# Algebra: Cantor sets, cyclotomic polynomials and Linear Multi-Polygon Relations (Lampreys)

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Based on works with Matt Bond, Izabella Laba

# 1. SSV property: structured sets of small values

$L = 3, 4, 5, \ldots$

$e^{-\frac{1}{4}\xi i} + e^{\frac{1}{4}\xi i} + e^{-t\frac{1}{4}i\xi} + e^{t\frac{1}{4}i\xi}$

**Definition**

We say that $\varphi$ has **the SSV property with SSV function** $\psi$ if there exist $c_1, c_2, c_3 > 0$ with $c_3 \gg c_2$ such that $SSV_\psi$ is contained in $L^{c_2 m}$ intervals of size $L^{-c_3 m}$. In decreasing order of strength:

- If $\psi(m) = L^{-c_1 m}$, we say that $\varphi$ has **the SSV property**.
- If $\psi(m) = L^{-c_1 m \log m}$, we say that $\varphi$ has **the log-SSV property**.
- If $\psi(m) = L^{-c_1 m^2}$, we say that $\varphi$ has **the square-SSV property**.

always

$e^{-c\sqrt{\log N}}$

# indep. $\rightarrow$ $\leftarrow$ SSV

of $C_1$ $\quad |\varphi(\xi)\,\varphi(L\xi)\ldots\varphi(L^m\xi)| \le L^{-C_1 m}$.

$L=4$

For $L = 3$ the SSV property turns out to be true for all angles. Let us see why SSV property holds for any set of non-collinear 4 points and for any angle.
When $L = 4$, some normalizations are possible. In fact, three out of four of the similarity centers $z_j$ can be mapped to arbitrary points by an affine map, leaving only one truly free parameter $z_4 = r_4 e^{i\theta_4}$. Without loss of generality, then, $z_1 = 0, z_2 = 1, z_3 = i$. Note that

$$\phi_\theta(\xi) = \frac{1}{4}\sum_{j=1}^{4} e^{ir_j \cos(\theta_j - \theta)\xi} =$$

$$\frac{1}{4}\left[1 + e^{i\cos(\theta)\xi} + e^{i\cos(\theta)\tan(\theta)\xi} + e^{ir_4\cos(\theta)[\cos(\theta_4)+\tan(\theta)\sin(\theta_4)]\xi}\right].$$

By a change of variable $\tan(\theta) \to t$, $\cos(\theta)\xi \to \xi$, we can write

$$\phi_t(\xi) := \frac{1}{4}(1 + e^{i\xi} + e^{it\xi} + e^{ig(t)\xi}),$$

where $t \in [-1, 1]$ and $g(t) = t\sin(\theta_4) + \cos(\theta_4)$; of course other $\theta$ are handled by symmetry. So for this $\phi_t = \phi$, we consider

$$\boxed{\prod_{k=0}^{m} \phi(4^k\xi)}\,.$$

$e^{-cm}$

It will be convenient to argue with "pseudo" trigonometric identities:

$$|e^{ix_1} + e^{ix_2} + e^{ix_3} + e^{ix_4}| \gtrsim \prod_{1\le j<k\le 4} \left|\cos\left(\frac{x_j - x_k}{2}\right)\right| \qquad (1)$$

For us, $x_1 = 0, x_2 = \xi, x_3 = t\xi, x_4 = g(t)\xi$. Letting $\alpha\xi = \frac{1}{2}(x_j - x_k)$, it is enough to prove the following claim.

**Claim:** *The function* $\psi(\xi) = \cos(\alpha\xi)$ *has the SSV property (with* $L = 4$*) for any* $\alpha \in \mathbb{R}$*. One may take* $c_3/c_2$ *arbitrarily large independent of* $\alpha$*.*

Consider the product of trigonometric functions. Repeating the double angle formula, one gets

$4^k$

$$2^m \sin(x) \cdot \prod_{k=0}^{m-1} \cos(2^k x) = \sin(2^m x)$$

Using the substitution $m \to 2m + 1$,

$4^m$

$$2 \cdot 4^m \sin(x) \cdot \prod_{k=0}^{2m} \cos(2^k x) = \sin(2 \cdot 4^m x)$$

Omitting even terms and reindexing,

$4^{-99m}$

$$\left|\prod_{k=0}^{m} \cos(4^k x)\right| \gtrsim 4^{-m} \left|\frac{\sin(2 \cdot 4^m x)}{\sin(x)}\right|$$

Now let $x = \alpha\xi$, where $\xi \in [0,1]$. Then

$$4^{-Cm} \geq \Big| \prod_{k=0}^{m} \cos(4^k \alpha\xi) \Big| \gtrsim 4^{-m} \Big| \frac{\sin(2 \cdot 4^m \alpha\xi)}{\sin(\alpha\xi)} \Big| \,.$$

The SSV of the right hand side are readily understood. Such $\xi \in \mathbb{R}$ are contained in this set:

$$(-c^m 4^{-m}\alpha^{-1}, c^m 4^{-m}\alpha^{-1}) + \pi \left[ 2^{-1} 4^{-m} \alpha^{-1}\mathbb{Z} \setminus \alpha^{-1}\mathbb{Z} \right] .$$

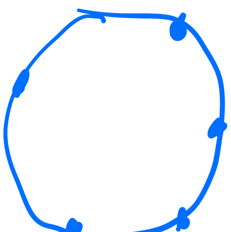
These intervals are exponentially small, and of the appropriate number. Their SSV property is achieved by making $C$ large and, hence, $c$ very small. This means that $c_3/c_2$ can be made arbitrarily large by making $c_1$ large and so $c$ small.
This proves the claim.

# 6. The SSV property can fail for $L = 5$ on a large set of angles

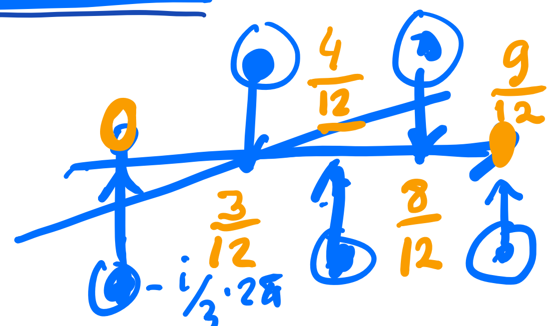
The self-similar set with $L = 5$ and $z_1 = 2\pi(0 - i/3)$, $z_2 = 2\pi(3/12 + i/3)$, $z_3 = 2\pi(4/12 - i/3)$, $z_4 = 2\pi(8/12 + i/3)$, $z_5 = 2\pi(9/12 - i/3)$. (The imaginary coordinates do not matter in this example other than to avoid collinearity.) Then $\phi$ function for the projection on the line with angle $\theta = 0$ will be

$$\phi_0(\xi) = \frac{1}{5}\left(1 + e^{i\frac{\pi}{2}\xi} + e^{i\frac{2\pi}{3}\xi} + e^{i\frac{4\pi}{3}\xi} + e^{i\frac{3\pi}{2}\xi}\right).$$

And for a fixed $m$,



$$P_{1,0}(\xi) = \prod_{k=1}^{m} \phi_0(5^k\xi).$$




We will see now that its set of small values (SSV) is not structured correctly, and that it **does not** satisfy SSV property. Moreover, this holds for all angles $\theta \in [-5^{-200\sqrt{m}}, 5^{-200\sqrt{m}}]$.

The reason for the SSV failure is that $\phi_0(5^k\xi)$ has a recurring zero at $\xi = 1$:

$5^{-200\sqrt{m}}$

$$\phi_0(1) = \phi_0(5^k) = 0, k = 1, \dots .$$

Therefore, for all $\xi \in [1 - 5^{-200\sqrt{m}}, 1]$ and $k = 0, 1, ..., \sqrt{m}$ we have

$$|\phi_0(5^k\xi)| = |\phi_0(5^k\xi) - \phi_0(5^k)| \le C\, 5^k |\xi - 1| \le C\, 5^k\, 5^{-200\sqrt{m}}.$$

Remind: $P_{1,0}(\xi) = \prod_{k=0}^m \phi_0(5^k\xi)$. Therefore:

$$|P_{1,0}(\xi)| \le |\prod_{k=0}^{\sqrt{m}} \phi_0(5^k\xi)|| \prod_{k=\sqrt{m}+1}^{m} \dots| \le |\prod_{k=0}^{\sqrt{m}} \phi_0(5^k\xi)|$$

$$\le C^{\sqrt{m}}\, 5^{1+2+\dots+\sqrt{m}}\, (5^{-200\sqrt{m}})^{\sqrt{m}} \le 5^{-100\, m}.$$

Hence the set of small values includes the entire interval $[1 - 5^{-200\sqrt{m}}, 1]$; **in particular, it cannot be covered by $5^{c_2 m}$ intervals of length at most $5^{-c_3 m}$, $0 < c_2 < c_3$.**

The existence of one "bad" direction $\theta = 0$ does not automatically make the SSV approach unviable. In fact, analysis of previous lectures confirms that if $\phi_\theta$ satisfies the uniform SSV property for all directions $\theta$ except for an exceptional set $\Theta_m$ of size $|\Theta_m| \lesssim e^{-c_4 m}$, then we can still get $Fav(S_n) \lesssim n^{-p}$, $p > 0$.

**But SSV property fails on a set of angles $\Theta_m$ of size $\gtrsim 5^{-c\sqrt{m}}$.** Let $\theta \in [0, 5^{-200\sqrt{m}}]$, then

$$|P_{1,\theta}(\xi)| \le |\prod_{k=0}^{\sqrt{m}} \phi_\theta(5^k \xi)| \le \prod_{k=0}^{\sqrt{m}} (|\phi_\theta(5^k\xi) - \phi_0(5^k\xi)| + |\phi_0(5^k\xi)|)\,.$$

The second term in each factor is at most $C\,5^k\,5^{-200\sqrt{m}}$. The first term can be estimated by differentiating in $\theta$ and using the mean value theorem: $|\phi_\theta(5^k\xi) - \phi_0(5^k\xi)| \le C\,5^k\,|\theta| \le C\,5^k\,5^{-200\sqrt{m}}$.

Hence each factor is at most $C\,5^k\,5^{-200\sqrt{m}}$, so again on the entire interval $[1-5^{-200\sqrt{m}}, 1]$ one has

$$|P_{1,\theta}(\xi)| \le C^{\sqrt{m}}\,5^{1+2+\dots+\sqrt{m}}\,(5^{-200\sqrt{m}})^{\sqrt{m}} \le 5^{-100\,m}\,.$$

$$\mathrm{Fav}(K_n) \le \boxed{e^{-c\sqrt{\log n}}}$$

$$\le \boxed{\frac{1}{n^c}}$$

Consider polynomial with integer coefficients $\varphi_0$

$$A(x) = 1 + x^3 + x^4 + x^8 + x^9 \in \mathbb{Z}[x] \,.$$

f deg $= 5$

Then $\phi_0(\xi) = \frac{1}{5} A(e^{\frac{2\pi i \xi}{12}})$, and $\phi_0(1) = 0$ means that $e^{\frac{2\pi i}{12}}$ is its root. This in its turn means that

$(12, 5) = 1$

primitive root of unity of order 12

$$\Phi_{12} | A \text{ in } \mathbb{Z}[x] \,,$$

where $\Phi_{12}(x) = 1 - x^2 + x^4$ is the 12-th cyclotomic polynomial. Its zeros are $\{e^{2\pi i \lambda}\}_{\lambda \in \Lambda}$, where $\Lambda := \{\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}\} + \mathbb{Z}$, and $\Lambda$ is invariant under multiplication by 5 as

$$(s, L) = (12, 5) = 1 \,.$$

So we have zero of multiplicity $m$ at $\xi = 1$ (corresponding to order 12 primitive root of unity, root $e^{\frac{2\pi i}{12}}$ of $\Phi_{12}$).

## 11. Cyclotomic polynomials

**Definition.** $\Phi_s(x) = \prod_{k \le s, (k,s)=1} \left(x - e^{\frac{2\pi i k}{s}}\right)$. Easy: $\Phi_s \in \mathbb{Z}[x]$.
Examples: $\Phi_1 = -1 + x, \Phi_2 = 1 + x, \Phi_3 = 1 + x + x^2, \ldots, \Phi_9 = 1 + x^3 + x^6, \ldots, \Phi_{30} = 1 + x - x^3 - x^4 - x^5 + x^7 + x^8, \ldots, \Phi_{105} = x^{48} + \ldots. - 2x^{41} - \ldots + 1$. The first one with coefficients not just $\pm 1, 0$.
**Properties:** $x^M - 1 = \prod_{s|M} \Phi_s(x)$. If order $s$ primitive root of $e^{\frac{2\pi i k}{s}}$ is a root of a polynomial $G \in \mathbb{Z}[x]$ then $\Phi_s | G$.
If $A = \{0, a_1, \ldots\}$ is a finite collection of integers we consider **a fewnomial** $A(x) = \sum_{a \in A} x^a$. Above $A(x) = 1 + x^3 + x^4 + x^8 + x^9$.
Any such $A$ with order $s$ root of unity such that

$$(s, |A|) = 1$$

will generate exactly the same type of example of Cantor set for which the SSV propert **does not hold**. What to do?

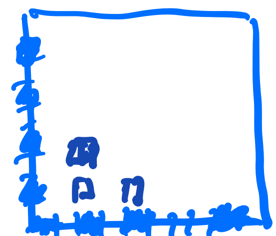
# 12. Rationally product Cantor sets

Let $L = L_1 L_2$, $L_i$ integers $\geq 2$. Divide sides of unit square to $LM$ equal segments. Now choose integers $A = \{a_0 = 0, a_1, \ldots\}$ such that $|A| = L_1$ and $(a_{i+1} - a_i) > M$. Do the same with vertical side and $B = \{b_0 = 0, b_1, \ldots\}$ such that $|B| = L_2$ and $(b_{i+1} - b_i) > M$. Then segments $I_i$ of length $1/L$ centered at segments numbered $a_i$ are disjoint, the same for the segments $J_i$ of length $1/L$ centered at segments numbered $b_i$. Consider $1/L \times 1/L$ squares that are descartes products $\{I_i \times J_j\}_{i=0,\ldots,|A|-1, j=0,\ldots,|B|-1}$.

**Figure:**

$A, B, \quad |A| = L_1 \quad L = L_1 L_2$

$|B| = L_2$



This is the first generation of rationally product Cantor set. Repeat picture inside each small square and iterate $N$ times. Get $\mathcal{K}_N$.

$A = \{0, 3, 5, 9\}$

# 13. Then SSV depends on the roots of $A(x), B(x)$

Remind $A(x) = \sum_{a\in A} x^a$, $B(x) = \sum_{x\in B} x^b$.

$e^{2\pi i \xi}$

$$\phi_t(\xi) = A(e^{2\pi i\xi})B(e^{2\pi it\xi}) = \left(1 + \sum_{a\in A\setminus 0} e^{2\pi i a\xi}\right)\left(1 + \sum_{b\in B\setminus 0} e^{2\pi i bt\xi}\right).$$

$e^{2\pi i \frac{k}{s}}$ $|A|$ $s$ $\Phi_s \mid A(x)$ $A, B$

## Theorem

*[Good rational phase roots] If all roots of $A, B$ on $\mathbb{T}$ have form $e^{\frac{2\pi ik}{s}}$, $(k, s) = 1$ and all of them are such that $(s, L) \neq 1$, then SSV property holds for $P_{1,t} = \prod_{j=0}^{m} \phi_t(L^j\xi)$ for any $t$.*
*$Fav(\mathcal{K}_N) \lesssim N^{-p}, p > 0$.*

good

$\boxed{(12, 5) = 1}$ bad

## Theorem (Irrational phase roots)

*If all roots of $A, B$ on $\mathbb{T}$ have either the form above or $e^{2\pi i\xi_0}$, where $\xi_0 \notin Q$, then log-SSV property holds for $P_{1,t} = \prod_{j=0}^{m} \phi_t(L^j\xi)$ for any $t$. $Fav(\mathcal{K}_N) \lesssim N^{-\frac{p}{\log\log N}}, p > 0$.*

Gelfond-Baker

# 14. Bad cyclotomic divisors

$\Phi_{12} \mid A(x)$ $(s, L) = 1$ bad cyclotomic roots

**Theorem (One bad cyclotomic divisor)**

*If among the roots of $A, B$ on $\mathbb{T}$ there is $e^{\frac{2\pi ik}{s}}$, $(k,s) = 1$, and $(s, L) = 1$, but only one such bad $s$ happens, then log-SSV happens for $P_{1,t} = \prod_{j=0}^{m} \phi_t(L^j \xi)$. $Fav(\mathcal{K}_N) \lesssim N^{-\frac{p}{\log\log N}}$, $p > 0$. If no root with irrational phase exists on $\mathbb{T}$ then SSV holds and $Fav(\mathcal{K}_N) \lesssim N^{-p}$, $p > 0$.*

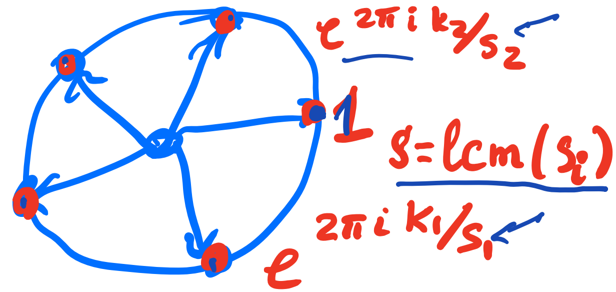
**Theorem (Some general estimate)**

*If among the roots of $A, B$ on $\mathbb{T}$ there are $e^{\frac{2\pi ik}{s}}$, $(k,s) = 1$, and $(s, L) = 1$, then only quadratic-SSV property may hold for $P_{1,t} = \prod_{j=0}^{m} \phi_t(L^j \xi)$. The following estimate always hold: $Fav(\mathcal{K}_N) \lesssim e^{-p\sqrt{N}}$, $p > 0$.*

LMPR

linear multi-polynomial relations

$\equiv$ Lampreys




**Theorem (Small size of A, B)**

*If $|A| \le 6, |B| \le 6$, then log-SSV happens for $P_{1,t} = \prod_{j=0}^m \phi_t(L^j \xi)$. $Fav(\mathcal{K}_N) \lesssim N^{-\frac{p}{\log\log N}}$, $p > 0$. If no root with irrational phase exists on $\mathbb{T}$ then SSV holds and $Fav(\mathcal{K}_N) \lesssim N^{-p}$, $p > 0$.*

Our enimies are Lampreys

with $(s, L) = 1$, where $s = \mathrm{lcm}(s_i)_{i=1}^{L}$

$:=$ determinant of Lamprey, $L = \#$ Lamprey

$=$ cardinality of Lamprey

# 16. Proof by example of Theorem "one bad cyclotomic divisor"

It turns out that one can work without SSV (structured set of small values), which is definitely not good (only quadratic) in the case of

$$A(x) = 1 + x^3 + x^4 + x^8 + x^9; \ |A| = 5, \ \phi(\xi) = \frac{1}{5} A(e^{\frac{2\pi i \xi}{12}}), \ (12, 5) = 1.$$

One can use instead SLV (structured set of large values). SLV structure of $P_1 := \prod_{j=0}^m \phi(L^j \xi)$ means that:

$\mathbb{R}/\mathbb{Z}$

$$\exists \Gamma \subset [0,1] : |P_1|_{\Gamma - \Gamma} \geq 5^{-C_1 m}, \ |\Gamma| \geq C_2 5^{-(1-\varepsilon)m}. \geq C\, 5^m \cdot K$$

$5^{\varepsilon m}$

We first construct a set $\Delta$ disjoint from the set of small values of $\phi$. Let $\Lambda = \{\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}\} + \mathbb{Z}$, so that $e^{2\pi i \lambda}$ for $\lambda \in \Lambda$ are exactly the zeroes of $\Phi_{12} = 1 - x^2 + x^4$. We want $\Delta$ to avoid a neighborhood of $\Lambda$. The key observation is that all points of $\frac{1}{6}\mathbb{Z}$ are at distance at least $1/12$ from $\Lambda$, hence we may take $\Delta$ to be a neighborhood of $\frac{1}{6}\mathbb{Z}$. We are using here that 6 divides 12, but $\phi$ does not vanish at any 6-th root of unity. Only one cyclotomic divisor.

$(\Gamma-\Gamma)\cap[-4^{-m},4^{-m}]$

High freq. $\int h|P_2|^2 \geq L^{-n}\cdot K$

$\geq \int_{\Gamma-\Gamma} \uparrow |P_1|^2$

$$\exists \Gamma \subset [0,1] : |P_1|_{\Gamma-\Gamma} \geq L^{-C_1 m},\ |\Gamma| \geq C_2 L^{-(1-\varepsilon)m} \geq CKL^{-n}.$$

$K >> 1$

High frequency polynomial $P_2$: $\mathbf{1}_{[0,1]} * \mathbf{1}_{[0,1]}$

$$P_2(\xi) = L^{-(n-m)} \sum_{\alpha\in\Lambda} e^{i\alpha\xi}, \quad |\Lambda| = L^{n-m}.$$

High frequency estimate: $\boxed{h := |\Gamma|^{-1}\mathbf{1}_\Gamma * \mathbf{1}_\Gamma.}$ Then $0 \leq h \leq 1$, $\hat{h} = |\Gamma|^{-1}|\hat{\mathbf{1}}_\Gamma|^2 > 0$. So,

Salem trick with Bohr set

Large

$|\Lambda| = L^{n-m}$

$$\int_{\Gamma-\Gamma \cap [-4^{-m},4^{-m}]} |P_2|^2 d\xi \geq \int_{\Gamma-\Gamma} h(\xi)|P_2(\xi)|^2 d\xi =$$

$|P_1|^2$

$$= L^{-2(n-m)}\Big(\sum_{\alpha\in\Lambda}\int_{\Gamma-\Gamma} h\, d\xi + \sum_{\alpha\neq\alpha'} \hat{h}(\alpha-\alpha')\Big) \geq L^{-n+m}\, |\Gamma| \geq CK\cdot L^{-n}$$

$L^{n-m}$, $>0$

Let consider Bohr set

$$\Delta_0 = \frac{1}{6}\mathbb{Z} + \left(-\frac{\eta}{12}, \frac{\eta}{12}\right)$$

To find $\Gamma$

for some $\eta \in (0,1)$. Then there is a constant $c = c(\eta) > 0$ such that

$$\phi(\xi) \geq c \text{ for } \xi \in \Delta_0.$$

$c(\eta) > 0$

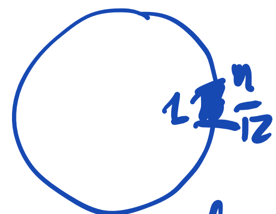
By scaling, we also have

$$\phi(5^j\xi) \geq c \text{ for } \xi \in \Delta_j := \frac{5^{-j}}{6}\mathbb{Z} + \left(-\frac{5^{-j}\eta}{12}, \frac{5^{-j}\eta}{12}\right).$$

Bohr sets

Bohr set

Let $\Delta = \bigcap_{j=0}^{m-1} \Delta_j$, then



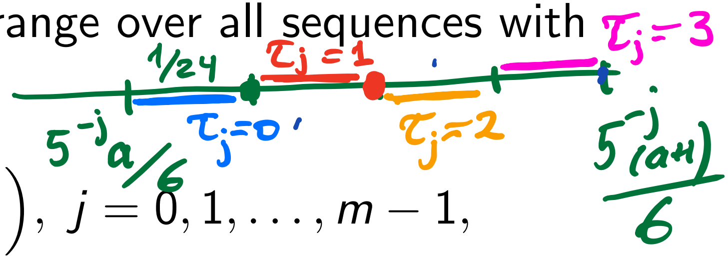

$$\prod_{j=0}^{m-1} |\phi_0(L^j\xi)|^2 \geq c^{2m} = 5^{-C_1 m} \text{ for } \xi \in \Delta$$

speed $5^j 6$

$\leq \frac{C}{N^p}$

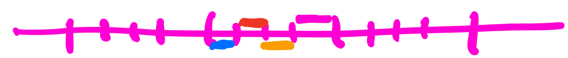
with $C_1 = \frac{\log 5}{2\log(1/c)}$.

It remains to prove that we can choose an $\eta \in (0,1)$ and a set $\Gamma \subset [0,1]$ of size at least $C_2 5^{-(1-\epsilon)m}$ so that $\Gamma - \Gamma \subset \Delta$. We fix $\eta = 1/2$, and let $\tau \doteq (\tau_0, \dots, \tau_{m-1})$ range over all sequences with $\tau_j \in \{0,1,2,3\}$. Define




$$\Gamma_{\tau,j} = \frac{5^{-j}}{6}\left(\frac{\tau_j}{4} + \mathbb{Z}\right) + \left(0, \frac{5^{-j}}{24}\right), \; j = 0, 1, \dots, m-1,$$

$$\Gamma_\tau := [0,1] \cap \bigcap_{j=0}^{m-1} \Gamma_{\tau,j}$$



$j \to j+1$

Then $\Gamma_{\tau,j} - \Gamma_{\tau,j} \subset \Delta_j$, so that $\Gamma_\tau - \Gamma_\tau \subset \Delta$. Moreover, we have

Lemma

$$\bigcup_{\tau \in \{0,1,2,3\}^m} \Gamma_\tau = [0,1]$$

Proof: Picture

$\tau_0, \tau_1, \dots, \tau_{m-1}$

except for the zero measure set of interval endpoints. Hence there is at least one $\tau$ such that $|\Gamma_\tau| \geq 4^{-m}$, which is greater than $5^{-(1-\epsilon)m}$ for $0 < \epsilon < 1 - \frac{\log 4}{\log 5}$.

# 19. More general results

$A(z) = 1 + z^3 + z^4 + z^8 + z^9$

Let $s = lcm(r : \Phi_r | A, \ (r, |A|) = 1\}$.

$s = 12, \ |A| = 5$

**Theorem**

*Suppose that we can write $s_A = s_{1,A}s_{2,A}$ with $s_{1,A}, s_{2,A} > 1$ so that:*

- $s_{2,A} < \frac{1}{2}|A|$,
- $\Phi_q(x)$ *does not divide* $A(x)$ *for any* $q | s_{1,A}$.

$s_2 = 2$

$s_1 = 6$

*Then there is a set* $\Gamma \subset [0, 1]$ *obeying*

$$\exists \Gamma \subset [0, 1] : |P_1|_{\Gamma - \Gamma} \geq |A|^{-C_1 m}, \ |\Gamma| \geq C_2 |A|^{-(1-\varepsilon)m}.$$

It has *precisely* the same proof as in example above, where $s = 12, s_{1,A} = 6, s_{2,A} = 2, |A| = 5$. Technically more difficult, but with the same idea is

# 20. More general result

the worst

bad

Same $s = lcm(r : \Phi_r | A, \ (r, |A|) = 1\}$.

**Theorem (Good factorization of $s$)**

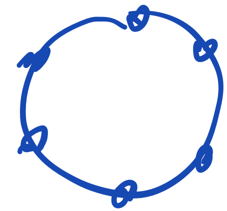
*Suppose that we can write $s_A = s_{1,A} s_{2,A}$ with $s_{1,A}, s_{2,A} > 1$ so that:*

- $s_{2,A} < |A|$,
- $\Phi_q(x)$ *does not divide* $A(x)$ *for any* $q | s_{1,A}$.

*Then there is a set* $\Gamma \subset [0,1]$ *obeying*

$$\exists \Gamma \subset [0,1] : |P_1|_{\Gamma - \Gamma} \geq |A|^{-C_1 m}, \ |\Gamma| \geq C_2 |A|^{-(1-\varepsilon) m}.$$

$n_j \in \mathbb{Z}$ $\quad \sum_j n_j$ roots of unity$_j$ $= 0$



$\mathbb{T}$

Given a fewnomial $A$ as above and an order $s$ primitive root of unity $e^{\frac{2\pi ik}{s}}$ such that $A(e^{\frac{2\pi ik}{s}}) = 0$, we get the example of a Lamprey (Linear multi-polygon relationships, LMPR)), that is **the sum of roots of unity (of different order) with integer coefficients such that this sum of those roots with those coefficients** $= 0$. Our Lamprey always contains $\{1\}$.
Lamprey has *cardinality* (weight) $n$=how many roots of unity it contains (counting multiplicity). And it has *denominator* $s$=least common multiple of rational phases of roots.

# 22. General Lampreys. Decomposition. Cardinality

$\mathcal{T}_p$ $z^p = 1$

**Theorem (Rédei–de Brujin–Schönberg–Mann)**

*Let Lamprey $\mathcal{T}$ has determinant $s$. Then it is of the form $\mathcal{T} = \sum_{j=1}^{J} n_j \eta_j \mathcal{T}_{p_j}$, where $\eta_j$ are some roots of unity, $n_j \in \mathbb{Z}$, $p_j | s$.*

**Theorem (Lam–Leung: Cardinality)**

*Let Lamprey $\mathcal{T}$ has determinant $s = p_1^{r_1} \dots p_J^{r_J}$. Then cardinality $n$ can be only of the type $n = \sum_j k_j p_j$, where $k_j$ are non-negative integers. In particular $n \geq \min_j p_j$.*

$\mathcal{T}_p$

A Lamprey is **irreducible** if no proper subset of it is a Lamprey. For example, any prime polygon $\mathcal{T}_p$ is irreducible. It is tempting to think that all irreducible Lampreys have this form; this is not true. Here is what Poonen–Rubinstein called $\mathcal{L}_{5:3}$ an irreducible relation between roots of unity of type $R(5:3)$:

$$\mathcal{L}_{5:3} := \{e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}, e^{5\pi i/3}, e^{7\pi i/3}\}.$$

$$0 = \sum_{\zeta\in\mathcal{T}_5} \zeta - \sum_{\zeta'\in\mathcal{T}_3} \zeta' + [e^{2\pi i/3} + e^{4\pi i/3}] \sum_{\zeta''\in\mathcal{T}_2} \zeta'' = \sum_{j=1}^{4} e^{2\pi ij/5} + e^{5\pi i/3} + e^{7\pi i/3}$$

The number of such rapidly grows beyond this point, though Poonen–Rubinstein classifies all such cases for sets having at most 12 points.

irreducible Lampreys
with $L \leq 12$.

B. Poonen and M. Rubinstein: Number of Intersection Points Made by the Diagonals of a Regular Polygon, SIAM J. Disc. Math. 11 (1998), 135–156.

J. H. Conway and A. J. Jones: Trigonometric Diophantine equations (On vanishing sums of roots of unity), Acta Arith. 30 (1976), 229–240.

# 25. Application of Theorem of Lam–Leung

We are going to prove Theorem (one bad cyclotomic divisor). We want to prove that if

$$A(x) = 1 + \sum_{x \in A \setminus 0} x^a, \; \phi(\xi) = \frac{1}{|A|} A(e^{2\imath \xi}), \; P_1 = \prod_{j=0}^{m} \phi(L^j \xi) \,.$$

has only one **bad cyclotomic divisor**, meaning $\Phi_s | A : (s, L) = 1$, then we have $\Gamma$ such that

$$\exists \Gamma \subset [0,1] : |P_1|_{\Gamma \setminus \Gamma} \geq |A|^{-C_1 m}, \; |\Gamma| \geq C_2 |A|^{-(1-\varepsilon)m} \,.$$

We want to reduce to Theorem (Good factorization of
$\boxed{s = lcm(r : \Phi_r | A, (r, |A|) = 1)}$
Consider first the case of $s = p = prime$. Then $\Phi_s(1) = p$ and $A(1) = |A|$–not divisible by $p$ as $(p, |A|) = (s, |A|) = 1$. So $s$ has several prime factors. Thus $\Phi_s$ cannot divide $A$, contradiction.

Consider now a root $\zeta$ of $\Phi_s$ and consider $\mathcal{T}_A = \{\zeta^a\}_{a\in A}$. It is a Lamprey as $\Phi_s|A$, and hence

$a$

$$1 + \sum_{a\in A\setminus 0} \zeta^a = 0.$$

$s = p_i \cdot \tilde{s}$

$q | \tilde{s}$

$\Phi_q \nmid A$

Its cardinality is $|A|$ and its determinant is $s$.

By Theorem of Rédei–de Brujin–Schönberg it is the sum of $\mathcal{T}_{p_i}$ with integer coefficients, where $p_i|s$. And by Theorem Lam–Leung $|p_1| \le |A|$, where $p_1$ is the smallest prime divisor of $s$. But $(s, |A|) = 1$, hence $(p_1, |A|) = 1$. Hence, $p_1 \ne |A|$, so $|p_1| < |A|$.

Put $s_{2,A} = p_1$. Then $s_{1,A} = s/p_1$. If $q|s_{1,A}$ then $q < s$ and $\Phi_s$ is the only cyclotomic poly that divides $A$. Hence $\Phi_q$ does not divide $A$ and we can apply Theorem (Good factorization of $s = lcm(r : \Phi_r|A, (r, |A|) = 1)$, using $s_{2,A} = p_1 < |A|$.

# 27. Now let us prove $|A| = 6$ case

Suppose first that $|A| = 6$. We will show that the cyclotomic roots of $A$ can only be zeroes of $\Phi_s$ for some $s$ divisible by 2 or 3, so that in particular $(s, L) \neq 1$. we do not have enemy.

Suppose that $\Phi_s$ divides $A$, and consider the lamprey $\mathcal{A}_s := \{\zeta^a\}_{a \in A}$. A 6-point lamprey can only take these forms:

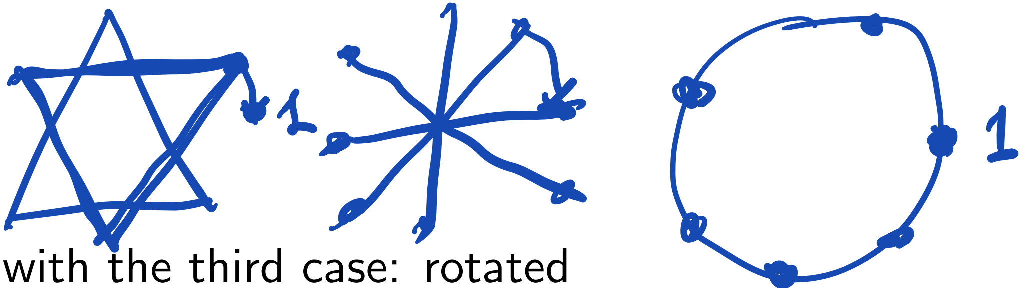
- $\mathcal{A}_s$ can be a union of triangles
- $\mathcal{A}_s$ can be a union of three line diameters (2-gons)
- $\mathcal{A}_s$ can be a rotation of $\mathcal{L}_{5:3}$

$1 + z^{n_1} + z^{n_2} + \dots z^{n_5}$

$s = 30$

$$\mathcal{L}_{5:3} := \{e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}, e^{5\pi i/3}, e^{7\pi i/3}\}.$$

But our Lampreys always contain $\{1\}$. So in the first case on of triangles contains $\{1\}$ and so the denominator $s$ in the first case should be divisible by 3. In the second case one diameter should contain $\{1\}$. Hence, the denominator $s$ in the second case should be divisible by 2. So $(s, |A|) = (s, 6) \neq 1$ in those cases.




We are left with the third case: rotated $\mathcal{L}_{5:3} := \{e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}, e^{2\pi i\cdot 5/6}, e^{2\pi i\cdot 7/3}\}$. It must be rotated as again $\{1\}$ should belong to it.
But if one rotates $\mathcal{L}_{5:3}$ to have $\{1\}$ in it, the denominator of rotated Lamprey obviously becomes divisible by 30 (in fact $s = 30\dot{g}cm(A)$). Thus again $(s, |A|) = (s, 6) \neq 1$.

This is the situation when SSV for $\phi = \frac{1}{6}A(e^{2\pi i\xi})$ holds.

Let $A = 1 + \sum_{a \in A \setminus} x^a$. Let cyclotomic $\Phi_s | A$. Then we can see the Lamprey $\mathcal{A}_s = \{e^{\frac{2\pi i a}{s}} : a \in A\}$.

We will be using repeatedly:

**Lemma (Algebra versus Geometry)**

*If cyclotomic $\Phi_s | A$ and cyclotomic $\Phi_{sM} | A$, then the following power relationship on Lampreys hold:*

$$(\mathcal{A}_{sM})^M = \mathcal{A}_s\,.$$

**Lemma**

*If we have $\mathcal{A}_{sM}$ and it is $D \cup T$ and $(M, 2) = 1, (M, 3) = 1$ then we also have $\mathcal{A}_s$ (equivalently $A(e^{\frac{2\pi i}{s}}) = 0$).*

# 30. Now let us prove $|A| = 5$ case

It is more complicated and we reduce this to SLV case, not to SSV case.

Let $\mathcal{A}$ be our Lampreys generated by set $A$, $|A| = 5$, they all have cardinality 5 and some denominators $s$ which obviously has $(s, 5) = 1$, they are $\mathcal{A}_s = \{e^{\frac{2\pi i a}{s}} : a \in A\}$.

As $1 \in \mathcal{A}$ and $\mathcal{A}$ has 5 points, it can be only consists of diameter $D$ and triangle $T$ and at least one of those have $\{1\}$ in it. Thus every $s$ is divisible by $2^{k(s)} 3^{\ell(s)}$, where $k, \ell$ are corresponding maximal powers. Theorem of Rédei–de Brujin–Schönberg.

Let us show that all $k(s) = k_0$, all $\ell(s) = \ell_0$.

Take any $s = 2^{k(s)}3^{\ell(s)}M$, where $M : (M, 2) = 1, (M, 3) = 1$. Raise $\mathcal{A}_{2^{k(s)}3^{\ell(s)}M}$ (corresponding to this $s$) to power $M$.

Raising to such power that $M : (M, 2) = 1, (M, 3) = 1$ does not change geometric picture of $D$ and $T$ (maybe rotate), so the power of that Lamprey is again Lamprey.

Hence, **we can see that primitive root** $e^{\frac{2\pi i}{2^{k(s)}3^{\ell(s)}}}$ **is a root of** $A$. So we get $\mathcal{A}_{2^{k(s)}3^{\ell(s)}}$.

Therefore, we can consider only Lampreys built by polynomial $A$ such that $s = 2^k 3^\ell M$ with $M = 1$. Call them for brevity $A_{k,\ell}$.

## 32. Continuation of the proof for $|A| = 5$

Let us have $s_1 = 2^k 3^\ell$ and $s_2 = 2^\kappa 3^\lambda$ among denominators of primitive roots of unity on which $A$ vanishes. In other words $\Phi_{2^k 3^\ell}|A$, $\Phi_{2^\kappa 3^\lambda}|A$, or

$$A(e^{\frac{2\pi i}{2^k 3^\ell}}) = 0, \quad A(e^{\frac{2\pi i}{2^\kappa 3^\lambda}}) = 0\,.$$

Let $(k, \ell) \le (\kappa, \lambda)$, $\kappa = k + r, \lambda = \ell + t$. And we have $\mathcal{A}_{k+r,s+t}, \mathcal{A}_{k,\ell}$.

If we act on $\mathcal{A}_{k+r,\ell+t}$ by raising to $2^r 3^t$ then diameter collapses if $1 \in D$ (if $r > 0$) or triangle collapses if $1 \in T$ $(t > 0)$.
But nothing can collapse as (using Lemma (Algebra versus Geometry))

$$(\mathcal{A}_{k+r,\ell+t})^{2^r 3^t} = \mathcal{A}_{k,r}$$

and the RHS still have diameter and triangle, nothing can collapse.

Let $(k,\ell),(\kappa,\lambda)$, $\kappa=k-r, \lambda=\ell+t$, $r>0, t>0$. Consider

$$(\mathcal{A}_{k-r,\ell+t})^{3^t}=\mathcal{T},\quad (\mathcal{A}_{k,\ell})^{2^r}=\mathcal{T},$$

where $\mathcal{T}$ is some Lamprey that might not have anything to do with $A$. But we have

$$(\mathcal{A}_{k-r,\ell+t})^{3^t}=(\mathcal{A}_{k,\ell})^{2^r}\,.$$

The LHS collapses triangle to one point and (maybe) rotates the diameter (does not rotate if $1\in D$). The RHS collapses diameter and (maybe) rotates triangle.

Point and diameter cannot be qual to point and triangle (our triangle cannot have diameter as one side, as all its degrees are $\pi/3$). Contradiction.

**Figure:**

**Theorem**

*Let $s$ and $L$ have a common divisor. Then $\Phi_s(e^{2\pi i \cdot})$ has the SSV property; equivalently, $\varphi(\xi) = e^{2\pi i \xi} - e^{2\pi i k/s}$ has the SSV property for all $(k, s) = 1$.*

We need to prove that the set where $|P_1| := |\prod_{j=0}^{m} \varphi)L^j\xi)| \le L^{-c_1 m}$ is well-structured: it can be put to $L^{c_2 m}$ intervals each having very small length $L^{-c_3 m}$, $c_3 = c_3(c_1) << c_2$.

$(s, L) \neq 1$

$\Downarrow$

SSV

Let $s$ and $L$ have a common divisor. Then $s = ML_1$, where $L_1 | L^a$ for some $a$ and $(M, L) = 1$. Let also

$$F(x) = \prod_{k \in \kappa} (x - e^{2\pi i k/L^a}),$$

where $\kappa \subset [1, L^a - 1]$ is chosen so that $e^{2\pi i k/L^a}$ runs through all primitive $L_1$-th roots of unity.

Notice that if $F$ has SSV property and $F = \Phi \cdot H$, where $|H| \le C(L)$, then $\Phi$ has SSV property! This is obvious.
The important thing to note is that

$$\Phi_s(x)|F(x^M), \quad \text{that is } F(x^M) = \Phi_s(x)H(x),\ |H| \le C(L)\,.$$

Thus, it is enough to prove that $F = \prod_{k\in\kappa}(x - e^{2\pi ik/L})$, $e^{2\pi ik/L}$ runs through all primitive $L_1$-th roots of unity has SSV property.

For that it is enough to find $G$ such that

$$F \cdot G \text{ has SSV and, } |G| \le C(L)\,.$$

Consider the "complement" of $F$,

$$G(x) = \prod_{k=1, k\notin\kappa}^{L^a-1} (x - e^{2\pi ik/L^a})$$

It follows that $F(x) \cdot G(x) = \frac{x^{L^a}-1}{x-1}$.

Now maybe just consider the case $a = 1$, then it is trivial to see that polynomial $p(x)\frac{x^L-1}{x-1}$ has SSV property:

$$\prod_{j=0}^{m} p(L^j x) =^{telescopic} \frac{x^{L^{m+1}} - 1}{x - 1}.$$

$$\text{if } a > 1 \text{ then} \quad \prod_{j=0}^{m-1} F(x^{L^{aj+b}})G(x^{L^{aj+b}}) = \frac{x^{L^{am+b}} - 1}{x^{L^b} - 1},$$

so the set of $L^{-c_1 m}$ -values of the RHS below

$$\prod_{j=0}^{am-1} F(x^{L^j})G(x^{L^j}) = \prod_{b=0}^{a-1} \frac{x^{L^{am+b}} - 1}{x^{L^b} - 1}$$

lies in at most $aL^{m+a}$ intervals of very small measure $2L^{-c_1 m}$, and we can choose $c_1 >> 1$.

B. Poonen and M. Rubinstein: Number of Intersection Points Made by the Diagonals of a Regular Polygon, SIAM J. Disc. Math. 11 (1998), 135–156.

J. H. Conway and A. J. Jones: Trigonometric Diophantine equations (On vanishing sums of roots of unity), Acta Arith. 30 (1976), 229–240.

# 38. Poonen–Rubinstein: counting DISTINCT points of intersections of diagonals of regular polygon

$I(n) = \binom{n}{4}$ for generic $n$-gon, and for regular polygon with $n$ odd.
For regular polygon with $n$ even see below.
Fact: Never 8 diagonal of any regular polygon can meet–except at the center.

$$\text{Let} \quad \delta_m(n) = 1 \text{ if } n = 0 \ m|n, \quad 0 \text{ otherwise}\,.$$

**Theorem**

*For $n \geq 3$,*

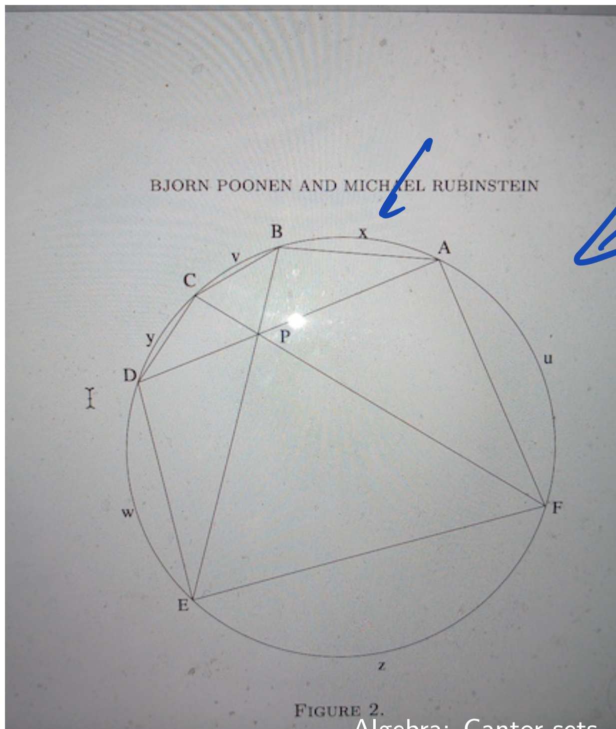
$$I(n) = \binom{n}{4} + (-5n^3 + 45n^2 - 70n + 24)/24 \cdot \delta_2(n) - (3n/2) \cdot \delta_{44}(n)$$

$$+ \ldots .$$

# 39. When 3 diagonals meet?

Let $A, B, C, D, E, F$ be points as on Figure, dividing to arcs $u, x, v, y, w, z$. If diagonals $AD$, $BE$, $CF$ intersects in one point then by similarity of triangles $AF \cdot BC \cdot DE = AB \cdot EF \cdot CD$,

$$\sin\frac{u}{2}\sin\frac{v}{2}\sin\frac{w}{2} = \sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2} \qquad (*)$$



But $u+v+w+x+y+z=2\pi$, putting $u/2 =: \pi U, \dots,$ we have $U+V+W+X+Y+Z=1$. Multiplying (*) through we get 8 terms in LHS, 8 terms in RHS, but 2 terms in LHS cancel 2 terms in *RHS* because of $U+V+W = 1-(X+Y+Z)$.

$$-e^{i\pi(V+W-U)} + -e^{-i\pi(V+W-U)} - \dots = -e^{i\pi(Y+Z-X)} + e^{-i\pi(Y+Z-X)} - \dots$$

Replace $-1 = e^{-i\pi}$. Denote $\alpha_1 = V+W-U-1/2, \alpha_2 = W+U-V-1/2, \alpha_3 = U+V-W-1/2, \alpha_4 = Y+Z-X+1/2, \dots \alpha_6 = \dots$. Then

$$\boxed{\sum_{j=1}^{6} e^{i\pi\alpha_j} + \sum_{j=1}^{6} e^{i\pi\beta_j} = 0}$$

$\beta_j = -\alpha_j$, $\sum_{j=1}^6 \alpha_j = U+V+W+X+Y+Z = 1$, and all $\alpha_j$ are rational, that is all terms above are roots of unity, maybe of huge order.

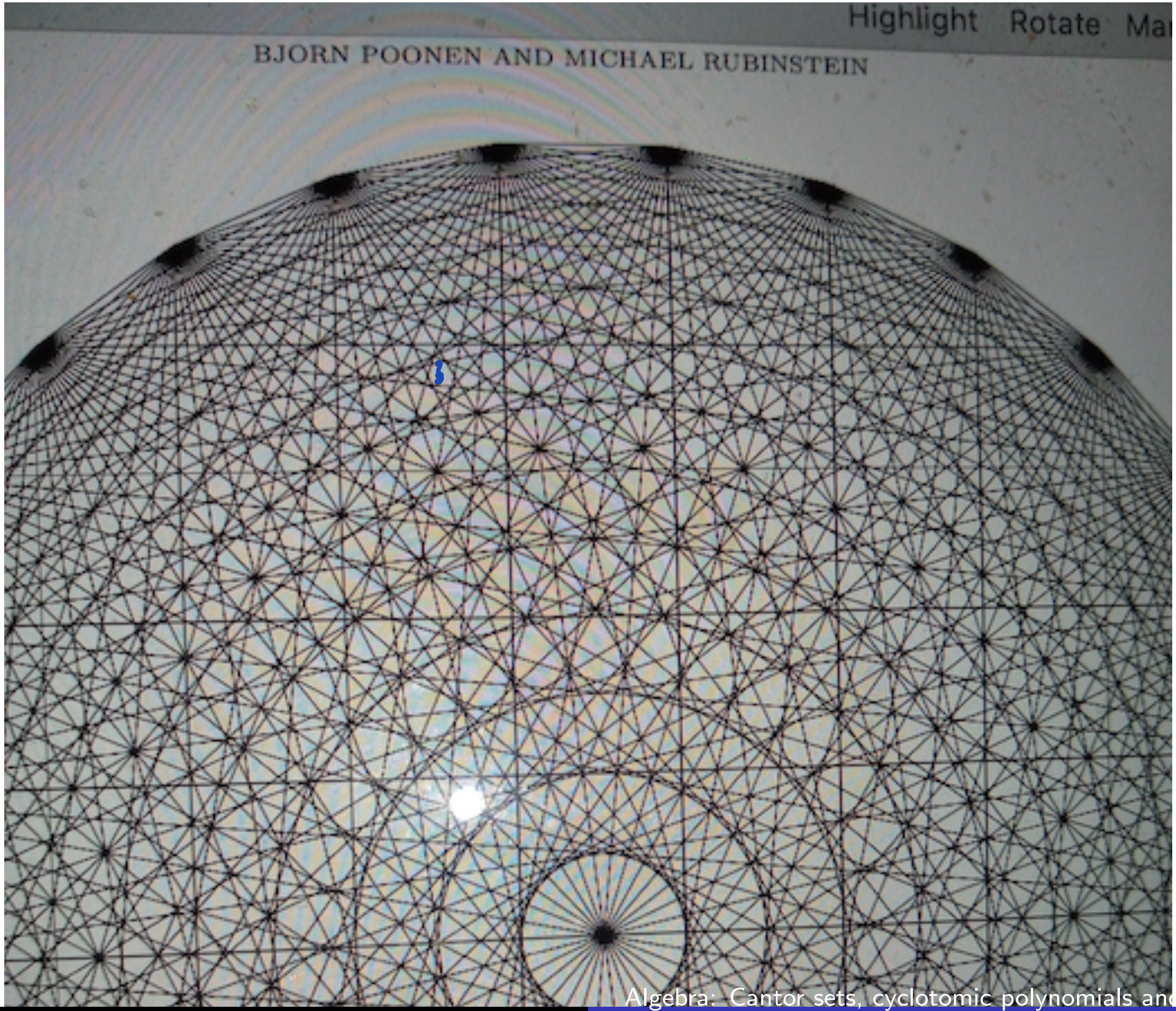
# 41. Minimal lampreys of different cardinality $\leq 12$

$$\sum_{i=1}^{k} n_i \eta_i = 0\,.$$

A relation: $n_i \in \mathbb{Z}_+$, $\eta_i$ distinct roots of unity.
Prime roots are minimal relations: $1 + \zeta_p + \cdots + \zeta_p^{p-1} = 0$ and cannot have a subset with 0 sum.
Schoenberg proved that all relations (with possible negative coef.) can be obtained as linear comb. with **integer** coef. by such $R_p$ relations.
But as here only positive coef. are allowed this becomes false for representing as $R_p$ with **positive integer** coef. Example:

$$\zeta_6 + \zeta_6^{-1} + \zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4 = 0\,.$$

There are 19 such primitive relations with 11 terms and 69 such primitive relations with 12 terms. This is up to rotation.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