Lecture 1. Estimates of Buffon needle probability from below

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Based on works with Michael Bateman

Table of content:

For bounded Borel set $K \subset \mathbb{R}^n$ and $s > 0$, consider

$$\mathcal{H}^s(K) = \lim_{\delta \to 0} \inf\{\sum_i r_i^s : K \subset \bigcup_i B(x_i, r_i), r_i \le \delta\}.$$

It can easily be ∞ or 0.

$$\dim K := \dim_H K := \inf\{s : \mathcal{H}^s(K) = 0\} = \sup\{s : \mathcal{H}^s(K) = \infty\}.$$

Figure:

For $0 < d > \frac{1}{2}$, let C_d be **the four corner Cantor set**:

$$C_d = \bigcap_{k=1}^{\infty} C_{d,k}, \quad C_{d,k} := \bigcup_{j=1}^{4^k} Q_k^j ,$$

where each Q_k^j is a closed square of side-length d^k, and they are defined as follows. First Q_1^j are 4 squares in the corners of the unit square $[0,1] \times [0,1]$, namely

$[0,d]\times[0,d]$, $[0,d]\times[1-d,1]$, $[1-d,1]\times[0,d]$, $[1-d,1]\times[1-d,1]$.

Figure:

Next generations are built the same way inside the previous ones.

C. Hausdorff dimension and measure of corner sets

Define s, $s = s_d$ as follows

$$4d^s = 1, \quad s = \frac{\log 4}{\log \frac{1}{d}}\,.$$

This is the Hausdorff dimension of C_d. Easy to see. Moreover,

$$s_d = \dim C_d, \quad 0 < \mathcal{H}^{s_d}(C_d) < \infty\,.$$

Measure H^{s_d} on C_d is called Cantor measure. It can be viewed as an analog of Lebesgue measure but on fractal set C_d.

D. Hausdorff dimension and Hausdorff measure of the support of projections

We are interested at "pushing forward" this measure to family of lines $\{\ell_\theta\}_{0\le\theta<\pi}$ in $\mathbb{R}^2$.
Pushing forward is done by orthogonal projection

$$Proj_\theta : \mathbb{R}^2 \to \ell_\theta \,.$$

But we are interested not so much in push forward measure itself, but rather in its **support**!

Questions: What is the Hausdorff dimension of this **support**?

If we find it, the next question is what is the Hausdorff measure (in that dimension) of the support?

Is it positive or 0?

E. Marstrand–Mattila theorem

Below V stand for m-dimensional linear subspace of $\mathbb{R}^n$, $G(n,m)$ the family of all such V, a grassmanian. It has a natural measure on it. P_V is the orthogonal projection $\mathbb{R}^n \to V$.

Theorem (Marstrand–Mattila)

Let $K \subset \mathbb{R}^n$ be a Borel set with $\dim A = s$. $0 < m \le n$. Then

- *If $s \le m$, $\dim P_V(K) = s$ for almost all $V \in G(n,m)$.*
- *If $s > m$, $\dim P_V(K) = m$ for almost all $V \in G(n,m)$.*

More subtle is the following. What happens with

$$\mathcal{H}^s(P_V(K)), s \le m; \quad \mathcal{H}^m(P_V(K)), s > m?$$

Consider $n = 2$, $m = s = 1$. We know when borderline case $0 < \mathcal{H}^1(Proj_\theta K)$ happens for a.e. θ for a given K: $\mathcal{H}^1(K) < \infty$.

Besicovitch–Federer theory.

F. Theorems of Kaufman and Falconer

Theorem (Kaufman)

Let $K \subset \mathbb{R}^2$, $\dim K = s$. If $s \leq 1$ then $\dim Proj_\theta K = s$ with a possible exceptional set of θ of dimension $\leq s$.

Interesting and more precise than Marstrand–Mattila theorem if $s < 1$, but worse than Marstrand–Mattila theorem if $s = 1$.

Theorem (Falconer)

Let $K \subset \mathbb{R}^2$, $\dim K = s$. If $s > 1$ then $Proj_\theta K = s$ has zero Lebesgue measure with a possible exceptional set of θ of dimension $\leq 2 - s$.

Again the borderline case $s = 1 = m$ is left over.

Left to Besicovitch–Federer theory of rectifiability.

G. Kaufman, Bourgain, Shmerkin, Huang Wang, Orponen

Kaufman's theorem can be reformulated as follows

Theorem (Kaufman, 1968)

Let $K \subset \mathbb{R}^2$, $\Lambda \subset \mathbb{T}$, $0 < \dim K < \dim \Lambda$. *Then there exists* $\theta \in \Lambda$ *such that* $\dim Proj_\theta K = \dim K$.

What if $\dim \Lambda > 0$ but $\dim \Lambda \leq \dim K$?

Theorem (Bourgain, additive combinatorics)

Let $\dim K = s, \dim \Lambda = t > 0$. *Then there exists* $\eta(s,t) > 0$ *and* $\theta \in \Lambda$ *such that* $\dim Proj_\theta K = \frac{1}{2} \dim K + \eta(s,t)$.

Shmerkin, Huang Wang. Not just for linear projections.

Orponen, $\dim K > 1$.

H. Let us return to Cantor sets C_d

Consider first $d \leq \frac{1}{4}$, so $s_d \leq 1$. We know by Marstrand–Mattila theorem that

$$\dim Proj_\theta C_d = s_d \quad \text{for almost all } \theta,$$

(a bit better by Kaufman). But what about $\mathcal{H}^{s_d}(Proj_\theta(C_d))$?

When $d = \frac{1}{4}$ Besicovitch theory claims

Theorem (Besicovitch)

$\mathcal{H}^1(Proj_\theta(C_{\frac{1}{4}})) = 0$ *for almost all* θ.

Peres–Simon–Solomyak: if $\frac{1}{6} < d < \frac{1}{4}$, then the same holds:

$$\mathcal{H}^{s_d}(Proj_\theta(C_d)) = 0 \quad \text{for almost all } \theta, \; s_d = \frac{\log 4}{\log \frac{1}{d}}.$$

Strangely enough

$$\text{if } d < \frac{1}{9} \text{ then } \mathcal{H}^{s_d}(Proj_\theta(C_d)) > 0 \quad \text{for almost all } \theta,\ s_d = \frac{\log 4}{\log \frac{1}{d}}\,.$$

Look at $C_d - C_d$. When $d \geq \frac{1}{3}$, it is a full square. When $d < \frac{1}{3}$ it is a self-similar Cantor set of dimension $\frac{\log 9}{\log \frac{1}{d}}$. Hence, if $d < \frac{1}{9}$, we have

$$\dim(C_d - C_d) < 1 \Rightarrow \mathcal{H}^1(C_d - C_d) = 0\,.$$

Then trivially the "angle projection" set $\{(x-y)/|x-y| : x, y \in C_d\} = \{X/|X| : X \in C_d - C_d\} \subset \mathbb{T}$ has zero Lebesgue measure. So almost every line does not intersect C_d in 2 points, only in 1 point or not at all. This shows that for a. e. θ, $Proj_\theta : C_d \to \ell_\theta$ is **injective**. And actually with Lipschitz inverse! Then for a. e. θ

$$\mathcal{H}^{s_d}(Proj_\theta(C_d)) > 0$$

J. Open question

1. What happens for $\mathcal{H}^{s_d}(Proj_\theta(C_d))$ for almost every θ if $\frac{1}{9} \leq d \leq \frac{1}{6}$?
2. Quantitative Besicovitch theorem. By Besicovitch theorem above we know that for $K = C_{\frac{1}{4}}$

$$Fav(K) := \int_0^\pi |Proj_\theta(K)| d\theta = 0 \,.$$

   Denote by $\mathcal{K}_n$ the n-th generation of Cantor construction: 4^n square of side-length 4^{-n}. Obviously $Fav(\mathcal{K}_n) := \int_0^\pi |Proj_\theta(\mathcal{K}_n)| d\theta = \varepsilon_n \to 0$. What is the rate of decrease of ε_n?
3. Let $\mathcal{B}_n$ be a family of closed disjoint 4^n square of side-length 4^{-n} inside the unit square. For which $\mathcal{B}_n$ $Fav\mathcal{B}_n$ is minimal? What is this minimum?

1. Notations

We have L^n closed discs of radius L^{-n}, disjoint, moreover **in Cantor order**. Meaning that this is the n-th generation of construction of Cantor set with base L. Call these discs $Q_n^j, j = 1, \dots, L^n$. $\mathcal{K}_n := \bigcup_{j=1}^{L^n} Q_n^j$.

Let $\theta \in [0, \pi]$ and

$$f_{\theta,n}(x) := \sum_{k=1}^{L^n} \mathbf{1}_{Proj_\theta(Q_n^j)}(x) =: \sum_{k=1}^{L^n} \mathbf{1}_{\theta, Q_n^j}(x)$$

where $Proj_\theta$ denotes orthogonal projection onto the line ℓ_θ having angle θ with $\mathbb{R}$.

$$\mathcal{L}_{\theta,n} := \text{supp } f_{n,\theta} \,.$$

2. A very simple estimate

$$\int_0^{\pi} \mathcal{L}_{\theta,n} d\theta \geq \frac{c}{n}\,.$$

Indeed, we have L^n characteristic functions of segments each of length $\asymp L^{-n}$, hence by Hölder in x and then in θ:

$$1 \asymp \int\int f_{\theta,n}(x) dx d\theta \leq \int |\mathcal{L}_{\theta,n}|^{1/2} \Big(\int f_{\theta,n}(x)^2 dx\Big)^{1/2} d\theta \leq$$

$$\Big(\int |\mathcal{L}_{\theta,n}| d\theta\Big)^{1/2} \Big(\int\int f_{\theta,n}(x)^2 dx d\theta\Big)^{1/2} \Rightarrow$$

$$Fav(\mathcal{K}_n) := \int |\mathcal{L}_{\theta,n}| d\theta \geq \frac{c}{\int\int f_{\theta,n}(x)^2 dx d\theta}\,.$$

$$\int\int f_{\theta,n}(x)^2 dx d\theta = \sum_{Q,Q'} \int \mathbf{1}_{\theta,Q}(x) \mathbf{1}_{\theta,Q'}(x) dx d\theta\,.$$

3. Pairs of discs interacting

Let $P \in \mathcal{P}$ denotes pairs of discs.

$$\int\int f_{\theta,n}(x)^2 dx d\theta = \sum_{Q,Q'} \int \mathbf{1}_{\theta,Q}(x)\mathbf{1}_{\theta,Q'}(x) dx d\theta =$$

$$\sum_{P\in\mathcal{P}} \int |Proj_\theta Q \cap Proj_\theta Q'| d\theta =: \sum_{k=1}^{n} \sum_{P\in\mathcal{P}_k} \rho_P,$$

where $\mathcal{P}_k$ are pairs that are together in the disc of radius L^{-k} (of k-th Cantor generation), but not together in the disc of radius L^{-k-1} (of $k+1$-th Cantor generation). The cardinality of $\mathcal{P}_k$ is at most $L^k \cdot (L^{n-k})^2$.

For every $P \in \mathcal{P}_k$ we have that ρ_P counts the measure of angles of lines $\ell_\theta^\perp$ intersecting both discs of P:

$$\rho_P = \int |Proj_\theta Q \cap Proj_\theta Q'| d\theta \le L^{-n} \cdot \frac{L^{-n}}{L^{-k}}$$

Hence,

$$\sum_{k=1}^{n} \sum_{P \in \mathcal{P}_k} \rho_P \lesssim \sum_{k=1}^{n} L^k \cdot (L^{n-k})^2 \cdot L^{-2n} L^k = \sum_{k=1}^{n} 1 = n \,.$$

Therefore,

$$Fav(\mathcal{K}_n) \geq \frac{c}{n} \,.$$

5. Energy approach

Theorem (Pertti Mattila)

Again the set is as before. Then $\int_0^\pi \frac{1}{|\mathcal{L}_{\theta,n}|} d\theta \le Cn$.

Notice

$$\frac{\varepsilon}{|z-\zeta|} \asymp |\{\theta : Proj_\theta z - Proj_\theta \zeta| \le \varepsilon\}| \,.$$

Let μ_n be uniform Cantor measure on $\mathcal{K}_n$. Then if $\Phi_\varepsilon(t) = 1, 0 \le t \le \varepsilon$ and $= 0$ $t > \varepsilon$:

$$\int\int \frac{\varepsilon}{|z-\zeta|} d\mu_n(z) d\mu_n(\zeta) = \int d\theta \int\int \Phi_\varepsilon(|Proj_\theta z - Proj_\theta \zeta|) d\mu_n(z) d\mu_n$$

$$\int d\theta \int\int \Phi_\varepsilon(|x-y|) d\mu_{\theta,n}(x) d\mu_{\theta,n}(y) \quad \text{and}$$

$$\int \Phi_\varepsilon(|x-y|) d\mu_{\theta,n}(y) = \mu_{\theta,n}(B(x,\varepsilon)), \quad d\mu_{\theta,n} = f_{\theta,n}(x) dx \,.$$

Therefore, for every ε

$$\int\int \frac{\mu_{\theta,n}(B(x,\varepsilon))}{\varepsilon} f_{\theta,n}(x)dxd\theta \le \int\int \frac{1}{|z-\zeta|} d\mu_n(z)d\mu_n(\zeta) \le^{easy} Cn$$

By Fatou's lemma ($\varepsilon \to 0$, $\liminf_{\varepsilon\to 0}$)

$$\int\int f^2_{\theta,n}(x)dxd\theta \le Cn.$$

By Hölder

$$\int \frac{1}{|\mathcal{L}_{\theta,n}|} d\theta \le \int d\theta \frac{\int_{\mathcal{L}_{\theta,n}} f_{\theta,n}(x)^2 dx}{(\int_{\mathcal{L}_{\theta,n}} f_{\theta,n}(x)dx)^2} \le Cn\,.$$

7. 4-corner Cantor set estimate from below is better

Theorem (Bateman–A.V)

Let cK_n is the n-th generation of the 4-corner Cantor set. Then $Fav(\mathcal{K}_n) = \int |\mathcal{L}_{\theta,n}| d\theta \geq c\frac{\log n}{n}$.

Tiling:

Projections of Cantor squares on line $\ell_{\arctan \frac{1}{2}}$ form precisely 4-adic lattice. Projections of each generation squares do not intersect and tile. We consider pairs $P = (Q, Q') \in \mathcal{P}_k$ as before and split $\mathcal{P}_k$ to $\mathcal{A}_{k,j}$, $j = 0, \dots, \log n$ as follows:

$$(Q_1, Q_2) \in \mathcal{A}_{k,j} \quad \text{iff projections } s_1, s_2 : \mathrm{dist}(s_1, s_2) \asymp 4^{-k-j} .$$

In other words four-adic distance is 4^{-k} but Euclidean distance is 4^{-k-j}. Geometrically this means that there exists the line crossing Q_1, Q_2 orthogonal to the line ℓ_θ such that

$$\text{angle}(\theta, \arctan\frac{1}{2}) \in (4^{-j-1}, 4^{-j}]\,.$$

Let the set of such angles θ be denoted by Θ_j.

Figure:

How many 4^{-n} intervals are such that four-adic distance is 4^{-k} but Euclidean distance is 4^{-k-j}? Those pairs of intervals should be separated by 4-adic point of generation $1, 2, \ldots, k$. There are 4^m 4-adic points of generation m (the ends of 4-adic intervals of length 4^{-m}). So

$$\sharp \mathcal{A}_{k,j} \le \sum_{m=0}^{k} 4^m (4^{-k-j}/4^{-n})^2 \le C 4^{2n-k-2j} \quad \text{if } k \le n-j;$$

$$\sharp \mathcal{A}_{k,j} \le \sum_{m=0}^{k} 4^m C \le C 4^k \quad \text{if } k \ge n-j\,.$$

In fact, if $k + j > n$ then $|s_1 - s_2| \le C4^{-k-j} \le C4^{-n}$. So 4^{-n} intervals s_1, s_2 are practically neighbors, so for every 4-adic point of generation $1, 2, \ldots, k$ there are at most C of them.

10. Continue bookkeeping

We saw that the angle between squares in $\mathcal{P}_k$ pair is $\leq C4^{-n}/4^{-k}$

Figure:

$$\text{Thus} \quad \rho_P = \int |Proj_\theta Q \cap Proj_\theta Q'| d\theta \leq C4^{-2n}4^k$$

Let $\mathcal{A}'_j := \cup_{k=1}^n \mathcal{A}_{k,j}$. Then $(j = 0, \dots \log n)$

$$\sum_{P \in \mathcal{A}'_j} \rho_P \leq C \sum_{k=1}^{n} 4^{-2n}4^k \cdot \sharp \mathcal{A}_{k,j} \leq C \sum_{k=1}^{n-j} 4^{-2n}4^k \cdot 4^{2n-k-2j} + \dots \leq$$

$$\frac{Cn}{4^{2j}} + C \leq \frac{Cn}{4^{2j}},$$

if $j \leq C'' \log n$.

11. Observation

Let $\mathcal{A}_j = \mathcal{A}'_j \cup_{Q \text{ of size } 4^{-n}} (Q, Q)$. Pair $P \in \mathcal{A}_j$ if and only if the lines intersection both squares of P are orthogonal to some $\theta \in \Theta_j$.

Figure:

Hence,

$$\int_{\Theta_j} \int f_{\theta,n}(x)^2 dx d\theta \le \sum_{P \in \mathcal{A}_j} \rho_P \le \frac{Cn}{4^{2j}} + \int_{\Theta_j} \int \sum_{P=(Q,Q)} \mathbf{1}_{\theta,Q} dx d\theta \le$$

$$\frac{Cn}{4^{2j}} + |\Theta_j| \le \frac{Cn}{4^{2j}} + 4^{-j} \le \frac{C'n}{4^{2j}},$$

if $j \le C'' \log n$. Then by Hölder again

$$\int_{\Theta_j} |\mathcal{L}_{\theta,n}| d\theta \ge \frac{\left(\int_{\Theta_j} f_{\theta,n}(x) dx d\theta\right)^2}{\int_{\Theta_j} f_{\theta,n}(x)^2 dx d\theta} \ge \frac{|\Theta_j|^2}{C'n4^{-2j}} \ge c\frac{4^{-2j}}{n4^{-2j}} \ge \frac{c}{n}.$$

$\int_0^\pi |\mathcal{L}_{\theta,n}| d\theta \ge C'' \log n \cdot \frac{c}{n}$. We proved that $Fav(\mathcal{K}_n) \ge \frac{c \log n}{n}$.

If $\dim \mathcal{K} = s > 1$ then by Frostman lemma there exists strictly positive measure on $\mathcal{K}$ such that

$$\mu(B(x, r)) \le r^s \,.$$

Theorem (Pertti Mattila)

If $\dim \mathcal{K} = s > 1$ then almost any projection of $\mathcal{K}$ has positive Lebesgue measure.

We already "almost" saw it. In fact, Frostman measure ensures that energy

$$\mathcal{E}[\mu] := \int_{\mathcal{K}} \int_{\mathcal{K}} \frac{1}{|z - \zeta|} d\mu(z) d\mu(zeta) \le C < \infty \,.$$

Let μ_θ is push forward of μ onto $\ell_t heta$ under the projection map.
Let $d\mu_\theta(x) = f_\theta(x) + d\mu_{\theta, sing}(x)$.
We saw above that $\int d\theta \int f_\theta(x) d\mu_\theta(x) \le \mathcal{E}[\mu] < \infty$. This means that abs. continuous part of μ_θ is L^2 integrable for almost every θ.
Actually $d\mu_{\theta, sing} = 0$ a.e. θ.

13. Buffon needle landing near a random Cantor set

We consider two models of randomness for Cantor sets. We are going to show that in both models

$$\mathbb{E} Fav(\mathcal{K}_n) \le \frac{C}{n}\,.$$

By Fatou' s lemma this implies immediately that

$$\liminf_{n\to\infty} n\, Fav(\mathcal{K}_n) < \infty \quad \text{almost surely}\,.$$

First model: $\mathcal{K}_0 = B(0,1)$, $\mathcal{K}_1 = \bigcup_{j=1}^4 B(\frac{3}{4}e^{i\frac{\pi}{2}j}, \frac{1}{4})$, $\mathcal{K}_2 = \bigcup_{j=1}^4 T_j(\mathcal{K}_1)$ .... . Self similar repetition. Here

$$T_j(z) = \frac{1}{4}z + e^{i\frac{\pi}{2}j}.$$

This would be a standard 4-corner Cantor set build on the square of side $1/\sqrt{2}$. We saw above that

$$Fav(\mathcal{K}_n) \ge \frac{c\,\log n}{n}\,.$$

But now introduce randomness: rotate all discs of $\mathcal{K}_1$ by angle θ_1 around the center of $\mathcal{K}_0$, rotate all disc of $\mathcal{K}_2$ by angle θ_2 around corresponding centers of $\mathcal{K}_1$, ... The probability spaces is $\prod_{k=1}^{\infty}[0, 2\pi)$ provided with product of normalized measures.

Figure:

Let $\omega := (\theta_1, \theta_2, \dots)$. And let $\mathcal{K}_k(\omega)$ is a corresponding collection of discs of radius 4^{-k}. Then we show that

$$\mathbb{E}_\omega Fav(\mathcal{K}_n) \le \frac{C}{n} \quad \text{proved by Shiwen Zhang}\,.$$

15. Main lemma

Figure:

Lemma

Consider $K_0 = B(0,r), K = \bigcup_{j=1}^4 B(\frac{3}{4}e^{i\frac{\pi}{2}j}r, b),\ b \le \frac{r}{4}$. *Then*

$$\mathbb{E}_\theta |Proj_x(e^{i\theta}\mathcal{K}_1)| \le 4b - \frac{c_0}{r} b^2 \,.$$

Notice that

$$\mathbb{E} Proj_x(e^{i\theta}\mathcal{K}_1)) = \sum_{j=1}^{4} Proj_x(B_j) - \mathbb{E}(\text{overlaps of projections}) \le$$

$$4b - \mathbb{E}_\theta(\text{overlaps of projections of } \ e^{i\theta}B_1 \text{ and } e^{i\theta}B_4)$$

16. The estimate of the overlap from below

On slides above, when we estimated $Fav(\mathcal{K})$ **from below** we needed the estimates of overlaps **from above**. Now we need a converse argument: we are estimating $Fav(\mathcal{K})$ **from above**, so we need to estimate overlaps **from below**.

Let I be interval $[-b, b]$ and J be interval $[\frac{3}{4}r - b, \frac{3}{4}r + b]$. Let $\rho := \frac{3}{4}r$.

$$Proj(e^{i\theta}B_1) = I + \rho\sin\theta, \quad Proj(e^{i\theta}B_4) = J - \rho\cos\theta\,.$$

$$\text{Overlap} = |(I+\rho\sin\theta)\cap(J-\rho\cos\theta)| = \int \mathbf{1}_I(x-\rho\sin\theta)\cdot\mathbf{1}_J(x+\rho\cos\theta)dx$$

$$\mathbb{E}_\theta(Overlap) = \int\int \mathbf{1}_I(x-\rho\sin\theta)\mathbf{1}_J(x+\rho\cos\theta)dxd\theta =$$

$$\int\int \mathbf{1}_I(u)\mathbf{1}_J(v)\det\left(\frac{\partial(x,\theta)}{\partial(u,v)}\right)dudv \geq \frac{1}{2\rho}\int \mathbf{1}_I(u)\int \mathbf{1}_J(v) = \frac{b^2}{2\rho} = \frac{c_0 b^2}{r}$$

17. The end of proof of Shiwen Zhang's theorem

Let us compare $|Proj_x \mathcal{K}_n|$ and $|Proj_x \mathcal{K}_{n-1}|$.

Collection $\mathcal{K}_n$ consists of dilated by $1/4$ $\mathcal{K}_{n-1}$-copies, rotated by θ_2 around the centers of B_j.

Those $\mathcal{K}_{n-1}$-copies are actually $\mathcal{K}_{n-1}(\bar{\theta}_n)$ copies, where $\bar{\theta}_n = (\theta_2, \dots, \theta_n)$ is independent of θ_2.

We are in the assumptions of Lemma above with $r = 1$, $b = 4^{-1}$. But instead of intervals I, J we have sets E, F that are projections of dilated by $1/4$ shifted sets $Proj_x(\mathcal{K}_{n-1}(\bar{\theta}_n))$. Hence,

$$\mathbb{E}_{\theta_1}|Proj_x \mathcal{K}_n(\theta_2, \bar{\theta}_n)| \le 4\frac{1}{4}|Proj_x \mathcal{K}_{n-1}(\bar{\theta}_n)| - \frac{c_0}{16}|Proj_x \mathcal{K}_{n-1}(\bar{\theta}_n)|^2 \Rightarrow$$

So by Hölder inequality

$$\mathbb{E}_{\theta_1,\theta_2,\dots,\theta_n}|Proj_x\mathcal{K}_n(\theta_1,\dots,\theta_n)| \le \mathbb{E}_{\theta_2,\theta_3,\dots,\theta_n}|Proj_x\mathcal{K}_{n-1}(\bar{\theta}_n)| -$$

$$c_1\left(\mathbb{E}_{\theta_1,\theta_2,\dots,\theta_n}|Proj_x\mathcal{K}_{n-1}(\bar{\theta}_n)|\right)^2 \Rightarrow$$

$$\mathbb{E}_\omega Fav\mathcal{K}_n(\omega) \le \mathbb{E}_\omega Fav\mathcal{K}_n(\omega) - c_1(\mathbb{E}_\omega Fav\mathcal{K}_n(\omega))^2\,.$$

This is by Fubini and because for such random Cantor sets $\mathbb{E}_{\theta_1,\theta_2,\dots,\theta_n}|Proj_\psi\mathcal{K}_n(\theta_1,\dots,\theta_n)|$ does not depend on ψ.

Let $a_{n-k} \leq a_{n-k-1} - c_1 a_{n-k-1}^2$ and $a_0 \leq 2$, then there exists C such that $a_{n-k} \leq \frac{C}{n-k}$. By induction.

In fact, $Ca_{n-k} \leq Ca_{n-k-1} - c_2(Ca_{n-k-1})^2$ with very small c_2 if C is large $c_2 = c_1/C$.

Function $x \to x - c_2x^2$ is increasing on $[0, 1/c_2] = [0, C/c_1]$. So $a_{n-k-1} \leq \frac{C}{n-k-1} \Rightarrow$

$$Ca_{n-k} \leq Ca_{n-k-1} - c_2(Ca_{n-k-1})^2 \leq \frac{C^2}{n-k-1} - c_2\left(\frac{C^2}{n-k-1}\right)^2 \leq$$

$$\frac{C^2}{n-k-1} - C^4\frac{c_1}{C}\left(\frac{1}{n-k-1}\right)^2 \Rightarrow \frac{a_{n-k}}{C} \leq \frac{1}{n-k-1} - Cc_1\left(\frac{1}{n-k-1}\right)^2$$

20. End of the proof

So, if C is so large that $Cc_1 > 1$ we have

$$\frac{a_{n-k}}{C} \leq \frac{1}{n-k-1} - \left(\frac{1}{n-k-1}\right)^2 \leq \frac{n-k-2}{(n-k-1)^2} \leq \frac{1}{n-k}\,.$$

Induction holds and Theorem is proved:

$$\mathbb{E}_\omega \mathcal{K}_n(\omega) \leq \frac{C}{n}\,.$$

Y. Peres and B. Solomyak "How likely is Buffon's needle to fall near planar Cantor set", Pacific J. Math. v. 204 (2002), N0. 2, pp. 473–496 had another model of randomness.

21. Peres–Solomyak random Cantor sets

Partition the unit square into 4 dyadic squares of side $1/2$. Partition each of them into 4 dyadic squares of side $1/4$, choose uniformly at random one $1/4$ square inside each $1/2$ square. Call the union of 4 randomly chosen dyadic squares of side $1/4$ $\mathcal{K}_1$.

Repeat this random procedure in each of squares forming $\mathcal{K}_1$, all the choices being independent. Get $\mathcal{K}_2$ consisting of 4^2 random squares of sides 4^{-2}. Inductively get $\mathcal{K}_k$ consisting of 4^k random squares of sides 4^{-k}.
It is a random 4 tree with children of even generations chosen randomly just 1 out of 4 and children of odd generations are just all 4.

Figure:

22. Theorem of Peres–Solomyak

Theorem (Peres–Solomyak, 2002)

$$\mathbb{E}Fav(\mathcal{K}_n) \leq \frac{C}{n}\,.$$

The proof in Y. Peres and B. Solomyak "How likely is Buffon's needle to fall near planar Cantor set" is full of tricks and not easy.

One can give another proof imitating Michael Bateman, Nets Katz "Kakeya sets in Cantor directions", Math. Res. Lett. 15 (2008), no. 1, 73–81 and percolation results á la Russell Lyons.

Or one can give yet another–the simplest proof–using the approach above.