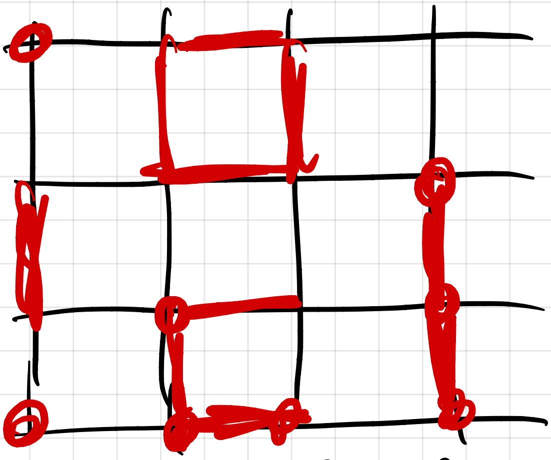
Lecture 8

Chapter II. Fortuin-Kasteleyn percolation

1. Definition and relation to Bernoulli percolation



[random-cluster]

As before: $G=(V,E)\subset \mathbb{Z}^d$ - finite

Config.: $\omega\in\{0,1\}^E$ (0: closed, 1: open)

FK percolation $\underbrace{\text{measure}}_{\text{probability}}$:

$$\varphi_{G,p,q}(\omega)=\frac{1}{Z_{G,p,q}}\,p^{\#\text{open}}(1-p)^{\#\text{closed}}\cdot\underbrace{q^{\#\text{clusters}}}_{\text{non-local}}$$

cluster = connected component

$q=1$: Bernoulli percolation

$q>0$, $p\in[0,1]$

$$Z_{G,p,q}=\sum_{\omega}\text{weight}(\omega)$$

Partition function

[normalizing constant]

How to extend $\varphi_{G,p,q}$ to $\mathbb{Z}^d$?

For percolation:
we used Kolmogorov's thm
(via independence)

Here:
- dependent model
- the measure on $\mathbb{Z}^d$ will be in fact non-unique!

Defining the model on $\mathbb{Z}^d$ is non-trivial and will rely on:

- FKG inequality (Fortuin–Kasteleyn–Ginibre):

$$\varphi_{G,p,q}(A \cap B) \geq \varphi_{G,p,q}(A) \cdot \varphi_{G,p,q}(B)$$

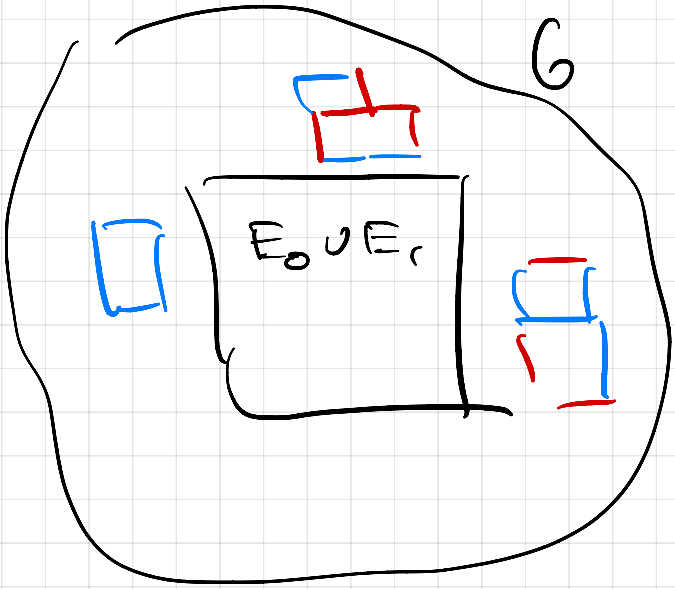
when $q \geq 1$ and $A, B \nearrow$.

- finite energy property (insertion toleration)

Let $E_0, E_1 \subseteq E(\mathbb{Z}^d)$ - disjoint
$|E_0|, |E_1| \leq n$.
Then, there exists $c = c(n) > 0$
s.t. $\varphi_{G,p,q}\left(\omega|_{E_0} = 0, \omega|_{E_1} = 1 \,\middle|\, \omega|_{(E_0 \cup E_1)^c} = T\right) \geq c$

for any $G=(V,E)$ - finite
with $E_0, E_1 \subset E$,
for any $T \in \{0,1\}^{E \setminus (E_0 \cup E_1)}$.

Proof:



Consider any $G=(V,E)$ and any config T on $E \setminus (E_0 \cup E_1)^c$.

This gives a probability measure

Denote it by φ^T.

For any $\omega_1, \omega_2 \in \{0,1\}^{E_0 \cup E_1}$

$$\frac{\varphi^T(\omega_1)}{\varphi^T(\omega_2)} = \frac{\varphi(\omega_1 \mid \omega|_{E\setminus(E_0\cup E_1)^c} = T)}{\varphi(\omega_2 \mid \omega|_{E\setminus(E_0\cup E_1)^c} = T)}$$

$$= \frac{\varphi(\omega_1 \circ T)}{\varphi(\omega_2 \circ T)}$$

$$= p^{o(\omega_1 \circ T) - o(\omega_2 \circ T)} \cdot (1-p)^{c(\omega_1 \circ T) - c(\omega_2 \circ T)} \cdot q^{k(\omega_1 \circ T) - k(\omega_2 \circ T)}$$

$$\boxed{\begin{array}{l} o(\omega) = \#\text{open} \\ c(\omega) = \#\text{closed} \\ k(\omega) = \#\text{clusters} \end{array}}$$

(82)

$$= p^{o(\omega_1)-o(\omega_2)}(1-p)^{c(\omega_1)-c(\omega_2)} \cdot q^{\#\text{clusters in } \omega_1^{o^T} \text{ that intersect } E_0 \cup E_1}$$

$$\overline{\qquad q^{\#\text{clusters in } \omega_2^{o^T} \; - \, " \, -} \qquad}$$

$$\geq \underbrace{p^{|E_0 \cup E_1|} \cdot (1-p)^{|E_0 \cup E_1|} \cdot \min(q, \tfrac{1}{q})^{2|E_0 \cup E_1|}}_{c''(|E_0|, |E_1|)}$$

$$\leq \frac{1}{c}$$

Sum over all ω_1:

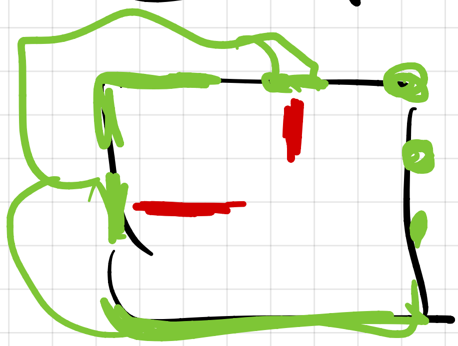
$$\frac{1}{\varphi^T(\omega_2)} \leq \underbrace{\frac{1}{c} \cdot 2^{\# |E_0 \cup E_1|}}_{c'(|E_0|, |E_1|)}$$

$\varphi^T(\omega_2) \geq c'$.

Remark: φ^T can be viewed as an FK percolation measure on $(E_0 \cup E_1)$ under different

boundary conditions

what matters is which boundary points belong to the same cluster of τ.



Def

Let ξ be a partition of ∂G.

Then, define

$$\varphi^{\xi}_{G,p,q}(\omega) := \frac{1}{Z^{\xi}_{G,p,q}}\, p^{o(\omega)}(1-p)^{c(\omega)}\, q^{k(\omega^{\xi})},$$

where ω^{ξ} is obtained from ω by identifying points belonging to the same element of the partition ξ.

Our original definition: ξ consists of singletons.

These are Free boundary conditions:

$\varphi^{0}_{G,p,q}$ free

Another important b.c.: trivial partition, where ∂G is one big group.

Wired boundary conditions:

$\varphi^{1}_{G,p,q}$ wired.

As we will see, $\varphi^{0}_{G,p,q}$ and $\varphi^{1}_{G,p,q}$ can lead to two different measures on $\mathbb{Z}^d$.

[In 2d, in fact each infinite volume (Gibbs) ~~measure~~ is a linear combination of free and wired.]

2. FKG inequality.

Thm (Fortuin-Kasteleyn-Ginibre '71)

Let $q \geq 1$, $p \in [0,1]$. Then,

$$\varphi^{\xi}_{G,p,q}(A \cap B) \geq \varphi^{\xi}_{G,p,q}(A) \cdot \varphi^{\xi}_{G,p,q}(B),$$

for any b.c. ξ and any $A, B \nearrow$.

Below, for simplicity:

$$\varphi = \varphi^{\xi}_{G,p,q}.$$

If $\varphi(B) = 0$, then trivial.
Assume below that

$$\varphi(B) \neq 0.$$

Write

$$\psi(A) = \varphi(A \mid B) = \frac{\varphi(A \cap B)}{\varphi(B)}$$

What we need to prove:

$$\psi(A) \geq \varphi(A),$$

for any increasing event A.

Def:

When this holds one says that ψ stochastically dominates φ.

To prove the domination, we will construct the following coupling $\mathbb{P}$:

- $\mathbb{P}$ on pairs $(\omega, \tilde{\omega}) \in \{0,1\}^E \times \{0,1\}^E$.
- the law of ω is φ
- the law of $\tilde{\omega}$ is ψ
- $\mathbb{P}(\omega \leq \tilde{\omega}) = 1$.

This would imply stochastic domination:

$$\varphi(A) = \mathbb{P}(\omega \in A) = \mathbb{P}(\omega \in A \text{ and } \omega \leq \tilde{\omega})$$

since $A \nearrow$.

$$\leq \mathbb{P}(\tilde{\omega} \in A) = \psi(A).$$

Here is a general construction of $\mathbb{P}$.

Lemma.

Let μ, ν be two probability measures on $\{0,1\}^E$.
Assume they are strictly positive:

$$\forall \omega \quad \mu(\omega), \nu(\omega) > 0.$$

Assume that, for any $e \in E$ and $\tau, \tau' \in \{0,1\}^{E \setminus \{e\}}$ s.t. $\tau \leq \tau'$, we have:

$$\mu\left(\omega_e = 1 \,\middle|\, \omega|_{E \setminus \{e\}} = \tau\right) \leq \nu\left(\omega_e = 1 \,\middle|\, \omega|_{E \setminus \{e\}} = \tau'\right)$$

Then, there exists a coupling $\mathbb{P}$ on $(\omega, \tilde{\omega})$:

- $\omega \sim \mu$
- $\tilde{\omega} \sim \nu$
- $\mathbb{P}(\omega \leq \tilde{\omega}) = 1$.

In particular ν dominates μ.

Proof:

We construct a continuous Markov chain $(\omega^t, \tilde{\omega}^t)$:

- at $t=0$, $\omega_e^0 = \tilde{\omega}_e^0 = 1$ $\forall e \in E$
- step:
  assign to every edge an exponential clock $(X_e) \sim exp(1)$
  When the clock rings,
  we do a step
- for $k \in \mathbb{N}$
  $$U_k \sim U[0,1]$$

All variables are indep.
Consider the k-th time when the clock rings:

- e - edge
- t - time

Let $(\omega^{t-}, \tilde{\omega}^{t-})$ be the config. just before.

$$\omega_e^t = \begin{cases} 1 & \text{if } U_k \leq \mu(\omega_e = 1 \mid \omega = \omega^{t-} \text{ on } E \setminus \{e\}) \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{\omega}_e^t = \begin{cases} 1 & \text{if } U_k \leq \nu(\omega_e = 1 \mid \omega = \tilde{\omega}^{t-} \text{ on } E \setminus \{e\}) \\ 0 & \text{otherwise} \end{cases}$$

(ω^t): continuous Markov chain

- irreducible (μ is strictly positive)
- the invariant measure is unique
- this is μ.

Hence: law of $\omega^t \to \mu$

Similarly, law of $\tilde{\omega}^t \to \nu$.

By our construction:

$\forall t \quad \omega^t \leq \tilde{\omega}^t$

Hence the limit of the joint law of $(\omega^t, \tilde{\omega}^t)$ is the coupling that we wanted

∎

Proof (FKG):

Want to the lemma to φ, and $\underbrace{\varphi(\cdot \mid B)}_{\text{not strictly positive}}$

Plan:

restrict only to the config. for which $\varphi(\cdot \mid B)$ is positive.

Start ~~by~~ $\omega_e^0 = \tilde{\omega}_e^0 = 1$ $\forall e$.

Then we update $(\omega^t, \tilde{\omega}^t)$ as in the lemma

(check the condition!)

We will only consider $\tilde{\omega}^t$, s.t. $\varphi(\tilde{\omega}^t \mid B) > 0$.

the set of such config. is connected

Hence, the law of $\tilde{\omega}_t \to \varphi(\cdot \mid B)$.

For ω: law $\to \varphi$

We will have $\omega^t \leq \tilde{\omega}^t$.

It remains to check the condition of the lemma.

Consider $T, T' \in \{0,1\}^{E \setminus \{e\}}$, s.t. $\varphi(T' \mid B) > 0$.

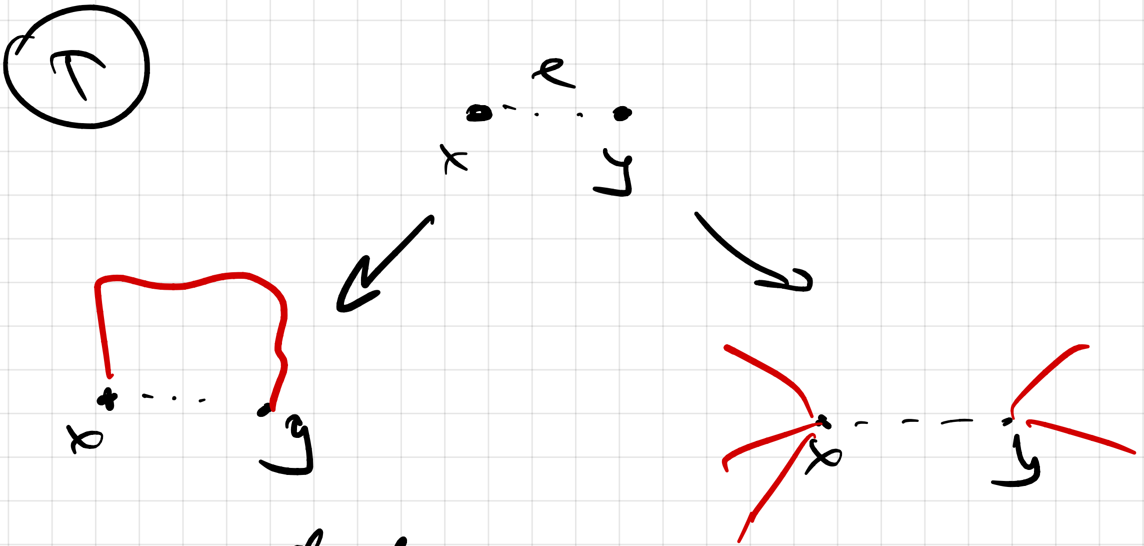
$$\varphi(\omega_e = 1 \mid T', B) =$$

$$= \begin{cases} 1 & \text{if } \omega_e = 1 \text{ is forced by } B \\ \varphi(\omega_e = 1 \mid T') & \text{if } B \text{ is satisfied by } T' \end{cases}$$

First case: clear.

Second case:

show $\varphi(\omega_e = 1 \mid T) \le \varphi(\omega_e = 1 \mid T')$



connected outside

not connected

$\varphi(\omega_e = 1 \mid T) = p$

$\varphi(\omega_e = 1 \mid T) = \dfrac{p}{p + (1-p)q}$

Same for T'.

Crucially:

- $x \overset{T}{\longleftrightarrow} y$ then $x \overset{T'}{\longleftrightarrow} y$
- $q \geq 1$, then $p \geq \dfrac{p}{p+(1-p)q}$

3 cases:

- $x \overset{T}{\not\longleftrightarrow} y$, $x \overset{T'}{\not\longleftrightarrow} y$:

$$\varphi(\omega_e = 1 \mid T) = \varphi(\omega_e = 1 \mid T') = \frac{p}{p+(1-p)q}$$

- $x \overset{T}{\longleftrightarrow} y$, $x \overset{T'}{\longleftrightarrow} y$:

$$\varphi(\omega_e = 1 \mid T) = \varphi(\omega_e = 1 \mid T') = p$$

- $x \overset{T}{\not\longleftrightarrow} y$, $x \overset{T'}{\longleftrightarrow} y$:

$$\varphi(\omega_e = 1 \mid T) = \frac{p}{p+(1-p)q}$$

$$\varphi(\omega_e = 1 \mid T') = p$$

$\blacksquare$