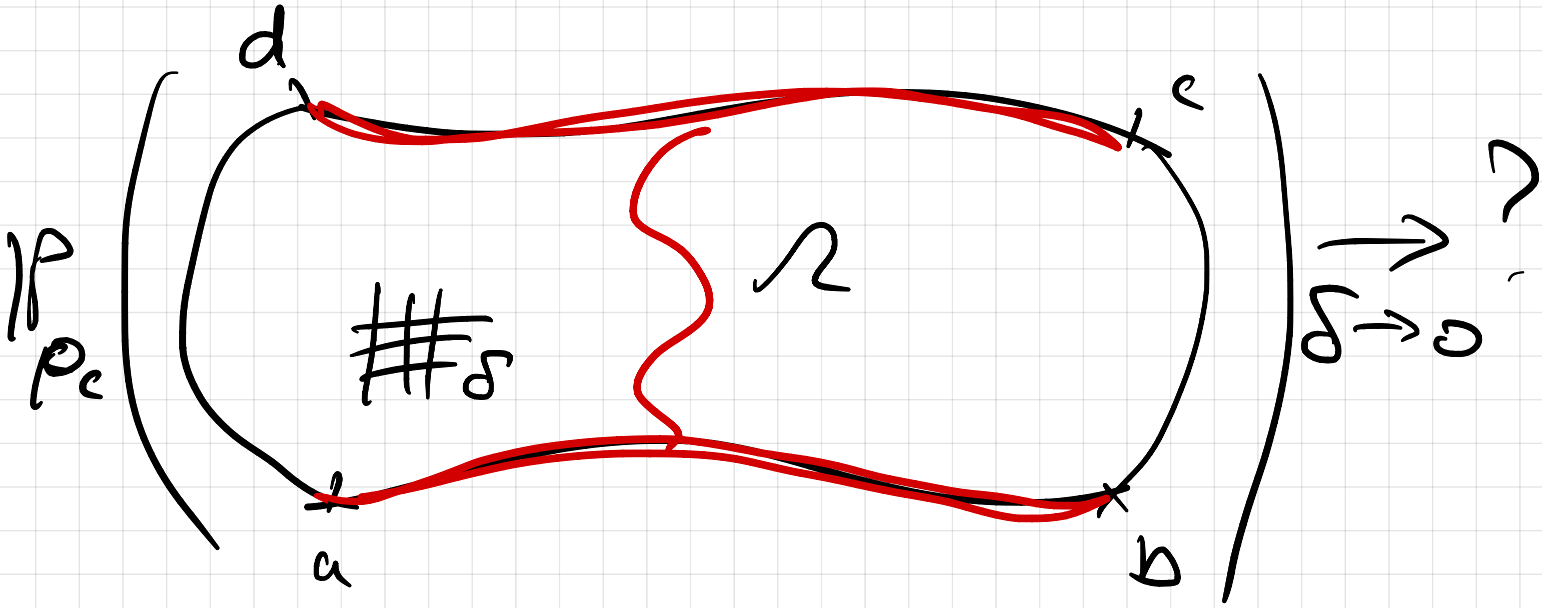
Lecture 7.

11. Conformal invariance.



Aizenman conjecture that the limit is invariant to conformal transformations:

if $\varphi : (\Omega, a, b, c, d) \to (\Omega', a', b', c', d')$, conformal

then the limits are the same for (Ω, a, b, c, d) and $(\Omega', a', b', c', d')$

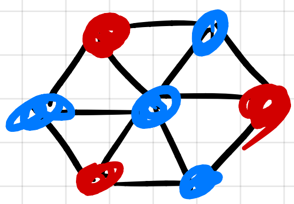
In 1992:

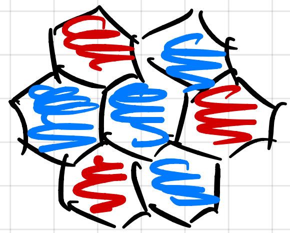
- Langlands – Pouliot – Saint-Aubin: numerical evidence
- Cardy: exact value.

In 2001, Smirnov proved Cardy's formula for the site percolation on the triangular lattice.
[implies also that the interfaces between open/closed sites converge to Schramm-Loewner Evolution - random fractal curves]

Let $\mathbb{T}$ be a triangular lattice
Let $\mathbb{H}$ be the hexagonal lattice dual to $\mathbb{T}$.

Consider a site percolation on $\mathbb{T}$:





Proposition:

The same behavour as in the bond percolation on $\mathbb{Z}^2$:

- sharp phase transition
- $p_c = \frac{1}{2}$
- at p_c: RSW estimates

Thm (Smirnov '2001)

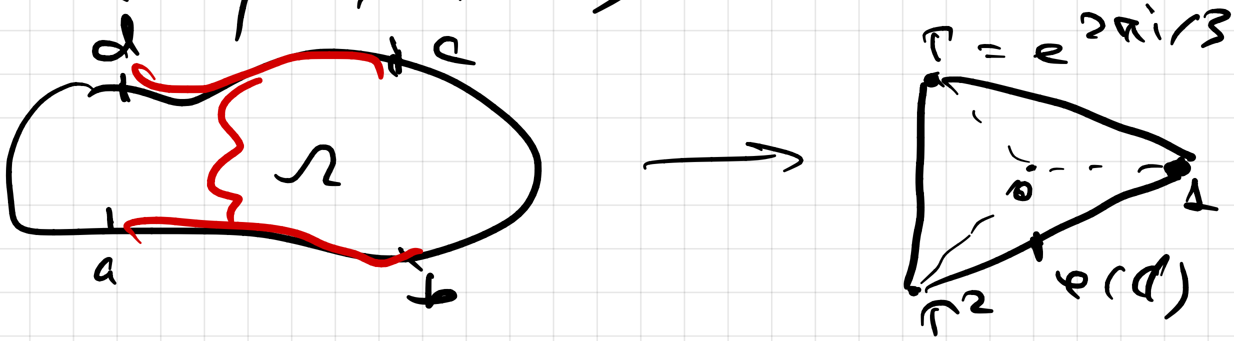
Let $\Omega \subset \mathbb{C}$ be finite, simply-connected, with a smooth bdry.

Pick $a, b, c, d \in \partial\Omega$ (in this cyclic order)

For $\delta > 0$, consider $(\Omega^\delta, a^\delta, b^\delta, c^\delta, d^\delta)$ - approximation of (Ω, a, b, c, d) by $\delta \cdot \mathbb{H}$: $a^\delta, b^\delta, c^\delta, d^\delta$ - centers of faces.

Note that there exits a unique conformal map

$$\varphi : (\Omega, a, b, c) \to (T, 1, \tau, \tau^2)$$



Clearly d is mapped to a boundary point $\varphi(d)$:

$$\varphi(d) = x + \tau^2(1-x),$$

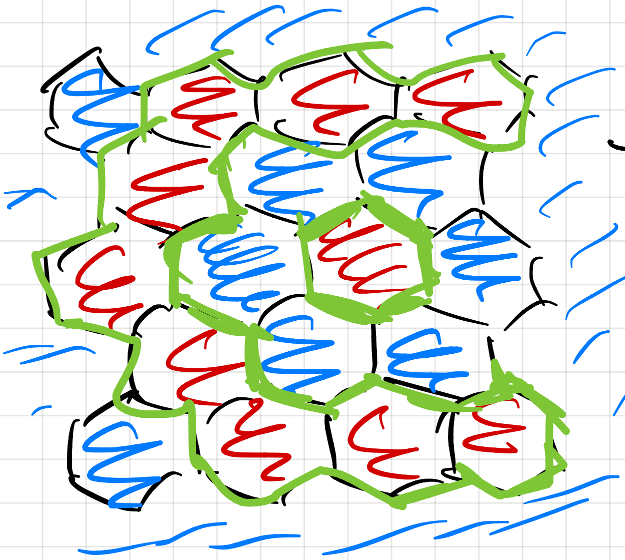
where $x \in [0, 1]$.

Then

$$\boxed{\mathbb{P}_{1/2}\left((a^\delta b^\delta) \to (c^\delta d^\delta)\right) \xrightarrow[\delta \to 0]{} x}$$

$\frac{1}{2}$ - percolation on the faces of $\delta\mathbb{H}$

Proof by Mikhail Khristoforov.



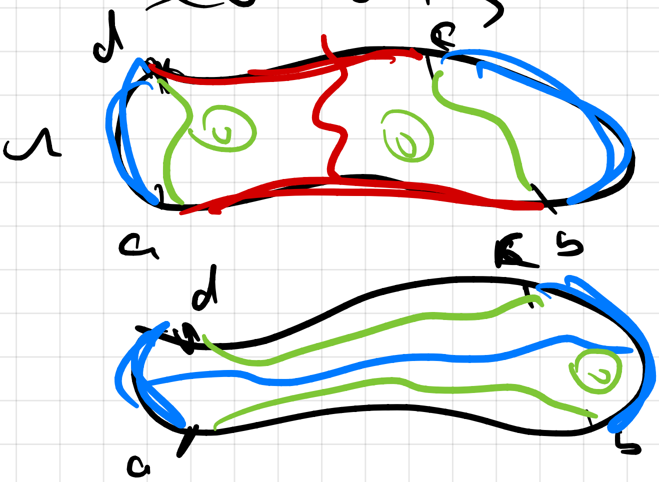
$\mathbb{P}_{1/2}$ – uniform of colorings of the faces into  and 

bijection

boundaries (domain walls) separating blue from red. They have degree 0 or 2 at every vertex — they split into simple cycles (loops)

We have a uniform measure on collections of loops.

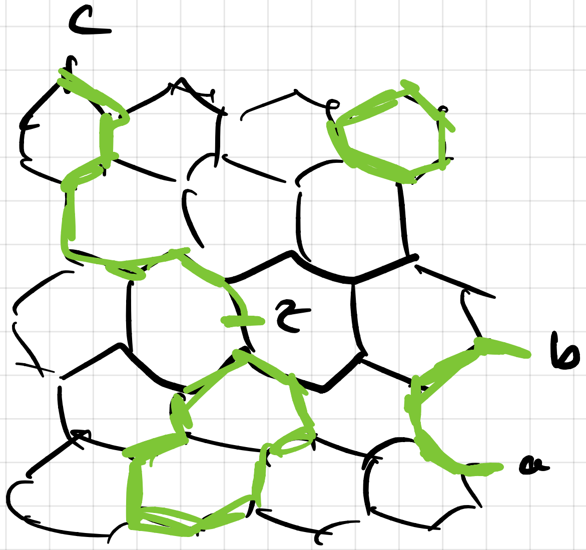
We're interested in crossing events



Consider configurations with loops AND paths: $a \leftrightarrow d$, $c \leftrightarrow b$

Trick:

Consider paths ending inside the domain.



The paths start and end at mid-edges.

Notation:

$\mathcal{E}^{ab,cz}$ - set of all config-s with loops and paths $a \leftrightarrow b$, $c \leftrightarrow z$.

Let $N := \#\{\text{faces in } \Omega\}$

Key in the proof: Smirnov's parafermionic observable

Def

$$F_a(z) := \frac{|\mathcal{E}^{az,bc}|}{2^N}$$

$$F_b(z) := |\mathcal{E}^{bz,ac}| / 2^N$$

$$F_c(z) := |\mathcal{E}^{cz,ab}| / 2^N$$

$2^N = \#\{\text{colorings in blue and red}\}$

Lemma $F_a(z) + F_b(z) + F_c(z) = 1$

Proof:

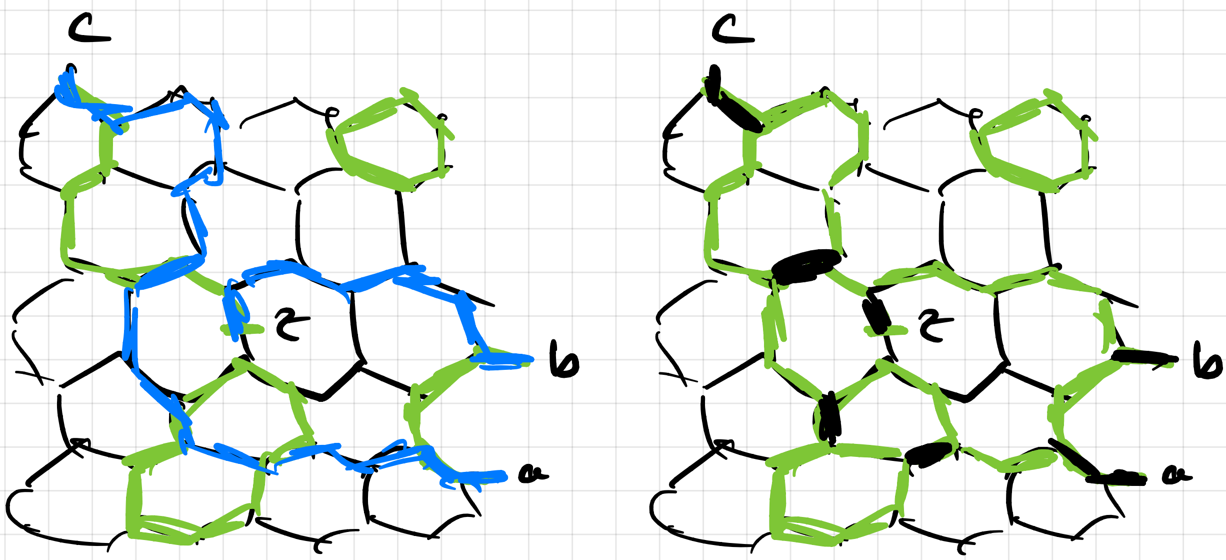
$|\mathcal{E}^{az,bc}| + |\mathcal{E}^{bz,ac}| + |\mathcal{E}^{cz,ab}|$

$= \#\{$config. with loops and 2 paths with endpoints at $a, b, c, z\}$

$\overset{?}{=} |\mathcal{E}^{\emptyset}| = 2^N$

do XOR with any fixed config. ω_0 with endpoints at a, b, c, z.

ω_0 XOR ω



$\blacksquare$

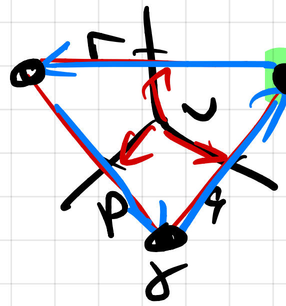
Combine these three projections:

$$F(z) := F_a(z) + \tau \cdot F_b(z) + \tau^2 \cdot F_c(z)$$

parafermionic observable.

(complex-valued)

Proposition (discrete holomorphicity)

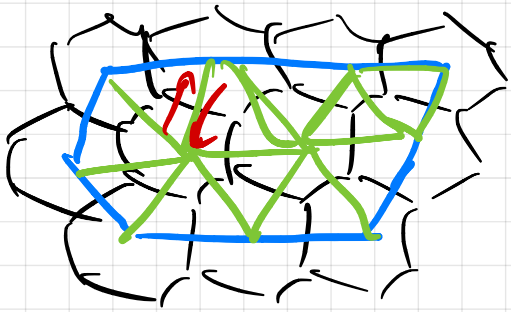


$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0$$

This means that the contour integral over ∇ is 0.

$$\oint_\gamma F(z)\,dz = \sum_i F\left(\frac{\gamma_i + \gamma_{i+1}}{2}\right) \cdot (\gamma_{i+1} - \gamma_i)$$

Any dual contour can be split into triangles:



Write the sum of contour integrals over these triangles.

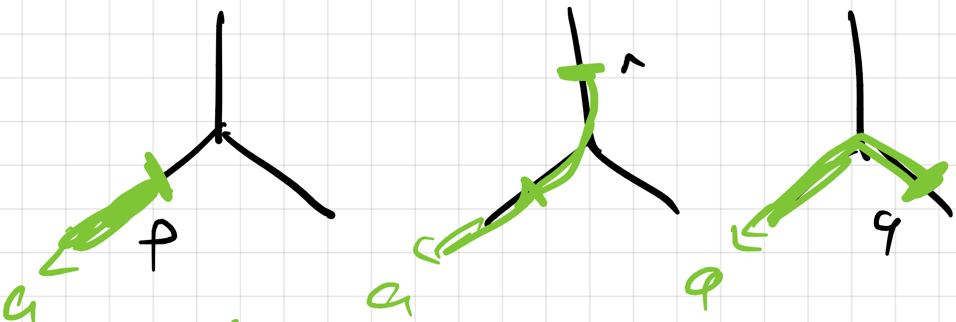
Inner edges will cancel out.

In conclusion:

contour integral over any closed contour is 0.

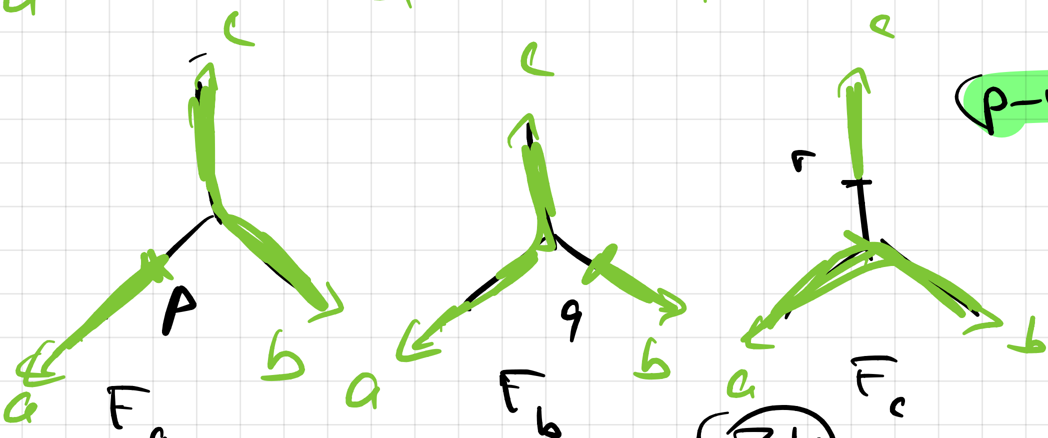
Proof (Proposition):

Split all contig.-s (that contribute) into triples that agree outside of v:



all in $\mathcal{E}^{a?,bc}$ contribution:

$(p-V)+(q-V)+(r-V)=0$.



$(p-V)+T(q-V)+T^2(r-V)=0$

Lemma (Precompactness)

Extend F^δ to the interior of Ω^δ by convexity.

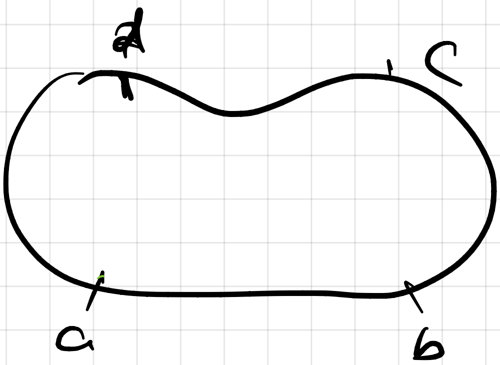
Family of $(F^\delta)_\delta$ has subsequential limits that converge uniformly on every compact subset of Ω.

Proof:

Use RSW (exercise) to prove equicontinuity. $\blacksquare$

Proof (Smirnov's thm)

Let $z \in (c^\delta, a^\delta)$



$F^\delta_b(z) = 0$

Recall that:

$$F^\delta_a(z) + \underbrace{F^\delta_b(z)}_{=0} + F^\delta_c(z) = 1 \quad (*)$$

Hence:

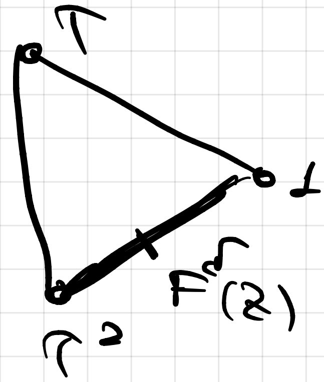
$$F^\delta_a(z) + F^\delta_c(z) = 1$$

$$F^\delta(z) = F_a^\delta(z) + \tau^2 F_c^\delta(z)$$

Then $F^\delta(z) \in [1, \tau^2]$.

Conclusion:

$$F^\delta : (c^\delta a^\delta) \longrightarrow (\tau^2, 1)$$



Similarly:

$$F^\delta : (a^\delta b^\delta) \longrightarrow (1, \tau)$$

$$F^\delta : (b^\delta c^\delta) \longrightarrow (\tau, \tau^2)$$

Overall:

$$F^\delta : \partial \Omega^\delta \to \partial T$$

Moreover, $(*)$ gives that

$$F^\delta : \Omega \longrightarrow T.$$

Consider any subsequential limit of F^δ: function F.

- The above also holds for F.
- Contour integrals of F are 0

Morera's thm:

this implies that F is holomorphic.

$$F : \Omega \to T$$

$$\partial\Omega \to \partial T$$

$$(a, b, c) \mapsto (1, \tau, \tau^2)$$

Then $F = \varphi$ - the unique conformal map

$$(\Omega, a, b, c) \to (T, 1, \tau, \tau^2)$$

Hence F is an actual limit.

What about d^δ:

- it's on $(c^\delta a^\delta)$
- $\underbrace{F_a^\delta(d^\delta) + F_b^\delta(d^\delta)}_{=0} + F_c^\delta(d^\delta) = 1$
- $F^\delta(d^\delta) = F_a^\delta(d^\delta) + \tau^2 F_c^\delta(d^\delta)$

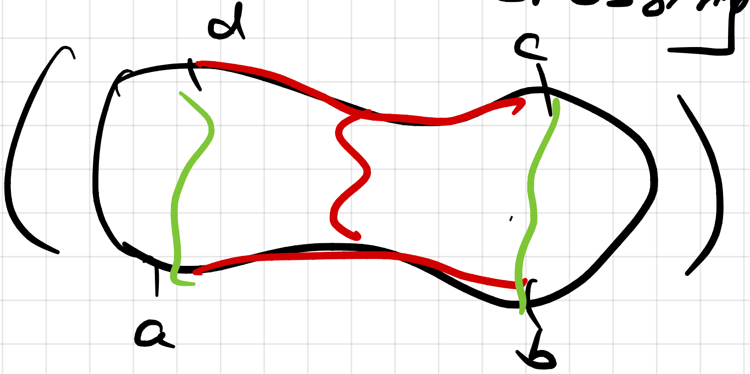
$$\downarrow$$

$$\varphi(d) = x + \tau^2(1 - x) \qquad F_a(d^\delta) \to x$$

77

$$F_a^{\delta}(d^{\delta}) = \frac{|\mathcal{E}^{ad,bc}|}{\underbrace{2^N}_{\underbrace{|\mathcal{E}^{ad,bc}|}_{\text{crossing}} + \underbrace{|\mathcal{E}^{ab,cd}|}_{\text{no crossing}}}}$$

$= \mathbb{P}_{1/2}\Big($

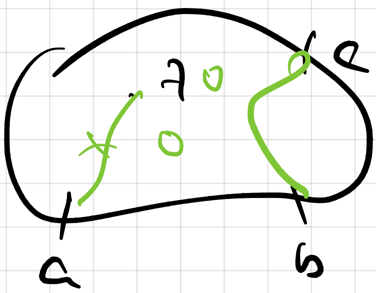


$\Big)$

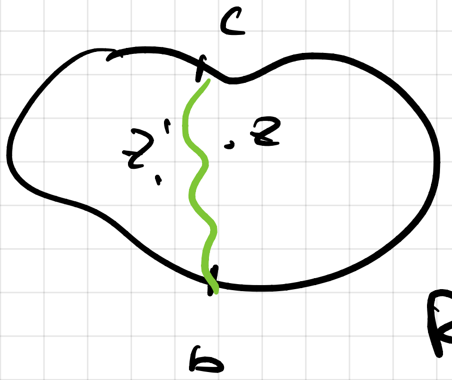
Proof (precompactness):

$$F_a(z) = \frac{|\mathcal{E}^{az,bc}|}{2^N} = \frac{|\mathcal{E}^{az,bc}|}{|\mathcal{E}^{bc}|}$$

$= \mathbb{P}_{1/2}\big($ a and z are not disconnected by $\gamma : b \to c$ $\big)$



Look at $F^\delta(z) - F^\delta(z')$:



$$|F^\delta(z) - F^\delta(z')|$$

$$\le \mathbb{P}(\gamma \text{ disconnects } z \text{ and } z')$$

RSW γ_2

$$\le c \cdot |z - z'|^\alpha$$

(details are missing)

Then (F^δ) are equicontinuous

By Arzela-Ascoli, you can select a convergent subsequence.

$\boxed{\text{RC}}$