always $\frac{1}{2}$!] S1 [later]

10. Russo-Seymour-Welsh estimates. $\mathbb{Z}^2$

Next question:
what happens at p_c?
This is in fact more important than the value of p_c — this behavior will not depend on the lattice
(it is universal)
So far we know only that

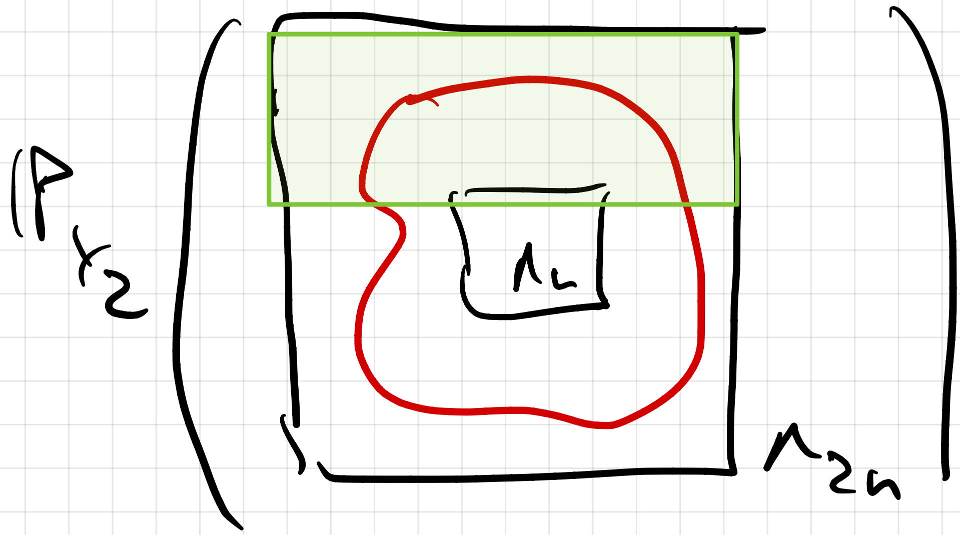
$$\theta(p_c) = 0.$$

Thus, a.s., there exists a dual circuit around 0.
Apply the same to ω^*.

Apply iteratively and obtain infinitely many primal and dual circuits around 0

Our goal: show that these circuits appear at every scale.

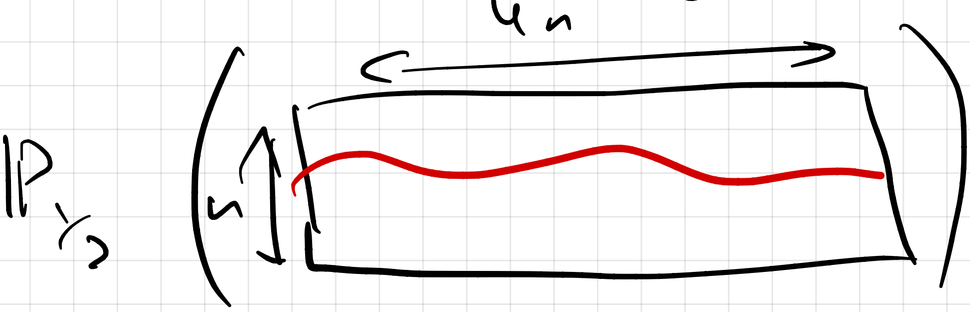
How to prove that



$\overset{?}{\geq} c > 0$

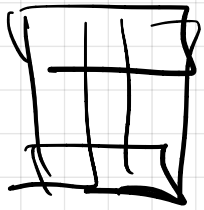
does not depend on n.

It would suffice to cross the rectangle:

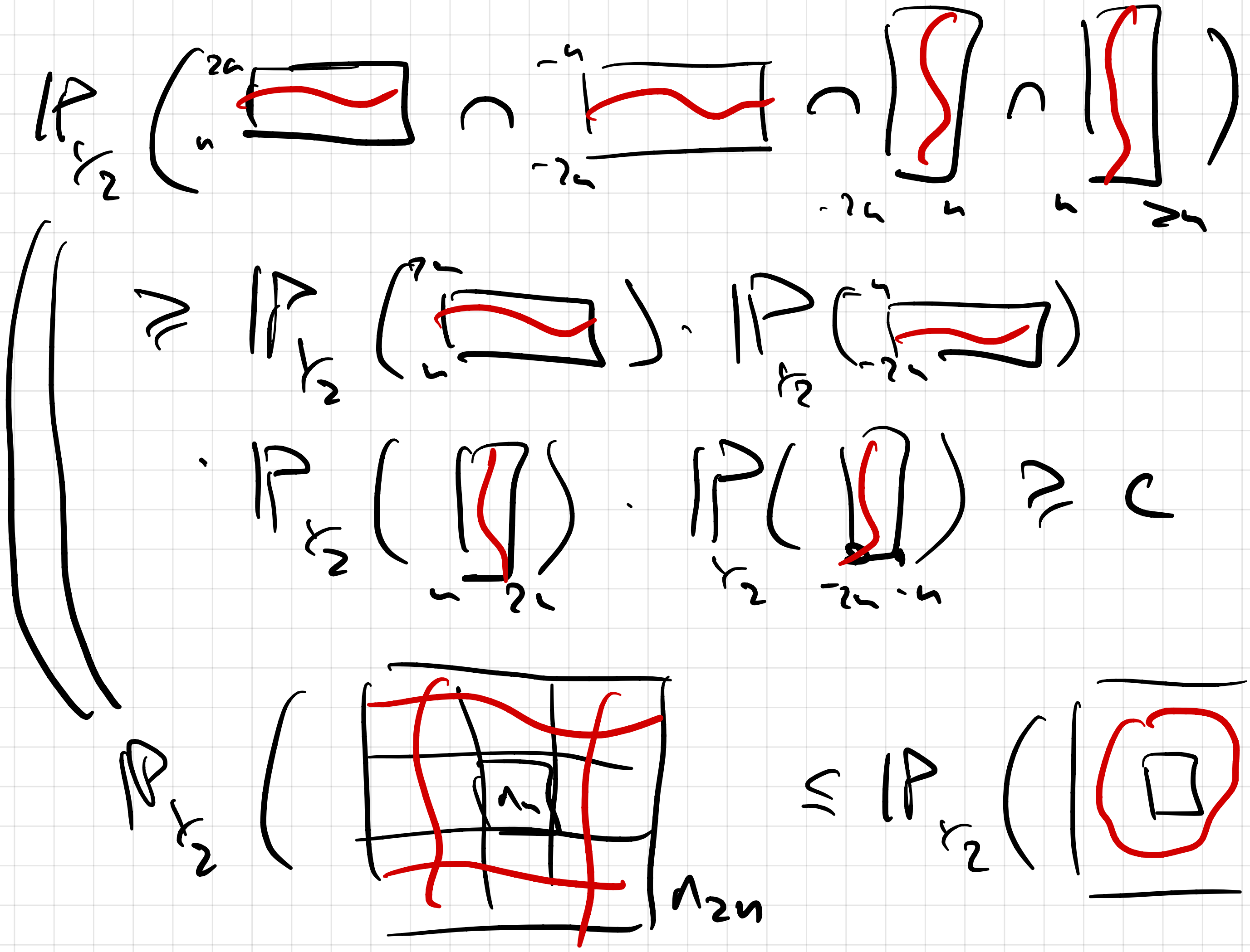


$\geq c^{1/4}$

Indeed, this is an increasing event. Consider its rotation by $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$:



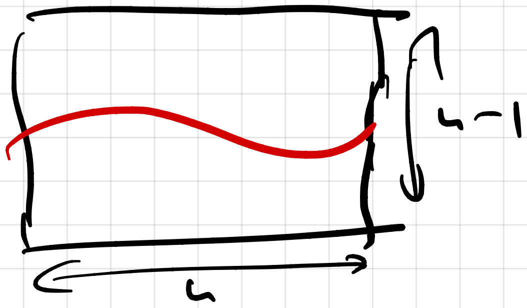
Apply Harris' inequality:



Conclusion:

it remains to cross a long rectangle.

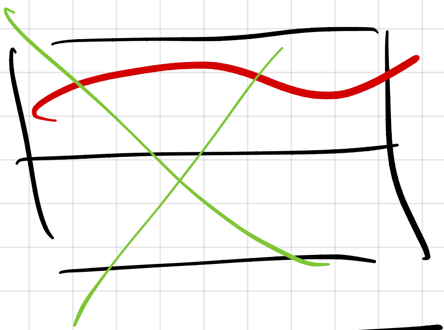
So far we know only how to cross a square:

$\mathbb{P}_{1/2}\Bigg($  $\Bigg) = \frac{1}{2}$

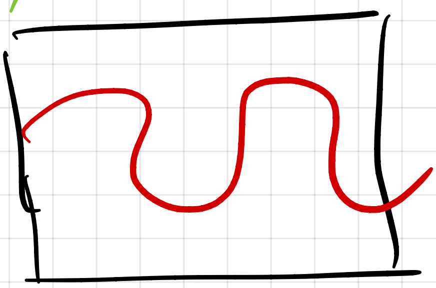
Going from a square to a long rectangle is very non-trivial.

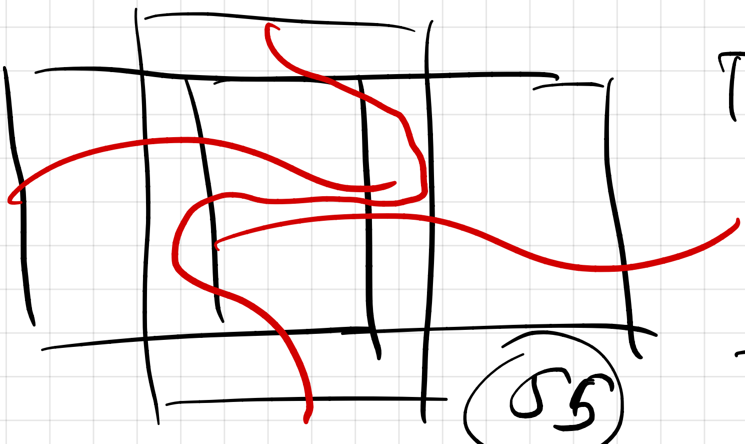
Wrong ideas:

- cut the square:



Q: why would it stay above?



- 

There is many counter examples...

Thm (Russo '78, Seymour-Welsh '78)

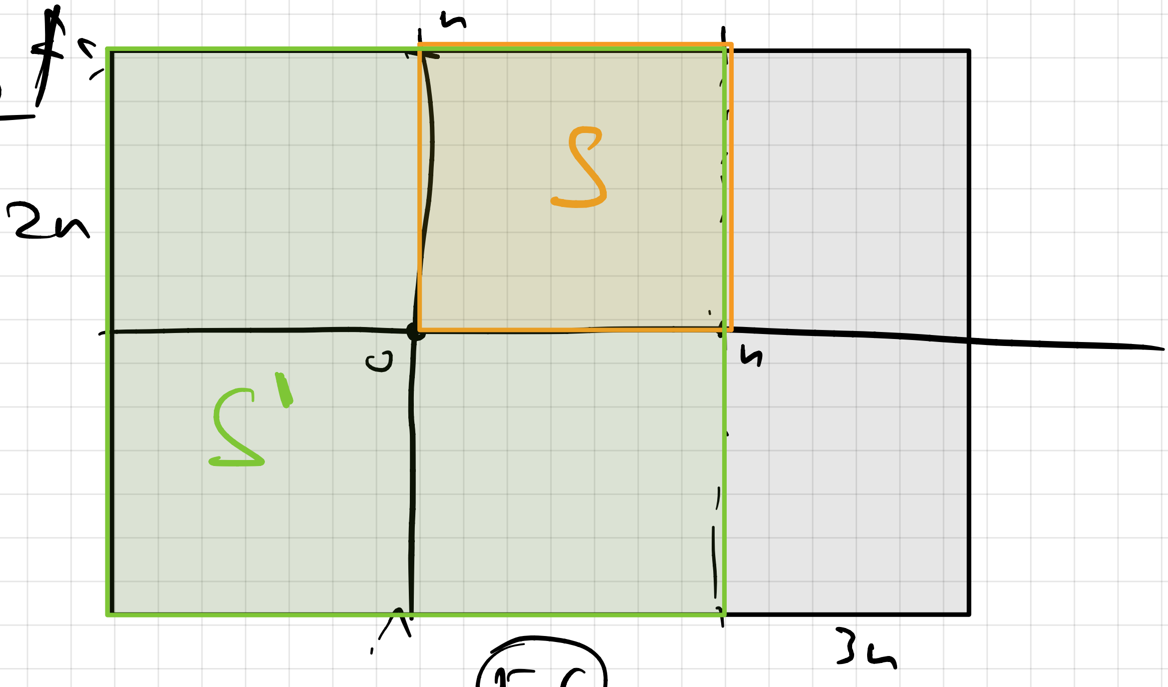
There exists $c > 0$, s.t. for any $n \in \mathbb{N}$,

$$\mathbb{P}_{1/2}\left(\vphantom{\int_0^1} \right) \geq c.$$

In fact, it's enough to take $c = \frac{1}{128}$.

We will discuss a more recent proof due to Bollobás-Riordan from 2006.

Proof:



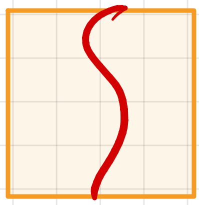
$S := [0,n] \times [0,n]$

$S' := [-n,n] \times [-n,n]$

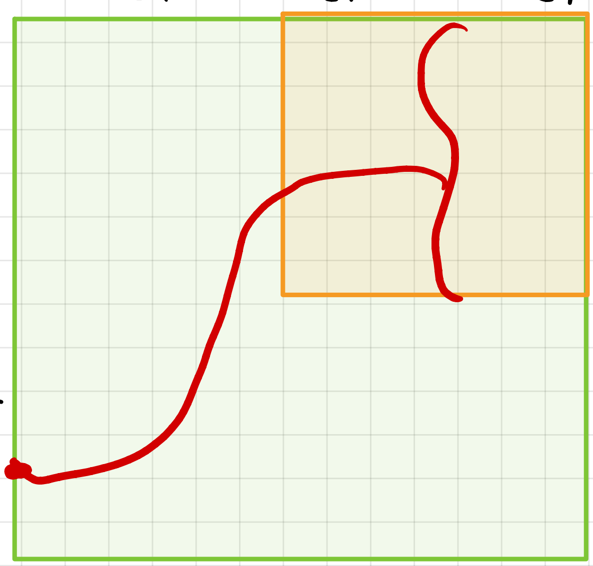
$R := [-n,2n] \times [-n,n]$.

Define the events:

$A := \{ S$ is crossed vertically $\}$

$= \{$  $\}$

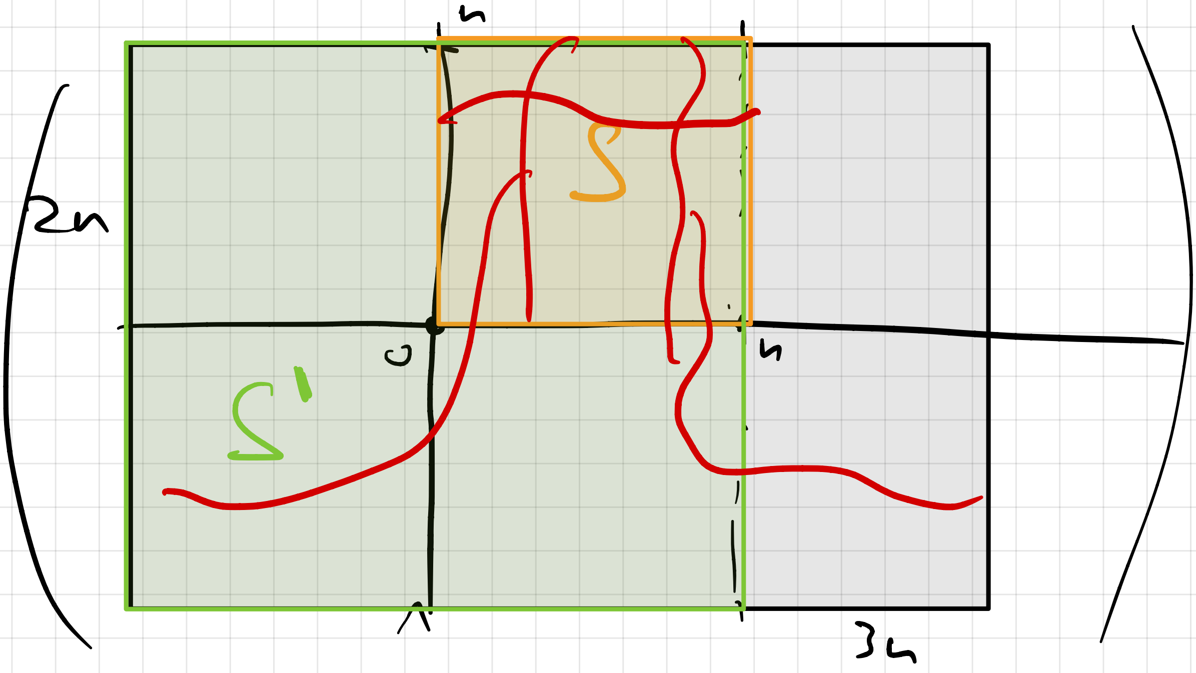
$B := \{$ exists a vertical crossing of S that is linked to the left side of $S' \}$

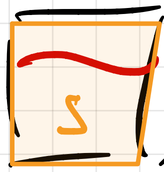
$= \{$  $\}$

Once we prove that

$\mathbb{P}_{1/2}(B) \geq c$ we are done:

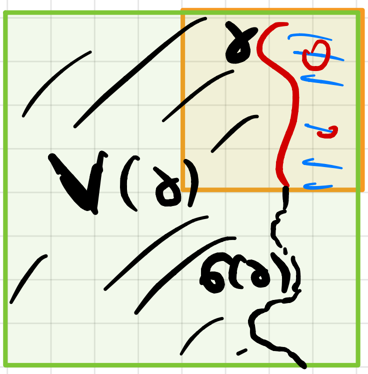
$\geq \frac{1}{8}$ (57)

$\mathbb{P}_{1/2}\Bigg($  $\Bigg)$

$\geq \underbrace{\mathbb{P}_{1/2}(B)}_{\text{Harris}} \cdot \underbrace{\mathbb{P}_{1/2}(B')}_{=\text{ reflection of } B} \cdot \underbrace{\mathbb{P}_{1/2}\Big(}_{} $  $\Big) = \frac{1}{2}$

$\geq \frac{1}{64} \cdot \frac{1}{2} = \frac{1}{128}$

How to estimate B?
We'll consider a vertical crossing of S (exists w.p. $\frac{1}{2}$) and explore it.



We consider the right-most crossing. Denote this crossing by Γ

Let γ be its realization.
We define:

- $\sigma(\gamma)$ – reflection of γ.
- $V(\gamma)$ – the part of S' to the left of $\gamma \cup \sigma(\gamma)$.

Consider all possible values of γ:

$$\mathbb{P}_{\gamma_2}(B) = \sum_{\gamma \subset S} \mathbb{P}_{\gamma_2}(B \mid A \cap \{\Gamma = \gamma\}) \cdot \mathbb{P}_{\gamma_2}(A \cap \{\Gamma = \gamma\})$$

$$\geq \sum_{\gamma \subset S} \underbrace{\mathbb{P}_{\gamma_2}\left(\gamma \xleftrightarrow{V(\gamma)} \text{left of } S'\right)}_{\geq \frac{1}{2} \mathbb{P}_{\gamma_2}\left(\gamma \cup \sigma(\gamma) \xleftrightarrow{V(\gamma)} \text{left of } S'\right)} \cdot \mathbb{P}_{\gamma_2}(A \cap \{\Gamma = \gamma\})$$

$$\geq \frac{1}{2} \cdot \mathbb{P}_{1/2}\Bigg($$ 

$$\Bigg)$$

$= \frac{1}{2}$

Overall,

$$\mathbb{P}_{1/2}(B) \geq \frac{1}{2} \cdot \frac{1}{2} \cdot \sum_{\gamma \subset S} \mathbb{P}_{1/2}(A \cap \{\Gamma = \gamma\})$$

$$= \frac{1}{4} \cdot \mathbb{P}_{1/2}(A) \geq \frac{1}{8}.$$

We get the estimate that we wanted.

$\square$

Rem.

Note that $S \neq n \times (n-1)$ and

$$\mathbb{P}_{1/2}\Big(n \, \underbrace{\boxed{\quad S \quad}}_{n} \Big) \geq \frac{1}{2}.$$

Corollary (Box-crossing property)

Let $\rho > 0$. Then, there exists $c = c(\rho) > 0$, s.t. for any $n \in \mathbb{N}$,

$$\mathbb{P}_{1/2}\Big(n \, \overbrace{\boxed{\qquad\qquad\qquad}}^{\rho n} \Big) \geq c.$$

Proof:

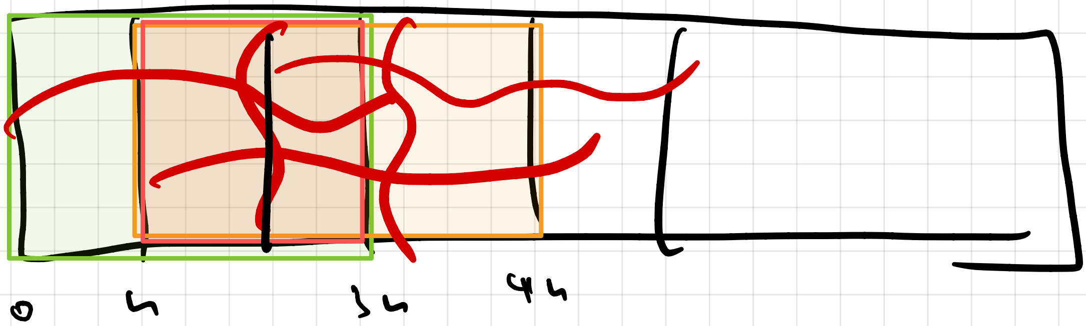
If $p \le 3/2$, then see the theorem.

Enough to consider only $p = k + \frac{1}{2}$, where $k \in \mathbb{N}$, $k > 1$.

Take the rectangles:

$$R_i := [(i-1)n, (i+2)n] \times [-n, n],$$

$$i = 1, \dots, 2k-1.$$



For every R_i:

$$\mathbb{P}_{1/2}(H(R_i)) \ge c$$

horizontal crossing of R_i

Take the squares S_i:

$$S_i := [in, (i+2)n] \times [-n, n]$$

Combine

In total,

$$\mathbb{P}_{1/2}\Big(H\big([0,\underbrace{(2k+1)n}_{\rho\cdot 2n}]\times[-n,n]\big)\Big)$$

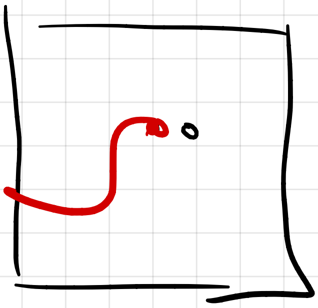
$$\geq \mathbb{P}_{1/2}\Big(\Big[\bigcap_{i=1}^{2k-1} H(R_i)\Big]\cap\Big[\bigcap_{i=1}^{2k-2} V(S_i)\Big]\Big)$$

Harris' ineq.

$$\geq c^{2k-1}\cdot\left(\tfrac{1}{2}\right)^{2k-2}$$

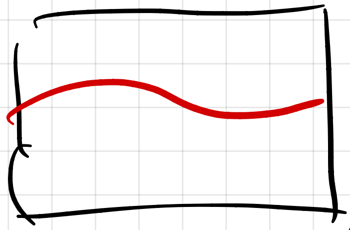
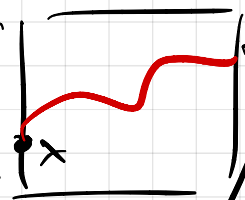
Corollary (one-arm exponent)

There exists $0<\alpha\leq 1$, s.t., for any $n\geq 1$,

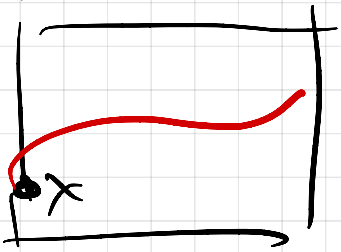
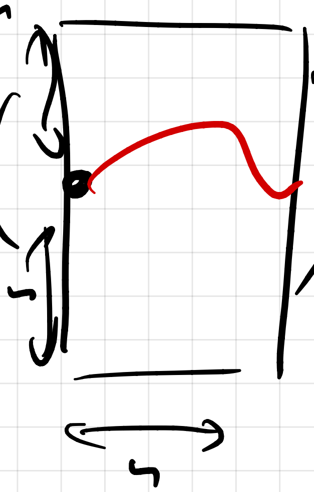
$$\frac{1}{2n} \leq \mathbb{P}_{1/2}\Big($$  $$\Lambda_n\Big) \leq \frac{1}{n^\alpha}$$

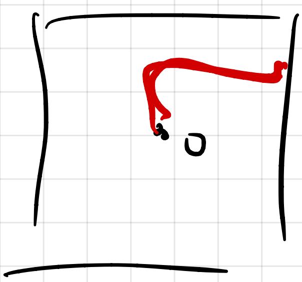
Proof:

Lower bound:

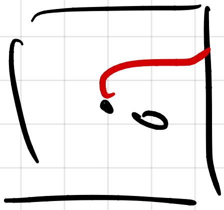
$$\frac{1}{2} \leq \mathbb{P}_{\frac{1}{2}}\Big($$  $$\Lambda_{n/2}\Big) \leq \sum_{x=-n/2}^{n/2} \mathbb{P}_{1/2}\Big($$  $$\Big)$$

For any x,

$\mathbb{P}_{1/2}\Big($  $\Big) \leq \mathbb{P}_{1/2}\Big($  $\Big)$

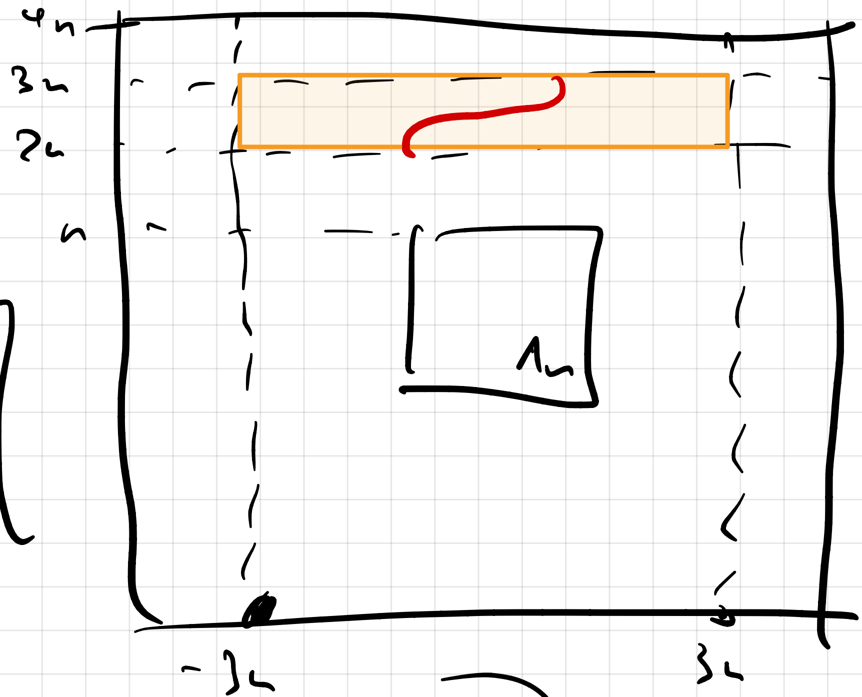
$\leq \mathbb{P}_{1/2}\Big($  $\Big)$

Thus,

$\frac{1}{2} \leq n \cdot \mathbb{P}_{1/2}\Big($  $\Big)$

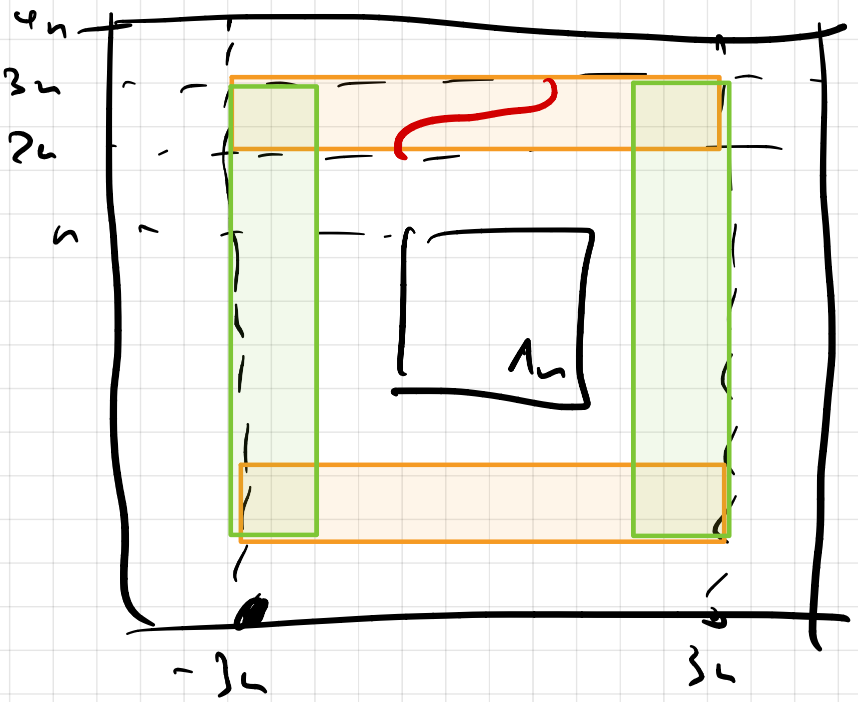
Upper bound:

Consider the event:

$A := \Big\{$  $\Big\}$

Consider also the tree rotations of this event:

$R_{\frac{\pi}{2}} \circ A, \; R_{\pi} \circ A, \; R_{\frac{3\pi}{2}} \circ A.$



If all these rectangles are crossed in a long direction in ω^*,

then $0 \not\leftrightarrow \partial\Lambda_{4n}$.

Hence,

$$\{\Lambda_n \leftrightarrow \partial\Lambda_n\} \subset A \cup (R_{\frac{\pi}{2}} \circ A) \cup (R_{\pi} \circ A) \cup (R_{\frac{3\pi}{2}} \circ A)$$

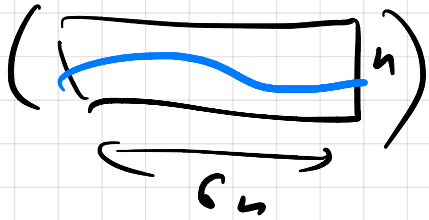
By the square-root trick:

(67)

$$\mathbb{P}(\Lambda_n \longleftrightarrow \partial\Lambda_{4n}) \le 1 - \Big(\underbrace{1 - \mathbb{P}_{p_c}(A)}_{}\Big)^{1/4}$$

RSW

$$c \le \mathbb{P}\Big(\text{crossing of } 6n \times n \text{ rectangle}\Big)$$



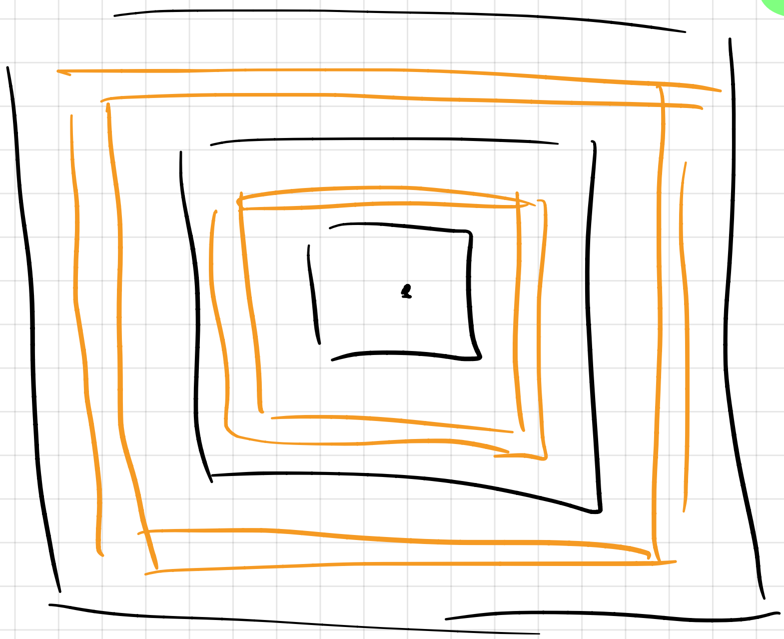
Then, for any n:

$$\mathbb{P}(\Lambda_n \longleftrightarrow \partial\Lambda_{4n}) \le 1 - c^{1/4}$$

Assume $4^k \le n < 4^{k+1}$.
Apply the estimate on these k scales:

$$\{0 \longleftrightarrow \partial\Lambda_n\} \subseteq \bigcap_{i=0}^{k-1} \{\Lambda_{4^i} \longleftrightarrow \Lambda_{4^{i+1}}\}$$

independent



Then,

$$\mathbb{P}(0 \longleftrightarrow \partial \Lambda_n) \leq (1-c'/4)^{k} \qquad k = \underbrace{\lfloor \log_4 n \rfloor}_{n^{-\alpha}}$$

$\blacksquare$