Lecture 3. Complex analysis, Riesz products for the Buffon needle's quasi singular directions

Alexander Volberg,
Based on works with Fedor Nazarov, Yuval Peres

1. Lebesgue measure of quasi singular directions is small

Quasi singular direction is t such that

$$|\{x \in \ell_t : F_{t,*}(x) > K\}| \le \frac{1}{K^3}\,.$$

A difficult combinatorial result (later) claims

Theorem

(Main combinatorics: quasi singular is almost like singular) If $t \in quasiS_N$, that is if $|\{t : |\{x : F_{t,*} > K\}| \le \frac{1}{K^3}\}|$, then $\|f_{t,n}\|_2^2 \le 2K$ for all $n \le N$. So they are like singular ones but with $\|\cdot\|_\infty$ replaced by $\|\cdot\|_2$.

For a while we assume even more:

$$\|f_{t,n}\|_\infty \le 2K \quad \forall n \le N\,. \qquad (\infty K)$$

Plan: Preparation and Averaging
KLM lemmas
Perforation by SSV and averaging

2. Preparation

As $\|f_{t,N}\|_2^2 \le 2K$ we have

$$2K \ge \int_{\mathbb{R}} \hat{f}_{t,N}(y)^2 dy = \int \frac{\sin^2(4^{-N}y)}{(4^{-N}y)^2} |\hat{\nu}_{t,N}(y)|^2 dy\,.$$

Thus

$$\int_0^{4^N} |\hat{\nu}_{t,N}(y)|^2 dy \le CK,$$

where $\nu_{t,N} = *_{k=0}^N \frac{\delta_{-4^{-k}}+\delta_{4^{-k}}+\delta_{-t4^{-k}}+\delta_{t4^{-k}}}{4}$, hence,

$$|\hat{\nu}_{t,N}(y)| = \prod_{k-0}^N \left| \frac{\cos 4^{-k}y + \cos 4^{-k}ty}{2} \right|.$$

Therefore for m to be chosen later $(m \asymp K)$

$$\sum_{n=0}^N \int_{4^{n-m}}^{4^n} |\hat{\nu}_{t,N}(y)|^2 dy \le CKm\,.$$

Therefore, for each $m < \frac{N}{2}$ there exists $n \in [m, \frac{N}{2}]$ such that

$$\int_{4^{n-m}}^{4^n} |\hat{\nu}_{t,N}(y)|^2 dy \leq \frac{CKm}{N} .$$

Let E be the set of quasi singular directions, then we could have done the same

$$\frac{1}{|E|} \int_E dt \int_{4^{n-m}}^{4^n} |\hat{\nu}_{t,N}(y)|^2 dy \leq \frac{CKm}{N} .$$

We assume here that $|E| \geq K$. We will come to contradiction on slide 19.

4. Estimate $\int_{4^{-m}}^1 |P_t(y)|^2 dy$ from above

For domain of integration $y \in [4^{n-m}, 4^n]$ we can forget about super-slow frequencies+

$$|\hat{\nu}_{t,N}(y)|^2 = \prod_{k=0}^{N} \left| \frac{\cos 4^{-k} y + \cos 4^{-k} \cdot ty}{2} \right|^2 \asymp \prod_{k=0}^{n} \left| \frac{\cos 4^{-k} y + \cos 4^{-k} \cdot ty}{2} \right|^2$$

$$=: \mathcal{P}_t(y) \quad \Rightarrow 4^n \int_{4^m}^1 |\mathcal{P}_t(4^n y)|^2 dy \le \frac{2CKm}{N}.$$

Denote $P_t(y) = \mathcal{P}_t(4^n y) = \prod_{\ell=0}^n \left| \frac{\cos 4^\ell y + \cos 4^\ell \cdot ty}{2} \right|$. Thus,

$$\int_{4^{-m}}^1 |P_t(y)|^2 dy = \int_{4^{-m}}^1 |P_{t,1}(y)|^2 |P_{t,2}(y)|^2 dy \le 4^{-n} \frac{2CKm}{N},$$

where

$$P_{t,1}(y) := \prod_{\ell=0}^{m} \frac{\cos 4^\ell y + \cos 4^\ell \cdot ty}{2}; \; P_{t,2}(y) := \prod_{\ell=m+1}^{n} \frac{\cos 4^\ell y + \cos 4^\ell \cdot ty}{2}$$

5. Estimate of $\int_{4^{-m}}^1 |P_t(y)|^2 dy$ from below

Plan: 1) Estimate of $\int_{4^{-m}}^1 |P_{t,2}(y)|^2 dy$ from below;

2) SSV and estimate of $\int_{SSV} |P_{t,2}(y)|^2 dy$ from above (averaging);

3) Hence, estimate of $\int_{[4^{-m},1]\setminus SSV} |P_{t,2}(y)|^2 dy$ from below;

4) Estimate of $|P_{t,1}(y)|^2$ on $[4^{-m}, 1] \setminus SSV$ from below;

5) Hence, estimate of
$\int_{4^{-m}}^1 |P_t(y)|^2 dy = \int_{4^{-m}}^1 |P_{t,1}(y)|^2 |P_{t,2}(y)|^2 dy$ from below.

6) Come to contradiction with slide 3.

6. Estimate $\int_{4^{-m}}^{1} |P_{t,2}(y)|^2 dy$ from below

First let us estimate $\int_0^1 |P_{t,2}(y)|^2 dy$ from below. Salem trick.

Let g be a triangle function on $[-2, 2]$, $g(0) = 1$, then

$$\int_0^1 |P_{t,2}(y)|^2 dy = \frac{1}{2}\int_{-1}^{1} |P_{t,2}(y)|^2 dy \geq \frac{1}{4}\int_{-1}^{1} g(y)|P_{t,2}(y)|^2 dy =$$

$$\frac{1}{4}\int_{-1}^{1} g(y)\Big|\frac{1}{4^{n-m}}\sum_{\lambda\in\Lambda_{m,n}} e^{i\lambda y}\Big|^2 dy,$$

where $\Lambda_{m,n} = \{\sum_{k=m}^{n} \varepsilon_k t_k 4^k\} = 4^n\{\sum_{s=0}^{n-m} \varepsilon_s t_s 4^{-s}\}$, where $\varepsilon_s = \pm 1,\ t_s = 1$ or t.

Notice that $\sum_{s=0}^{n-m} \varepsilon_s t_s 4^{-s}\}$ are centers of characteristic functions of intervals such that the sum of those characteristic functions is precisely $f_{t,n-m}$.

Figure:

The first use of assumption: $\|F_{t,*}\|_\infty \le K$ means that, in particular, $f_{t,n-m}(x) \le K$ uniformly in x, and this means that in any interval of length $\asymp 4^{-n+m}$ there **at most** K points from $\{\sum_{s=0}^{n-m} \varepsilon_s t_s 4^{-s}\}$.

By scaling, this means that in each interval I, $|I| \asymp 4^m$, we have

$$|I \cap \Lambda_{m,n}| \le K\,.$$

Figure:

8. Salem trick continues

Salem trick:

$$\int_{-1}^{1} \Big| \sum_{\lambda \in \Lambda_{m,n}} e^{i\lambda y} \Big|^2 dy = |\Lambda_{m,n}| \int_{-1}^{1} g + \sum_{\lambda,\mu \in \Lambda_{m,n},\, \lambda \neq \mu} \hat{g}(\lambda - \mu)$$

Easy: $\hat{g}(\xi) = c\frac{1-\cos(\xi/2)}{\xi^2}$, $c > 0$. Hence, $\hat{g}(\lambda - \mu) > 0$. So,

$$\int_0^1 |P_{t,2}(y)|^2 dy \geq c\frac{|\Lambda_{m,n}|}{(4^{n-m})^2} \geq c4^{-n}4^m$$

9. KLM lemmas. Complex analysis, Carleson embedding

But we want the same type of estimate for $int^1_{4^{-m}}|P_{t,2}(y)|^2dy$.
Let us use for that the geometric information:

$$|I| \asymp 4^m \to |I \cap \Lambda_{m,n}| \leq K\,.$$

Lemma (KLM 1)

Let Λ be real frequencies, and $|\Lambda| = M$ (lemma will be applied with $M = 4^{n-m}$) and we have

$$\|\sum_{\lambda\in\Lambda} \mathbf{1}_{[\lambda-\frac{1}{4}L,\lambda+\frac{1}{4}L]}\|_\infty \leq K$$

(Lemma will be applied with $L = 4^m$). Then

$$\int_0^{1/L} |\sum_{\lambda\in\Lambda} e^{i\lambda y}|^2 dy \leq A\,\frac{KM}{L}.$$

whenever $K \leq \varepsilon_0 L$ with small positive absolute ε_0.

10. The proof of KLM 1 lemma

Scaling first : $\int_0^{1/L} |\sum_{\lambda\in\Lambda} e^{i\lambda y}|^2 dy = \frac{1}{L}\int_0^1 |\sum_{\lambda\in\Lambda} e^{i\frac{\lambda}{L}\xi}|^2 d\xi$. By assumption we know that

$$\|\sum_{\mu\in\frac{1}{L}\Lambda} \mathbf{1}_{[\mu-\frac{1}{4},\lambda+\frac{1}{4}]}\|_\infty \leq K\,.$$

Hence, measure $\sigma = \sum_{\mu \in \frac{1}{L}\Lambda} \delta_{\mu+i}$ is Carleson measure with absolute Carleson constant $A_0 K$. Then

$$\sup_{f \in L^2(0,1), \|f\|_2 \le 1} \sum_{\mu \in \frac{1}{L}\Lambda} |\hat{f}(\mu + i)|^2 \le A_0 K .$$

On the other hand,

$$\int_0^1 |\sum_{\mu \in \frac{\Lambda}{L}} e^{i\mu\xi}|^2 d\xi \le e^2 \int_0^1 |\sum_{\mu \in \frac{\Lambda}{L}} e^{i(\mu+i)\xi}|^2 d\xi =$$

$$e^2 \Big(\sup_{f \in L^2(0,1), \|f\|_2 \le 1} \Big| \sum_{\mu \in \frac{\Lambda}{L}} \hat{f}(\mu+i) \Big| \Big)^2 \le e^2 |\Lambda| \sup_{f \in L^2(0,1), \|f\|_2 \le 1} \sum_{\mu \in \frac{\Lambda}{L}} |\hat{f}(\mu+i)|^2 \le$$

$$A_0 e^2 K M$$

Lemma KLM 1 is proved

Let $M=4^{n-m}, L=4^m, K+\varepsilon_0 L$, the we have

$$\int_{1/L}^1 |P_{t,2}(y)|^2 dy \ge \int_0^1 \dots - \int_0^{1/L} \ge (4^{n-m})^{-2}(cM - A\frac{KM}{L}) =$$

$$(4^{n-m})^{-2}(cM - \frac{\varepsilon_0 LM}{L}) \ge c4^{-n}4^m\,.$$

This is what we wanted to prove.

Lemma (KLM 2)

Let Λ be real frequencies, and $|\Lambda| = M$ (lemma will be applied with $M = 4^{n-m}$) and we have

$$\|\sum_{\lambda\in\Lambda} \mathbf{1}_{[\lambda-\frac{1}{4}L,\lambda+\frac{1}{4}L]}\|_2^2 \le KLM$$

(Lemma will be applied with $L = 4^m$). Then

$$\int_0^{1/L} |\sum_{\lambda\in\Lambda} e^{i\lambda y}|^2 dy \le A\,\frac{KM}{L}.$$

whenever $K \le \varepsilon_0 L$ with small positive absolute ε_0.

Let us believe. It is close to KLM 1. Strictly stronger than KLM 1:

$$\|\sum_{\lambda\in\Lambda} \mathbf{1}_{[\lambda-\frac{1}{4}L,\lambda+\frac{1}{4}L]}\|_2^2 \le \|\sum_{\lambda\in\Lambda} \mathbf{1}_{[\lambda-\frac{1}{4}L,\lambda+\frac{1}{4}L]}\|_\infty \int \sum \le K\cdot L\cdot M$$

14. SSV, $\int_{SSV} |P_{2,t}(y)|^2 dy$ from above

We can do this only "in the average".
We estimates $4^n \int_{4^{-m}}^1 |P_{2,t}(y)|^2 dy \geq c4^m$ from below. But multiplication by $|P_{t,1}|^2$ can spoil this completely.

So consider **the set of small values, SSV:**

$$SSV_t := \{x \in \ell_t : |P_{1,t}| \leq \delta := e^{-Am}\} .$$

It is structured: $|P_{t,1}(y)| =$

$$\prod_{k=0}^{m} \left| \frac{\cos(4^k y) + \cos(4^k t y)}{2} \right| = \prod_{k=0}^{m} \left| \cos\left(\frac{4^k(y+ty)}{2}\right) \cos\left(\frac{4^k(y-ty)}{2}\right| ,\right.$$

$$|\cos(x/2)\cos(x) \cdot \cos(2^{2m}x/2)| = 2^{-2m} \left| \frac{\sin(2^{2m}x)}{\sin(x/2)} \right| \Rightarrow$$

$$|P_{t,1}(y)| \leq 4^{-m} |\sin(4^m(y+ty)| |\sin(4^m(y-ty)|.$$

Consider

$$\Omega_\delta := \bigcup_{k=0}^{4^m} \left[\frac{\pi k}{4^m} - \frac{\delta\pi}{4^m}, \frac{\pi k}{4^m} + \frac{\delta\pi}{4^m}\right].$$

$$SSV_t := \omega(t,\delta) := \{y \in [4^{-m}, 1] : y + ty \in \Omega_\delta \text{ or } y - ty \in \Omega_\delta\}.$$

We choose $\delta = 4^{-9m}$. Notice that SSV_t is covered by 4^m intervals of size $\delta 4^{-m} = 4^{-10m}$. And SSV contains all points on $[4^{-m}, 1]$, where $|P_{t,1}(y)| \leq 4^{-11m}$.

Figure:

Perforation: *perfo* $:= [4^{-m}, 1] \setminus \omega(t,\delta)$. Assume that $|E| \geq \frac{1}{K}$ (we wish to go to contradiction).

16. $\int_{\omega(t,\delta)} |P_{t,2}(y)|^2 dy \le \dots$

$$A := \frac{1}{|E|}\int_E dt \int_{\omega(t,\delta)} |P_{2,t}(y)|^2 dy \le$$

$$K\int_0^1 \int_{\omega(t,\delta)} \prod_{k=m+1}^n \cos^2\left(\frac{4^k(y+ty)}{2}\right) \prod_{k=m+1}^n \cos^2\left(\frac{4^k(y-ty)}{2}\right) dt dy$$

$$=^{u=y+ty, v=y-ty} K4^m \int_{\Omega_\delta} \prod_{k=m+1}^n \cos^2\left(\frac{4^k u}{2}\right) du \int_0^1 \prod_{k=m+1}^n \cos^2\left(\frac{4^k v}{2}\right) dv +$$

$$K4^m \int_0^1 \prod_{k=m+1}^n \cos^2\left(\frac{4^k u}{2}\right) du \int_{\Omega_\delta} \prod_{k=m+1}^n \cos^2\left(\frac{4^k v}{2}\right) dv$$

Jacobian is bounded by 4^m

$$A \le 2K4^m 2^{2(m-n)} \int_0^1 \prod_{k=m+1}^n (1+\cos 4^k u)du \int_{\Omega_\delta} \prod_{k=m+1}^n (1+\cos 4^k u)du\,.$$

It is well known that for Riesz product $R(u) := \prod_{k=m+1}^n (1+\cos 4^k u)$ we have $\int_0^1 R(u) \le \int_{-\pi}^{\pi} R(u) \le 2\pi$.

$$A \le 4\pi K 4^{m-n} \int_{\Omega_\delta} R(u)du\,.$$

Set Ω_δ consists of 4^m intervals J of length $2\pi 4^{-11m}$. So if $n-m \le 10m$, $\int_J R(u) \le 2\pi 2^{n-m} 4^{-11m} = 2\pi 4^{-6m}$.Thus $A \le CK4^{m-n}4^m 4^{-6m}$.

If $n-m > 10m$, we split $R(u) = R_1(u)R_2(u) = \prod_{k=m+1}^{11m}(1+\cos 4^k u)\prod_{k=11m+1}^{n}(1+\cos 4^k u)$. $R_1(u) \le 4^{5m}$. $R_2(u)$ is $2\pi 4^{-11m-1}$-periodic.
$\int_J R_2(u) \le 4^{-11m-1}\int_{-\pi}^{\pi} R_2 \le 2\pi 4^{-11m-1}$. $A \le CK4^{m-n}4^m 4^{-6m}$.

18. Finally, estimate $\int_{4^{-m}}^1 |P_t|^2 dy$ from below

So

$$A := \frac{1}{|E|}\int_E dt \int_{\omega(t,\delta)} |P_{2,t}(y)|^2 dy \le CK4^m 4^{-6m} \cdot 4^{m-n}.$$

On the other hand, we saw that (without any averaging, in fact)

$$B := \frac{1}{|E|}\int_E dt \int_{[4^m,1]} |P_{2,t}(y)|^2 dy \ge c4^{m-n}$$

Thus

$$\frac{1}{|E|}\int_{perfo} |P_{2,t}(y)|^2 dy = \int_{[4^{-m},1]\setminus\omega(t,4^{-10m})} \ge 4^{m-n}(c - CK4^m 4^{-6m})$$

Thus $\frac{1}{|E|}\int_{perfo} |P_{2,t}(y)|^2 dy \ge c_1 4^{m-n}$ if we have $K \asymp 4^m$. Hence,

$$\int_{4^{-m}}^1 |P_t|^2 dy = \int_{4^{-m}}^1 |P_{t,1}P_{t,2}|^2 dy \ge c4^{-22m}4^{m-n}.$$

19. Compare estimate from below and from above

Compare estimate from above on slide 4 by $\leq 4^{-n}\frac{CKm}{N}$ and the estimate from below $\geq c4^{-22m}4^{m-n}$ from slide 18. We get

$$4^{-21m} \leq \frac{C4^m m}{N} \Rightarrow N \leq CK^{22}\log K\,.$$

So if $N > CK^{22}\log K$ we come to contradiction proving that $|E| \leq \frac{1}{|K|}$.